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ANGULAR LIMITS OF DOUBLE LAYER POTENTIALS

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Dedicated to the memory of Professor Jan Mařík

We use the standard notation $\mathbb{R}$ for the real line, $\mathbb{R}^m$ for the Euclidean space of dimension $m \geqslant 2$ equipped with the usual scalar product $x \cdot y = \sum_{i=1}^{m} x_i y_i$ for $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ and the Euclidean norm $|x| = (x \cdot x)^{1/2}$. For $M \subset \mathbb{R}^m$ the symbols ∂M, $\operatorname{cl} M$ and $\operatorname{int} M$ will stand for the boundary, closure and interior of M, respectively. $\mathbb{N}$ is the set of natural numbers. If $k \in \mathbb{N}$, then λ_k denotes the outer k-dimensional Hausdorff measure with its natural normalization (so that $\lambda_k([0,1]^k) = 1$). The open ball with center $z \in \mathbb{R}^m$ and radius $r > 0$ will be denoted by

$$B(z,r) \equiv \{x \in \mathbb{R}^m; |x - z| < r\},$$
$$\sigma_m \equiv \lambda_{m-1}\big(\partial B(0,1)\big) = 2\pi^{\frac{1}{2}m}/\Gamma(\tfrac{1}{2}m)$$

is the area of the unit sphere in $\mathbb{R}^m$. If $z \in \mathbb{R}^m$ is fixed, then the fundamental harmonic function with pole at z is given by

$$h_z(x) \equiv \frac{1}{(m-2)\sigma_m}|x - z|^{2-m}, \quad x \in \mathbb{R}^m \setminus \{z\}$$

if $m > 2$, while for $m = 2$

$$h_z(x) \equiv \frac{1}{2\pi}\ln|x - z|^{-1}, \quad x \in \mathbb{R}^2 \setminus \{z\}.$$

For $M \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$ the upper density of M at x is defined by

$$\bar{d}(M,x) \equiv \limsup_{r \downarrow 0} \lambda_m\big(M \cap B(x,r)\big)/\lambda_m\big(B(x,r)\big).$$

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The so-called essential boundary of M will be denoted by

$$\partial_e M \equiv \{x \in \mathbb{R}^m;\ \bar{d}(M,x) > 0, \bar{d}(\mathbb{R}^m \setminus M, x) > 0\}.$$

If $U \subset \mathbb{R}^m$ is open, then $\mathscr{C}_0^{(1)}(U)$ is the class of all continuously differentiable functions ψ with a compact support $\operatorname{spt}\psi$ contained in U. We will fix a Borel set $A \subset \mathbb{R}^m$ with a compact boundary and put $G = \mathbb{R}^m \setminus A$. Let

$$\mathscr{C}^{(1)}(\partial A) \equiv \{f|\partial A;\ f \in \mathscr{C}_0^{(1)}(\mathbb{R}^m)\}$$

be the class of all restrictions to ∂A of functions in $\mathscr{C}_0^{(1)}(\mathbb{R}^m)$. Given $f \in \mathscr{C}^{(1)}(\partial A)$ and $z \in \mathbb{R}^m \setminus \partial A$ we choose a $\varphi_f \in \mathscr{C}_0^{(1)}(\mathbb{R}^m)$ such that

$$z \notin \operatorname{spt}\varphi_f, \quad \varphi_f|\partial A = f$$

and define (compare [1], [7], [11], [15])

$$W^A f(z) \equiv \int_G \operatorname{grad}\varphi_f(x) \cdot \operatorname{grad} h_z(x)\ \mathrm{d}\lambda_m(x).$$

It is easily verified that this quantity does not depend on the choice of φ_f with the above properties (cf. [11], Lemma 2.1). The function

$$W^A f \equiv z \mapsto W^A f(z)$$

is harmonic on $\mathbb{R}^m \setminus \partial A$ (cf. [11], Lemma 2.4) and will be called the double layer potential of the density f. If $\varphi \in \mathscr{C}_0^{(1)}(\mathbb{R}^m)$ and $z \in \mathbb{R}^m \setminus \operatorname{spt}\varphi$, then

$$\int_{\mathbb{R}^m} \operatorname{grad}\varphi(x) \cdot \operatorname{grad} h_z(x)\ \mathrm{d}\lambda_m(x) = 0,$$

which shows that $W^A f(z) = -W^G f(z)$. Since one of the sets A, G is bounded, we may assume without loss of generality that G is bounded when we investigate $W^A f \equiv Wf$. If 1_M denotes the constant function equal to 1 on $M \subset \mathbb{R}^m$, then easy calculation shows that

$$W 1_{\partial A}(x) = \begin{cases} -1 & \text{for } x \in \operatorname{int} G, \\ 0 & \text{for } x \in \operatorname{int} A \end{cases} \tag{1}$$

(compare [10], p. 21). We will fix a point $\eta \in \partial A$ and a lower-semicontinuous function $q\colon \partial A \to [0, +\infty]$ which is bounded and strictly positive on $\partial A \setminus \{\eta\}$. Let $\mathscr{C}(\partial A, q)$ be the linear space of all continuous functions $f\colon \partial A \to \mathbb{R}$ satisfying the condition

$$f(\xi) - f(\eta) = o(q(\xi)) \quad \text{as} \quad \xi \to \eta,\ \xi \in \partial A.$$

We define the norm in $\mathscr{C}(\partial A, q)$ by

$$\|f\|_q = \max\left\{ \sup_{\xi\in\partial A\setminus\{\eta\}} \frac{|f(\xi)-f(\eta)|}{q(\xi)}, \sup_{\xi\in\partial A} |f(\xi)| \right\}.$$

In the subspace $\mathscr{C}_0(\partial A, q) = \{f \in \mathscr{C}(\partial A, q); f(\eta) = 0\}$ this norm is equivalent to

$$\|f\|_{q,0} = \sup_{\xi\in\partial A\setminus\{\eta\}} \frac{|f(\xi)|}{q(\xi)}.$$

Clearly, $\mathscr{C}(\partial A, q)$ and $\mathscr{C}_0(\partial A, q)$ (when normed as shown above) are Banach spaces.

We will see below (in Lemma 1) that

$$\int_{\partial_e A} q \, \mathrm{d}\lambda_{m-1} < \infty \tag{2}$$

is a necessary and sufficient condition for the operator $W^A \equiv W\colon f \mapsto Wf$ to be continuous from $\mathscr{C}^{(1)}(\partial A) \cap \mathscr{C}(\partial A, q)$ into the space of all harmonic functions on $\mathbb{R}^m \setminus \partial A$ equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}^m \setminus \partial A$. We will always assume (2) which permits to extend the operator W continuously to the whole space $\mathscr{C}(\partial A, q)$. For any $f \in \mathscr{C}(\partial A, q)$ we thus have the corresponding double layer potential Wf which is a harmonic function on $\mathbb{R}^m \setminus \partial A$. We will be engaged in the existence of angular limits of double layer potentials Wf at η. In order to be able to formulate some sample results we adopt the following notation. The contingent of a set $M \subset \mathbb{R}^m$ at a point $\xi \in \mathbb{R}^m$ (cf. [16], chap. IX, §2), to be denoted by $\operatorname{contg}(M, \xi)$, consists of all half-lines

$$H(\xi, \theta) = \{\xi + t\theta; t > 0\} \tag{3}$$

for which there exists a sequence of points $z_n \in M \setminus \{\xi\}$ such that

$$\lim_{n\to\infty} \frac{z_n - \xi}{|z_n - \xi|} = \theta.$$

Introducing for $\theta \in \partial B(0,1)$ and $\xi \in \mathbb{R}^m$ the sum

$$n^q(\theta, \xi) \equiv \sum q(x), \quad x \in \partial_e A \cap H(\xi, \theta), \tag{4}$$

we obtain a non-negative extended real-valued function of the variable $\theta \in \partial B(0,1)$ which is λ_{m-1}-measurable (cf. Lemma 3 below), so that we may put

$$v^q(\xi) = \frac{1}{\sigma_m} \int_{\partial B(0,1)} n^q(\theta, \xi) \, \mathrm{d}\lambda_{m-1}(\theta). \tag{5}$$

Let

$$L(\xi,\omega) = \{\xi\} \cup H(\xi,\omega) \cup H(\xi,-\omega) \tag{6}$$

be the line of direction $\omega \in \partial B(0,1)$ passing through the point $\xi \in \mathbb{R}^m$. If $\omega \in \partial B(0,1)$ is fixed then for any open set $U \subset \mathbb{R}^m$ the sum

$$n^q(\omega,\xi;U) = \sum q(x), \quad x \in U \cap \partial_e A \cap L(\xi,\omega), \tag{7}$$

is a λ_{m-1}-measurable extended real-valued function of the variable ξ on $\{\xi \in \mathbb{R}^m;\ \xi \cdot \omega = 0\} \equiv N(\omega)$ and we may define

$$\mu^q_\omega(U) = \int_{N(\omega)} n^q(\omega,\xi;U)\ \mathrm{d}\lambda_{m-1}(\xi). \tag{8}$$

With this notation we may formulate the following result.

Theorem 1. *Let $S \subset \mathbb{R}^m \setminus \partial A$ be a connected set, $\eta \in \operatorname{cl} S \cap \partial A$,*

$$\operatorname{contg}(\partial A,\eta) \cap \operatorname{contg}(S,\eta) = \emptyset.$$

If $\theta \in \partial B(0,1)$ and

$$\left|\theta - \frac{(z-\eta)}{|z-\eta|}\right| = O(|z-\eta|^{m-1}) \quad \text{as} \quad z \to \eta,\ z \in S,$$

then a necessary and sufficient condition for the existence of a finite limit

$$\lim_{\substack{z\to\eta\\ z\in S}} Wf(z) \tag{9}$$

for every $f \in C(\partial A, q)$ consists in

$$v^q(\eta) + \sup_{r>0} r^{1-m}\mu^q_\theta\big(B(\eta,r)\big) < \infty. \tag{10}$$

Defining further

$$\mu^q(M) = \int_{M\cap\partial_e A} q\ \mathrm{d}\lambda_{m-1} \tag{11}$$

for any Borel set $M \subset \mathbb{R}^m$ we have also the following result.

Theorem 2. *Suppose that $S_j \subset \mathbb{R}^m \setminus \partial A$ are connected sets such that*

$$\eta \in \operatorname{cl} S_j \cap \partial A, \quad \lim_{\substack{z\to\eta\\ z\in S_j}} (z-\eta)/|z-\eta| = \theta_j \quad (j = 1,\dots,m).$$

If the vectors $\theta_1,\dots,\theta_m$ are linearly independent, then the condition

$$v^q(\eta) + \sup_{r>0} r^{1-m}\mu^q\big(B(\eta,r)\big) < \infty \tag{12}$$

is necessary for the existence of finite limits

$$\lim_{\substack{z\to\eta\\ z\in S_j}} Wf(z) \quad (j = 1,\dots,m) \tag{13}$$

for all $f \in C(\partial A, q)$; if

$$\operatorname{contg}(\partial A,\eta) \cap \operatorname{contg}(S_j,\eta) = \emptyset \quad (j = 1,\dots,m),$$

then (12) *is also sufficient.*

Proofs of these theorems depend on a series of auxiliary results. We will extend q from ∂A to $\mathbb{R}^m$ defining $q = \sup q(\partial A)$ on $\mathbb{R}^m \setminus \partial A$; thus $q \geqslant 0$ is bounded and lower-semicontinuous on $\mathbb{R}^m$, $q > 0$ on $\mathbb{R}^m \setminus \{\eta\}$. If $\emptyset \neq U \subset \mathbb{R}^m$ is open, then $\mathscr{H}(U)$ denotes the space of all harmonic functions on U equipped with the topology of uniform convergence on compact subsets of U.

Lemma 1. *The operator $W\colon f \mapsto Wf$ acting from $\mathscr{C}^{(1)}(\partial A) \cap \mathscr{C}(\partial A, q)$ into $\mathscr{H}(\mathbb{R}^m \setminus \partial A)$ is continuous (where $\mathscr{C}(\partial A, q)$ is equipped with the norm $\|\dots\|_q$) iff* (2) *holds.*

Proof. Assuming (2) we conclude that, for each bounded open set $U \subset \mathbb{R}^m$ with $\operatorname{cl} U \subset \mathbb{R}^m \setminus \{\eta\}$, $\lambda_{m-1}(U \cap \partial_e G) < \infty$. This implies that G has locally finite perimeter in $\mathbb{R}^m \setminus \{\eta\}$ (cf. [5], chap. 4 and [19], section 5.8) which means that the distributional first order partial derivatives of the indicator function χ_G of G are locally in $\mathbb{R}^m \setminus \{\eta\}$ representable by signed Radon measures. Let us recall that a unit vector $n \in \partial B(0,1)$ is termed Federer's exterior normal of G at $\xi \in \partial G$ if the symmetric difference of G and the half-space $P_n(\xi) = \{x \in \mathbb{R}^m; (x-\xi)\cdot n < 0\}$ has vanishing density at ξ:

$$\bar{d}(G \setminus P_n(\xi), \xi) = 0 = \bar{d}(P_n(\xi) \setminus G, \xi).$$

Obviously, there is at most one vector $n \in \partial B(0,1)$ with this property which will then be denoted by $n^G(\xi)$; we put $n^G(\xi) = 0$ ($\in \mathbb{R}^m$) if no Federer's normal of G at ξ exists in the above described sense. Then

$$\widehat{\partial G} = \{\xi \in \partial G; |n^G(\xi)| = 1\}$$

is a Borel set (cf. [4]) which is called the reduced boundary of ∂G. Clearly, $\widehat{\partial G} = \widehat{\partial A} \subset \partial_e A = \partial_e G$, $n^G = -n^A$. Since G has locally finite perimeter in $\mathbb{R}^m \setminus \{\eta\}$,

$$\lambda_{m-1}(\partial_e G \setminus \widehat{\partial G}) = 0 \tag{14}$$

and for any vector-valued function $v = (v_1, \dots, v_m)$ with components $v_j \in \mathscr{C}_0^{(1)}(\mathbb{R}^m \setminus \{\eta\})$ the divergence formula holds

$$\int_{\widehat{\partial G}} v \cdot n^G \, \mathrm{d}\lambda_{m-1} = \int_G \Big(\sum_{j=1}^m \partial_j v_j\Big) \, \mathrm{d}\lambda_{m-1}, \tag{15}$$

where ∂_j denotes the partial derivative with respect to the j-th variable. Using this formula we get for $g \in \mathscr{C}^{(1)}(\partial A)$ vanishing near η and any $z \in \mathbb{R}^m \setminus \partial A$

$$Wg(z) = \int_{\widehat{\partial G}} g n^G \cdot \operatorname{grad} h_z \, \mathrm{d}\lambda_{m-1},$$

and this formula extends easily to any $g \in \mathscr{C}^{(1)}(\partial A) \cap \mathscr{C}_0(\partial A, q)$. Using (1) we conclude that for any $f \in \mathscr{C}^{(1)}(\partial A) \cap \mathscr{C}(\partial A, q)$,

$$Wf(z) = -f(\eta)\chi_G(z) + \int_{\widehat{\partial G}} [f(\xi) - f(\eta)] n^G(\xi) \cdot \operatorname{grad} h_z(\xi) \, \mathrm{d}\lambda_{m-1}(\xi). \tag{16}$$

Denoting by

$$\operatorname{dist}(z, M) = \inf \{|z - \xi|; \xi \in M\}$$

the distance from z to $M \subset \mathbb{R}^m$ we get

$$\begin{aligned} |Wf(z)| &\leqslant |f(\eta)| + [\operatorname{dist}(z, \partial G)]^{1-m} \cdot \|f\|_q \cdot \frac{1}{\sigma_m} \int_{\widehat{\partial G}} q \, \mathrm{d}\lambda_{m-1} \\ &\leqslant \|f\|_q \Big\{1 + \big[\operatorname{dist}(z, \partial G)\big]^{1-m} \cdot \frac{1}{\sigma_m} \int_{\widehat{\partial G}} q \, \mathrm{d}\lambda_{m-1}\Big\} \end{aligned} \tag{17}$$

which shows that W maps $\mathscr{C}^{(1)}(\partial A) \cap \mathscr{C}(\partial A, q)$ continuously into $\mathscr{H}(\mathbb{R}^m \setminus \partial A)$.

Conversely, let W act continuously from $\mathscr{C}^{(1)}(\partial A) \cap \mathscr{C}(\partial A, q)$ into $\mathscr{H}(\mathbb{R}^m \setminus \partial A)$. Using the argument from [11] (cf. the proof of Thm. 2.12) we shall first show that G

has locally finite perimeter in $\mathbb{R}^m \setminus \{\eta\}$. For this purpose we fix affinely independent points $z^1, \dots, z^{m+1} \in \mathbb{R}^m \setminus \partial A$. There is a $c \in [0, \infty[$ such that

$$(18) \qquad f \in \mathscr{C}^{(1)}(\partial A) \cap \mathscr{C}_0(\partial A, q) \implies |Wf(z^k)| \leqslant c\|f\|_{q,0}, \quad 1 \leqslant k \leqslant m+1.$$

Consider an arbitrary bounded open set U with $\operatorname{cl} U \subset \mathbb{R}^m \setminus \{\eta\}$ and denote by $\partial_\theta = \theta \cdot \operatorname{grad}$ the derivative in the direction of $\theta \in \partial B(0,1)$. We wish to show that, for any fixed $\theta \in \partial B(0,1)$,

$$\sup\left\{ \int_G \partial_\theta \psi \, \mathrm{d}\lambda_m \,;\, \psi \in \mathscr{C}_0^{(1)}(U),\ |\psi| \leqslant 1 \right\} < \infty.$$

Let Π_j be the hyperplane containing all points in $\{z^k; k \neq j\}$ and notice that $\bigcup_{j=1}^{m+1} (\mathbb{R}^m \setminus \Pi_j) = \mathbb{R}^m$. There are infinitely differentiable functions α_j with compact supports $\operatorname{spt} \alpha_j \subset \mathbb{R}^m \setminus (\{\eta\} \cup \Pi_j)$ $(j = 1, \dots, m+1)$ which form a decomposition of unity near $\operatorname{cl} U$ in the sense that

$$\alpha \equiv \sum_{j=1}^{m+1} \alpha_j$$

equals 1 in a neighborhood of $\operatorname{cl} U$. We then have

$$\int_G \partial_\theta \psi \, \mathrm{d}\lambda_m = \int_G \alpha(x) \partial_\theta \psi(x) \, \mathrm{d}\lambda_m(x)$$

for $\psi \in \mathscr{C}_0^{(1)}(U)$ so that it suffices to verify

$$(19) \qquad \sup\left\{ \int_G \alpha_j(x) \partial_\theta \psi(x) \, \mathrm{d}\lambda_m(x) \,;\, \psi \in \mathscr{C}_0^{(1)}(U),\ |\psi| \leqslant 1 \right\} < \infty$$

for $j = 1, \dots, m+1$. We will do this, e.g., for $j = m+1$. Fix $\psi \in \mathscr{C}_0^{(1)}(U)$ such that $|\psi| \leqslant 1$. For $x \in \operatorname{spt} \alpha_{m+1}$ the vectors $x - z^1, \dots, x - z^m$ are linearly independent so that, in a neighborhood of $\operatorname{spt} \alpha_{m+1}$,

$$\theta = \sum_{k=1}^{m} a_k(x) \operatorname{grad} h_{z^k}(x)$$

with infinitely differentiable functions $a_k(x)$. Hence

$$\int_G \alpha_{m+1} \partial_\theta \psi \, \mathrm{d}\lambda_m = \sum_{k=1}^{m} \int_G \alpha_{m+1}(x) a_k(x) \operatorname{grad} \psi(x) \cdot \operatorname{grad} h_{z^k}(x) \, \mathrm{d}\lambda_m(x).$$

Fix $k \in \{1,\dots,m\}$ and put $F(x) = \alpha_{m+1}(x)a_k(x)$. Then $F \in \mathscr{C}_0^{(1)}(\mathbb{R}^m \setminus \{z^k\})$ and

$$\begin{aligned}\int_G F(x)\,\mathrm{grad}\,\psi(x)\cdot \mathrm{grad}\,h_{z^k}(x)\;\mathrm{d}\lambda_m(x)\\ = \int_G \mathrm{grad}\,\big(F(x)\psi(x)\big)\cdot \mathrm{grad}\,h_{z^k}(x)\;\mathrm{d}\lambda_m(x)\\ - \int_G \psi(x)\,\mathrm{grad}\,F(x)\cdot \mathrm{grad}\,h_{z^k}(x)\;\mathrm{d}\lambda_m(x).\end{aligned}$$

Clearly,

$$\begin{aligned}\left|\int_G \psi(x)\,\mathrm{grad}\,F(x)\cdot \mathrm{grad}\,h_{z^k}(x)\;\mathrm{d}\lambda_m(x)\right|\\ \leqslant \frac{1}{\sigma_m}\int_G |\,\mathrm{grad}\,F(x)|\cdot|x-z^k|^{1-m}\,\mathrm{d}\lambda_m(x) < \infty.\end{aligned}$$

Noting that q is strictly positive on $\partial A \cap \mathrm{spt}\,\alpha_{m+1} \supset \partial A \cap \mathrm{spt}\,F$ we can choose $a \in [0,\infty[$ such that $|F| \leqslant aq$ on ∂A and (18) yields

$$\left|\int_G \mathrm{grad}\,\big(F(x)\psi(x)\big)\cdot \mathrm{grad}\,h_{z^k}(x)\;\mathrm{d}\lambda_m(x)\right| \leqslant ac.$$

Thus (19) has been verified for $j = m+1$ and, of course, the same reasoning works also for $j = 1,\dots,m$. We now know that G has locally finite perimeter in $\mathbb{R}^m \setminus \{\eta\}$ and, consequently, (16) holds for $f \in \mathscr{C}^{(1)}(\partial A) \cap \mathscr{C}(\partial A, q)$ vanishing near η, $z \in \mathbb{R}^m \setminus \partial A$. It follows easily that

$$(20)\quad \begin{aligned}\int_{\widehat{\partial G}} q(\xi)\big|n^G(\xi)\cdot \mathrm{grad}\,h_z(\xi)\big|\;\mathrm{d}\lambda_{m-1}(\xi)\\ = \sup\big\{Wf(z);\ f \in \mathscr{C}^{(1)}(\partial A) \cap \mathscr{C}_0(\partial A, q),\ \|f\|_{q,0} \leqslant 1\big\},\quad z \in \mathbb{R}^m \setminus \partial A.\end{aligned}$$

Since the points $z^1,\dots,z^{m+1}$ are affinely independent, there is a constant $b \in\,]0,\infty[$ such that

$$\sum_{k=1}^{m+1}\big|n^G(\xi)\cdot \mathrm{grad}\,h_{z^k}(\xi)\big| \geqslant b,\quad \xi \in \partial A.$$

Combining this with (18), (20) (where we set $z = z^k$, $k = 1,\dots,m+1$) we arrive at

$$\int_{\widehat{\partial G}} q\;\mathrm{d}\lambda_{m-1} \leqslant b^{-1}\sum_{k=1}^{m+1}\int_{\widehat{\partial G}} q|n^G\cdot \mathrm{grad}\,h_{z^k}|\;\mathrm{d}\lambda_{m-1} \leqslant b^{-1}(m+1)c,$$

which together with (14) proves (2). □

Remark 1. In what follows we always assume (2) which guarantees that G has locally finite perimeter in $\mathbb{R}^m \setminus \{\eta\}$ and (14) holds. The operator W extends, by continuity, from $\mathscr{C}^{(1)}(\partial A) \cap \mathscr{C}(\partial A, q)$ to $\mathscr{C}(\partial A, q)$. For $f \in \mathscr{C}(\partial A, q)$ the corresponding function $Wf \in \mathscr{H}(\mathbb{R}^m \setminus \partial A)$ is given by (16) for $z \in \mathbb{R}^m \setminus \partial A$.

Notation. We fix an infinitely differentiable function φ on $\mathbb{R}$ with $\operatorname{spt} \varphi \subset]-1, 1[$ such that

$$\int_{\mathbb{R}} \varphi \, \mathrm{d}\lambda_1 = 1, \quad \varphi(-r) = \varphi(r), \quad r \in \mathbb{R},$$

and define for each locally integrable function $g\colon \mathbb{R} \to \mathbb{R}$ and each $n \in \mathbb{N}$

$$A_n g(t) = n \int_{\mathbb{R}} g(t-r)\varphi(nr) \, \mathrm{d}\lambda_1(r).$$

Then $A_n g$ is infinitely differentiable and for each integrable function ψ with compact support in $\mathbb{R}$ we have

$$\int_{\mathbb{R}} \psi A_n g \, \mathrm{d}\lambda_1 = \int_{\mathbb{R}} g A_n \psi \, \mathrm{d}\lambda_1.$$

Suppose now that Z is a non-void set. For each $f\colon \mathbb{R} \times Z \to \mathbb{R}$ and each $z \in Z$ define $f_z\colon \mathbb{R} \to \mathbb{R}$ by

$$f_z(t) = f(t, z), \quad t \in \mathbb{R}.$$

If f_z is locally integrable for each $z \in Z$ then we define $A_n f\colon \mathbb{R} \times Z \to \mathbb{R}$ by

$$(A_n f)_z = A_n f_z, \quad z \in Z, \ n \in \mathbb{N}.$$

If the finite derivative $(f_z)'$ exists on $\mathbb{R}$ for each $z \in Z$ then ∂f denotes the partial derivative on $\mathbb{R} \times Z$ given by

$$(\partial f)_z = (f_z)', \quad z \in Z.$$

Assuming that $\partial f\colon \mathbb{R} \times Z \to \mathbb{R}$ is well defined as above and, for each $z \in Z$, both f_z and $(f_z)'$ are locally integrable on $\mathbb{R}$, we have

$$A_n \partial f = \partial A_n f, \quad n \in \mathbb{N}.$$

Let now $\mathbf{A}$ be a σ-algebra of subsets in Z and $\mathbf{B}$ the σ-algebra of Borel sets in $\mathbb{R}$. If $h\colon \mathbb{R} \times Z \to \mathbb{R}$ is $\mathbf{B} \times \mathbf{A}$-measurable and h_z is integrable for each $z \in Z$ then

$$z \mapsto \int_{\mathbb{R}} h(t, z) \, \mathrm{d}\lambda_1(t)$$

is $\mathbf{A}$-measurable on Z. Hence it follows (cf. [9], Remarks 1.2) that $A_n f$ is $\mathbf{B} \times \mathbf{A}$-measurable provided $f\colon \mathbb{R} \times Z \to \mathbb{R}$ is $\mathbf{B} \times \mathbf{A}$-measurable and f_z is locally integrable for each $z \in Z$. Consequently, for such an f also $\partial A_n f$ is $\mathbf{B} \times \mathbf{A}$-measurable. Keeping this notation we can formulate the following slight modification of Lemma 1.3 in [9].

Proposition 1. *Let $\lambda \geqslant 0$ be a measure on* **A**. *For each $k \in \mathbb{N}$ let Ψ_k be a class of* **B** $\times$ **A**-*measurable functions on $\mathbb{R} \times Z$ such that the following conditions* (P_1)–(P_6) *are satisfied:*

(P_1) $\Psi_k \subset \Psi_{k+1}$, $k \in \mathbb{N}$.

(P_2) $\psi \in \Psi_k \implies -\psi \in \Psi_k$.

(P_3) *For each $\psi \in \Psi \equiv \bigcup\limits_{k\in\mathbb{N}} \Psi_k$ and each $z \in Z$, ψ_z is a continuously differentiable function with compact support in $\mathbb{R}$; besides, both ψ and $\partial\psi$ are integrable with respect to $\lambda_1 \times \lambda$ on $\mathbb{R} \times Z$.*

(P_4) *For each $k \in \mathbb{N}$ there is a $n_k \in \mathbb{N}$ such that*

$$(\psi \in \Psi_k,\ n \geqslant n_k) \implies A_n\psi \in \Psi.$$

(P_5) *Given $k \in \mathbb{N}$, there is a* **B** $\times$ **A**-*measurable function $p_k\colon \mathbb{R} \times Z \to [0,\infty[$ such that, for each bounded* **B** $\times$ **A**-*measurable $h\colon \mathbb{R} \times Z \to \mathbb{R}$,*

$$\sup\left\{ \int_{\mathbb{R}\times Z} h\psi \,\mathrm{d}(\lambda_1 \times \lambda)\,;\ \psi \in \Psi_k \right\} = \int_{\mathbb{R}\times Z} |h| p_k \,\mathrm{d}(\lambda_1 \times \lambda).$$

(P_6) *If $g\colon \mathbb{R} \to \mathbb{R}$ is a bounded* **B**-*measurable function then, for each $z \in Z$ and $k \in \mathbb{N}$,*

$$\sup\left\{ \int_{\mathbb{R}} g\psi_z \,\mathrm{d}\lambda_1\,;\ \psi \in \Psi_k \right\} = \int_{\mathbb{R}} |g| p_{kz} \,\mathrm{d}\lambda_1,$$

where $p_{kz} = (p_k)_z$ and p_k occurs in (P_5).

Suppose now that $f\colon \mathbb{R} \times Z \to \mathbb{R}$ is a bounded **B** $\times$ **A**-*measurable function and let*

$$F(z) = \sup\left\{ \int_{\mathbb{R}} f_z(\partial\psi)_z \,\mathrm{d}\lambda_1\,;\ \psi \in \Psi \right\}, \quad z \in Z.$$

Then $F\colon Z \to [0,\infty]$ is **A**-*measurable and*

$$\int_Z F \,\mathrm{d}\lambda = \sup\left\{ \int_{\mathbb{R}\times Z} f\partial\psi \,\mathrm{d}(\lambda_1 \times \lambda)\,;\ \psi \in \Psi \right\}.$$

Proof. We will follow the ideas employed in the proof of Lemma 1.3 in [9]; for convenience of the reader we include the details. Having fixed $z \in Z$ we get from (P_4) for $k \in \mathbb{N}$ and $n \geqslant n_k$

$$\sup\left\{ \int_{\mathbb{R}} f_z(\partial A_n\psi)_z \,\mathrm{d}\lambda_1\,;\ \psi \in \Psi_k \right\} \equiv F_{kn}(z) \leqslant F(z),$$

so that

$$\underline{F}_k(z) \equiv \liminf_{n\to\infty} F_{kn}(z) \leqslant \limsup_{n\to\infty} F_{kn}(z) \equiv \overline{F}_k(z) \leqslant F(z).$$

In view of (P_1) we have for any $k \in \mathbb{N}$

$$\underline{F}_k(z) \leqslant \underline{F}_{k+1}(z), \quad \overline{F}_k(z) \leqslant \overline{F}_{k+1}(z) \tag{21}$$

and

$$\lim_{k\to\infty} \overline{F}_k(z) \leqslant F(z).$$

On the other hand, for each $c < F(z)$ there is a $\psi \in \Psi$ such that

$$\int_{\mathbb{R}} f_z(\partial\psi)_z \, \mathrm{d}\lambda_1 > c.$$

Thanks to (P_3) all functions $A_n(\partial\psi)_z$ have supports in a fixed compact subset of $\mathbb{R}$ and converge uniformly to $(\partial\psi)_z$ as $n \to \infty$; choosing $k \in \mathbb{N}$ such that $\psi \in \Psi_k$ we obtain

$$\underline{F}_k(z) \geqslant \int_{\mathbb{R}} f_z(\partial\psi)_z \, \mathrm{d}\lambda_1 > c.$$

We conclude that

$$F(z) = \lim_{k\to\infty} \underline{F}_k(z) = \lim_{k\to\infty} \overline{F}_k(z). \tag{22}$$

If $\psi \in \Psi_k$ and $n \geqslant n_k$ (cf. (P_4)), then

$$\int_{\mathbb{R}} f_z(\partial A_n\psi)_z \, \mathrm{d}\lambda_1 = \int_{\mathbb{R}} f_z(A_n\partial\psi)_z \, \mathrm{d}\lambda_1 = -\int_{\mathbb{R}} (\partial A_n f)_z \psi_z \, \mathrm{d}\lambda_1,$$

whence using (P_2), (P_6) we get

$$F_{kn}(z) = \int_{\mathbb{R}} |(\partial A_n f)_z| p_{kz} \, \mathrm{d}\lambda_1. \tag{23}$$

We see that F_{kn} is **A**-measurable and, consequently, the same holds of $\underline{F}_k = \liminf\limits_{n\to\infty} F_{kn}$ and $F = \lim\limits_{k\to\infty} \underline{F}_k$. The proof will be complete when we verify

$$\int_Z F \, \mathrm{d}\lambda \leqslant \sup\left\{ \int_{\mathbb{R}\times Z} f\partial\psi \, \mathrm{d}(\lambda_1 \times \lambda);\, \psi \in \Psi \right\} \equiv s,$$

because the opposite inequality follows from the definition of F by Fubini's theorem. Referring to (22), (21) we have

$$\int_Z F \, \mathrm{d}\lambda = \lim_{k\to\infty} \int_Z \underline{F}_k \, \mathrm{d}\lambda \leqslant \lim_{k\to\infty} \liminf_{n\to\infty} \int_Z F_{kn} \, \mathrm{d}\lambda.$$

The condition (P_5) together with (23) yields

$$\int_Z F_{kn}\,\mathrm{d}\lambda = \sup\left\{\int_{\mathbb{R}\times Z}(\partial A_n f)\psi\,\mathrm{d}(\lambda_1\times\lambda);\ \psi\in\Psi_k\right\}.$$

It remains to notice that, for $\psi\in\Psi_k$ and $n\geqslant n_k$, we have by (P_4), (P_2)

$$\int_{\mathbb{R}\times Z}(\partial A_n f)\psi\,\mathrm{d}(\lambda_1\times\lambda) = -\int_{\mathbb{R}\times Z} f\partial A_n\psi\,\mathrm{d}(\lambda_1\times\lambda)\leqslant s.$$

□

Definition 1. If $H, M\subset\mathbb{R}^m$ ($m\geqslant 1$) are Borel sets, then $z\in\mathbb{R}^m$ will be called a *hit* of H on M provided

$$\lambda_1\big(B(z,r)\cap H\cap M\big)>0 \text{ and } \lambda_1\big(B(z,r)\cap(H\setminus M)\big)>0$$

for each $r>0$ (compare 1.7 in [11] and [7]). We will use the symbol $H\odot M$ to denote the set of all hits of H on M.

Lemma 2. *Let $p\colon\mathbb{R}\to[0,\infty]$ be a lower-semicontinuous function, $U\subset\mathbb{R}$ an open set and $M\subset\mathbb{R}$ a Borel set. If*

$$U(p)\equiv\{\psi\in\mathscr{C}_0^{(1)}(U);\ |\psi|<p \text{ on } \operatorname{spt}\psi\},$$

then

$$\sup\left\{\int_M\psi'\,\mathrm{d}\lambda_1;\ \psi\in U(p)\right\} = \sum_{t\in U\odot M}p(t). \tag{24}$$

Proof. If $\psi\in U(p)$, then $\psi'=0$ almost everywhere on $p^{-1}(0)=\{t\in\mathbb{R}\colon p(t)=0\}$. If $I_1,\ldots,I_r$ are all components of $U\setminus p^{-1}(0)$ intersecting $\operatorname{spt}\psi$, then

$$\int_M\psi'\,\mathrm{d}\lambda_1 = \sum_{j=1}^r\int_{M\cap I_j}\psi'\,\mathrm{d}\lambda_1$$

and

$$\sum_{j=1}^r\sum_{t\in I_j\odot M}p(t)\leqslant\sum_{t\in U\odot M}p(t).$$

In order to prove

$$\left|\int_M\psi'\,\mathrm{d}\lambda_1\right|\leqslant\sum_{t\in U\odot M}p(t)$$

it suffices to verify

$$\left|\int_{M\cap I_j}\psi'\,\mathrm{d}\lambda_1\right|\leqslant\sum_{t\in I_j\odot M}p(t) \tag{25}$$

for any component $I_j =]a,b[$ for which the right-hand side in (25) is finite. There is a compact interval $[\bar{a},\bar{b}]\subset\,]a,b[$ such that $\operatorname{spt}\psi\cap\,]a,b[\,\subset[\bar{a},\bar{b}]$ and $\inf p([\bar{a},\bar{b}])>0$. Under these circumstances the set

$$[\bar{a},\bar{b}]\cap(I_j\odot M)=\{e_1,\dots,e_s\},$$

where $e_1<\dots<e_s$, must be finite. No component of $[\bar{a},\bar{b}]\setminus\{e_1,\dots,e_s\}$ can meet both M and $\mathbb{R}\setminus M$ in a set of positive λ_1-measure; consequently, these components in their natural order are alternatively almost entirely contained in one of the sets M and $\mathbb{R}\setminus M$ (cf. the reasoning in the proof of 1.8 in [11], p. 13) and

$$\left|\int_{M\cap I_j}\psi'\,\mathrm{d}\lambda_1\right|=\left|\int_{M\cap[\bar{a},\bar{b}]}\psi'\,\mathrm{d}\lambda_1\right|=\left|\sum_{i=1}^{s}(-1)^i\psi(e_i)\right|\leqslant\sum_{t\in I_j\odot M}p(t),$$

which proves (25). Denoting by $\alpha(p)$ and $\beta(p)$ the left-hand side and the right-hand side in (24), respectively, we have thus shown that $\alpha(p)\leqslant\beta(p)$. It remains to establish the opposite inequality

$$\alpha(p)\geqslant\beta(p). \tag{26}$$

For this purpose we denote by $\mathscr{P}$ the class of all lower-semicontinuous functions $p\colon\mathbb{R}\to[0,\infty]$ fulfilling (26) and proceed by checking validity of the following assertions (a)–(d):

(a) $\mathscr{P}$ contains the indicator function of any open interval in $\mathbb{R}$.

(b) If $p\in\mathscr{P}$, then also $cp\in\mathscr{P}$ for any $c\in[0,\infty[$.

(c) If $\{p_n\}$ is a non-decreasing sequence of functions in $\mathscr{P}$, then also $\lim\limits_{n\to\infty}p_n\in\mathscr{P}$.

(d) If p_1, $p_2\in\mathscr{P}$, then also $p_1+p_2\in\mathscr{P}$.

The assertion (a) follows from Lemma 1.8 in [11] and the proofs of (b), (c) are easy. In order to prove (d) choose arbitrary functions p_1, $p_2\in\mathscr{P}$ and real numbers $c_k<\beta(p_k)$ $(k=1,2)$. There are $\psi_k\in U(p_k)$ with

$$\int_M\psi'\,\mathrm{d}\lambda_1>c_k\quad(k=1,2).$$

It is easily seen that $\psi=\psi_1+\psi_2\in U(p_1+p_2)$. The relation

$$\int_M\psi'\,\mathrm{d}\lambda_1=\int_M\psi_1'\,\mathrm{d}\lambda_1+\int_M\psi_2'\,\mathrm{d}\lambda_1>c_1+c_2$$

shows that $\alpha(p_1+p_2) > c_1+c_2$. Since c_k can be chosen arbitrarily close to $\beta(p_k)$, we have $\alpha(p_1+p_2) \geqslant \beta(p_1)+\beta(p_2) = \beta(p_1+p_2)$ and (d) is established. It follows from (a)–(d) that $\mathscr{P}$ contains all lower-semicontinuous functions $p\colon \mathbb{R}\to[0,\infty]$ and the proof is complete (cf. also [10], pp. 22–24). □

Lemma 3. *For any fixed $\xi\in\mathbb{R}^m$, the function $\theta\mapsto n^q(\theta,\xi)$ defined by* (4) *is λ_{m-1}-measurable on $\partial B(0,1)$, so that we may define $v^q(\xi)$ by* (5). *Writing*

$$V^\xi \equiv \mathbb{R}^m\setminus\{\xi,\eta\},\quad V^\xi(q)\equiv\{\psi\in\mathscr{C}_0^{(1)}(V^\xi);\ |\psi|<q \text{ on } \operatorname{spt}\psi\},$$

we then have

(27)

$$v^q(\xi)=\int_{\widehat{\partial}G} q|n^G\cdot\operatorname{grad}h_\xi|\,\mathrm{d}\lambda_{m-1}=\sup\left\{\int_G \operatorname{grad}\psi\cdot\operatorname{grad}h_\xi\,\mathrm{d}\lambda_m\,;\ \psi\in V^\xi(q)\right\}.$$

Proof. Consider $\psi\in\mathscr{C}_0^{(1)}(V^\xi)$. Applying the divergence formula (15) to $v=\psi\operatorname{grad}h_\xi$ (with $v(\xi)=0\in\mathbb{R}^m$) we get

$$\int_G \operatorname{grad}\psi\cdot\operatorname{grad}h_\xi\,\mathrm{d}\lambda_m=\int_{\widehat{\partial}G}\psi n^G\cdot\operatorname{grad}h_\xi\,\mathrm{d}\lambda_{m-1},$$

whence it easily follows that

$$\sup\left\{\int_G \operatorname{grad}\psi\cdot\operatorname{grad}h_\xi\,\mathrm{d}\lambda_m\,;\ \psi\in V^\xi(q)\right\}=\int_{\widehat{\partial}G} q|n^G\cdot\operatorname{grad}h_\xi|\,\mathrm{d}\lambda_{m-1}. \tag{28}$$

Introducing the spherical coordinates and writing

$$\psi_\theta(t)\equiv\psi(\xi+t\theta),\quad G_\theta\equiv\{t>0;\ \xi+t\theta\in G\}$$

for $\theta\in\partial B(0,1)$ we obtain (cf. (18) in [11])

$$\int_G \operatorname{grad}\psi\cdot\operatorname{grad}h_\xi\,\mathrm{d}\lambda_m=-\frac{1}{\sigma_m}\int_{\partial B(0,1)}\left[\int_{G_\theta}\psi'_\theta(t)\,\mathrm{d}\lambda_1(t)\right]\mathrm{d}\lambda_{m-1}(\theta).$$

Next, use Proposition 1 with $Z=\partial B(0,1)$, $\lambda=\lambda_{m-1}$. For $\psi\in V^\xi(q)$ define $\tilde\psi$ on $\mathbb{R}\times Z$ by $\tilde\psi\colon(t,\theta)\mapsto\psi_\theta(t)$ and let $\Psi_k\equiv\{\tilde\psi;\ \psi\in V^\xi(q),\ |\psi|+\frac{1}{k}<q \text{ on } \operatorname{spt}\psi\}$, $\Psi\equiv\bigcup_{k\in\mathbb{N}}\Psi_k$. Proposition 1 tells us that

$$\theta\mapsto\sup\left\{\int_{G_\theta}\psi'_\theta(t)\,\mathrm{d}\lambda_1(t)\,;\ \psi\in V^\xi(q)\right\} \tag{29}$$

is a Baire function of the variable $\theta \in \partial B(0,1)$ whose integral (with respect to $\mathrm{d}\lambda_{m-1}(\theta)$) over $\partial B(0,1)$ is equal to (28). It follows from Lemma 2 that for any fixed $\theta \in \partial B(0,1)$ with $H(\xi,\theta) \subset \mathbb{R}^m \setminus \{\eta\}$ (cf. (3)), we have

$$\sup\left\{\int_{G_\theta} \psi'_\theta(t)\,\mathrm{d}\lambda_1(t)\,;\ \psi \in V^\xi(q)\right\} = \sum_{x \in H(\xi,\theta)\odot G} q(x).$$

M. Chlebík pointed out in [2] that methods of geometric measure theory (cf. [5]) permit to conclude that

$$\lambda_{m-1}\big(\{\theta \in \partial B(0,1)\,;\ H(\xi,\theta) \cap \partial_e G \neq H(\xi,\theta) \odot G\}\big) = 0,$$

which shows that $\theta \mapsto n^q(\theta,\xi)$ is λ_{m-1}-equivalent to (29) and (27) holds. □

Corollary 1. *The function $\xi \mapsto v^q(\xi)$ is lower-semicontinuous on $\mathbb{R}^m$.*

Proof. This follows easily from the formula (27). □

Lemma 4. *Let $S \subset \mathbb{R}^m \setminus \partial A$ be contained either in A or in $G = \mathbb{R}^m \setminus A$, $\eta \in \operatorname{cl} S \cap \partial A$. Then the limit (9) exists and is finite for every $f \in \mathscr{C}(\partial A, q)$ iff*

$$\limsup_{\substack{z \to \eta \\ z \in S}} v^q(z) < \infty. \tag{30}$$

Proof. We have seen in (17) that, for any fixed $z \in \mathbb{R}^m \setminus \partial A$, the linear functional

$$f \mapsto Wf(z)$$

is bounded on $\mathscr{C}(\partial A, q)$ and, according to (20), (27),

$$\sup\big\{Wf(z)\,;\ f \in \mathscr{C}_0(\partial A, q),\ \|f\|_{q,0} \leqslant 1\big\} = v^q(z). \tag{31}$$

The existence of the limit (9) implies that, for any sequence $z_n \in S$ tending to η, the sequence $Wf(z_n)$ is bounded for each $f \in \mathscr{C}_0(\partial A, q)$; by the Banach-Steinhaus theorem we conclude in view of (31) that $\sup\limits_n v^q(z_n) < \infty$ and (30) follows.

Now we shall prove the converse (compare [10]). Since χ_G is constant on S, it follows from (16) that it is sufficient to verify the existence of (9) for $f \in \mathscr{C}_0(\partial A, q)$ only. According to (30) we have for suitable $r > 0$

$$\sup\big\{v^q(z)\,;\ z \in B(\eta, r) \cap S\big\} \equiv K < \infty.$$

From Fatou's Lemma we conclude by (27) that also

$$\int_{\widehat{\partial G}} q|n^G \cdot \operatorname{grad} h_\eta| \, \mathrm{d}\lambda_{m-1} \leqslant K. \tag{32}$$

We shall show that, for any $f \in \mathscr{C}_0(\partial A, q)$,

$$\lim_{\substack{z \to \eta \\ z \in S}} Wf(z) = \int_{\widehat{\partial G}} fn^G \cdot \operatorname{grad} h_\eta \, \mathrm{d}\lambda_{m-1}. \tag{33}$$

Fix an arbitrary $\varepsilon > 0$, $f \in \mathscr{C}_0(\partial A, q)$ and choose $\delta > 0$ such that

$$|f(y)| \leqslant \varepsilon q(y), \quad y \in B(\eta, \delta) \cap \partial A.$$

By (16) we then have

$$Wf(z) = \int_{B(\eta,\delta)\cap\widehat{\partial G}} fn^G \cdot \operatorname{grad} h_z \, \mathrm{d}\lambda_{m-1} + \int_{\widehat{\partial G}\setminus B(\eta,\delta)} fn^G \cdot \operatorname{grad} h_z \, \mathrm{d}\lambda_{m-1}. \tag{34}$$

Clearly,

$$\lim_{\substack{z \to \eta \\ z \in S}} \int_{\widehat{\partial G}\setminus B(\eta,\delta)} fn^G \cdot \operatorname{grad} h_z \, \mathrm{d}\lambda_{m-1} = \int_{\widehat{\partial G}\setminus B(\eta,\delta)} fn^G \cdot \operatorname{grad} h_\eta \, \mathrm{d}\lambda_{m-1},$$

because $\operatorname{grad} h_z \to \operatorname{grad} h_\eta$ uniformly on $\widehat{\partial G} \setminus B(\eta, \delta)$ as $z \to \eta$. On the other hand,

$$\left| \int_{B(\eta,\delta)\cap\widehat{\partial G}} fn^G \cdot \operatorname{grad} h_z \, \mathrm{d}\lambda_{m-1} \right| \leqslant \varepsilon \int_{\widehat{\partial G}} g|n^G \cdot \operatorname{grad} h_z| \, \mathrm{d}\lambda_{m-1} \leqslant \varepsilon K, \qquad z \in B(\eta, r) \cap S,$$

and, in view of (32), this estimate holds for $z = \eta$ as well. We conclude from (34) that

$$\limsup_{\substack{z \to \eta \\ z \in S}} \left| Wf(z) - \int_{\widehat{\partial G}} fn^G \cdot \operatorname{grad} h_\eta \, \mathrm{d}\lambda_{m-1} \right|$$
$$\leqslant 2\varepsilon K + \limsup_{\substack{z \to \eta \\ z \in S}} \left| \int_{\widehat{\partial G}\setminus B(\eta,\delta)} fn^G \cdot (\operatorname{grad} h_z - \operatorname{grad} h_\eta) \, \mathrm{d}\lambda_{m-1} \right| = 2\varepsilon K$$

which proves (33), because $\varepsilon > 0$ was arbitrary. □

Lemma 5. *For any open $U \subset \mathbb{R}^m$ and $\omega \in \partial B(0,1)$ the function*

$$\xi \to n^q(\omega, \xi; U) \tag{35}$$

defined by (7) *(cf. also* (6)*) is λ_{m-1}-measurable on $N(\omega) \equiv \{\xi \in \mathbb{R}^m;\ \xi \cdot \omega = 0\}$, so that we may define $\mu^q_\omega(U)$ by* (8)*. Let*

$$U^\eta(q) \equiv \{\psi \in \mathscr{C}_0^{(1)}(U \setminus \{\eta\});\ |\psi| < q \text{ on } \operatorname{spt}\psi\}$$

and denote by $\partial_\omega \equiv \omega \cdot \operatorname{grad}$ the derivative in the direction of $\omega \in \partial B(0,1)$. Then

$$\mu^q_\omega(U) = \int_{U \cap \widehat{\partial G}} q|n^G \cdot \omega| \,\mathrm{d}\lambda_{m-1} = \sup\left\{ \int_G \partial_\omega \psi \,\mathrm{d}\lambda_m;\ \psi \in U^\eta(q)\right\}. \tag{36}$$

P r o o f. Fix $\psi \in \mathscr{C}_0^{(1)}(U \setminus \{\eta\})$ and apply the divergence formula (15) to $v = \psi\omega$. It follows that

$$\int_{\widehat{\partial G}} \psi\omega \cdot n^G \,\mathrm{d}\lambda_{m-1} = \int_G \partial_\omega \psi \,\mathrm{d}\lambda_m,$$

whence

$$\int_{U \cap \widehat{\partial G}} q|\omega \cdot n^G| \,\mathrm{d}\lambda_{m-1} = \sup\left\{ \int_G \partial_\omega \psi \,\mathrm{d}\lambda_m;\ \psi \in U^\eta(q)\right\}. \tag{37}$$

Let us agree to write for $\xi \in N(\omega)$

$$\psi^\xi(t) \equiv \psi(\xi + t\omega), \quad t \in \mathbb{R}; \quad G^\xi \equiv \{t \in \mathbb{R};\ \xi + t\omega \in G\}.$$

Using Fubini's theorem we obtain

$$\int_G \partial_\omega \psi \,\mathrm{d}\lambda_m = \int_{N(\omega)} \left[\int_{G^\xi} \frac{\mathrm{d}}{\mathrm{d}t} \psi^\xi(t) \,\mathrm{d}\lambda_1(t)\right] \mathrm{d}\lambda_{m-1}(\xi).$$

Next we will employ Proposition 1 with $Z = N(\omega)$ and $\lambda = \lambda_{m-1}$ on the σ-algebra $\mathbf{A}$ of all Borel subsets in $N(\omega)$. With any $\psi \in U^\eta(q)$ we associate $\tilde{\psi}$ defined on $\mathbb{R} \times Z$ by

$$\tilde{\psi}\colon (t, \xi) \mapsto \psi^\xi(t)$$

and let $\Psi_k = \{\tilde{\psi};\ \psi \in U^\eta(q),\ |\psi| + \frac{1}{k} < q \text{ on } \operatorname{spt}\psi\}$, $k \in \mathbb{N}$. Proposition 1 tells us that the function

$$\xi \mapsto \sup\left\{ \int_{G^\xi} \frac{\mathrm{d}}{\mathrm{d}t} \psi^\xi(t) \,\mathrm{d}\lambda_1(t);\ \psi \in U^\eta(q)\right\} \tag{38}$$

is Borel measurable on $N(\omega)$ and its integral $\mathrm{d}\lambda_{m-1}(\xi)$ over $N(\omega)$ is equal to (37). For fixed $\xi \in N(\omega)$ with $L(\xi,\omega) \subset \mathbb{R}^m \setminus \{\eta\}$ (cf. (6)), Lemma 2 yields

$$\sup\left\{\int_{G^\xi} \frac{\mathrm{d}}{\mathrm{d}t}\psi^\xi(t)\,\mathrm{d}\lambda_1(t)\,;\ \psi \in U^\eta(q)\right\} = \sum_t q(t) \quad t \in [L(\xi,\omega)\cap U]\odot G.$$

Now we refer to the following result of M. Chlebík from [2] based on geometric measure theory (cf. [5]):

$$\lambda_{m-1}\big(\{\xi \in N(\omega)\,;\ L(\xi,\omega)\cap U \cap \widehat{\partial G} \neq [L(\xi,w)\cap U]\odot G\}\big) = 0,$$

which implies that the function (35) is λ_{m-1}-equivalent to (38). Consequently, (35) is λ_{m-1}-measurable on $N(\omega)$ and its integral $\mathrm{d}\lambda_{m-1}(\xi)$ over $N(\omega)$, which has been denoted by $\mu^q_\omega(U)$, satisfies (36). □

Lemma 6. *Let $S \subset \mathbb{R}^m$, $\eta \in \operatorname{cl} S$. If* (30) *holds, then*

(39) $$v^q(\eta) + \sup\big\{r^{1-m}\mu^q_\omega\big(B(\eta,r)\big)\,;\ r>0,\ \omega \in \partial B(0,1),\ \eta + r\omega \in \operatorname{cl} S\big\} < \infty,$$

where $\mu^q_\omega(\ldots)$ is defined by (8).

Proof. Employing (30) we get by the lower-semicontinuity of $v^q(\cdot)$ that there is a $\delta > 0$ such that

(40) $$\sup\{v^q(x)\,;\ x \in B(\eta,\delta)\cap \operatorname{cl} S\} \equiv c < \infty.$$

Let $r > 0$, $\omega \in \partial B(0,1)$, $z = \eta + r\omega \in \operatorname{cl} S$. If $r \geqslant \delta$, then using the notation (11) we have

$$r^{1-m}\mu^q_\omega\big(B(\eta,r)\big) \leqslant \delta^{1-m}\mu^q(\mathbb{R}^m) < \infty.$$

Let now $r \in\,]0,\delta[$, $y \in \widehat{\partial G} \cap B(\eta,r)$. If $|(y-\eta)\cdot n^G(y)| \geqslant \frac{1}{2}r|n^G(y)\cdot\omega|$, then

$$\frac{|(y-\eta)\cdot n^G(y)|}{|y-\eta|^m} \geqslant \frac{1}{2}r^{1-m}|n^G(y)\cdot\omega|.$$

If $|(y-\eta)\cdot n^G(y)| < \frac{1}{2}r|n^G(y)\cdot\omega|$, then

$$\begin{aligned}\frac{|(y-z)\cdot n^G(y)|}{|y-z|^m} &\geqslant (2r)^{-m}\big[|(z-\eta)\cdot n^G(y)| - |(y-\eta)\cdot n^G(y)|\big]\\ &> (2r)^{-m}\big[r|\omega\cdot n^G(y)| - \frac{1}{2}r|\omega\cdot n^G(y)|\big]\\ &= 2^{-m-1}\cdot r^{1-m}|n^G(y)\cdot\omega|.\end{aligned}$$

Hence

$$r^{1-m}|n^G(y)\cdot\omega| \leqslant 2^{m+1}\sigma_m\left[|n^G(y)\cdot \operatorname{grad} h_\eta(y)| + |n^G(y)\cdot \operatorname{grad} h_z(y)|\right]$$

and (40), (27) give

$$\begin{aligned} r^{1-m}&\mu_\omega^q(B(\eta,r))\\ &\leqslant 2^{m+1}\sigma_m \int\limits_{B(\eta,\delta)\cap\widehat{\partial G}} q(y)\left[|n^G(y)\cdot \operatorname{grad} h_\eta(y)| + |n^G(y)\cdot \operatorname{grad} h_z(y)|\right]\,\mathrm{d}\lambda_{m-1}(y)\\ &\leqslant 2^{m+2}\sigma_m c. \end{aligned}$$

□

Remark 2. It follows from Lemma 5 that $\mu_\omega^q(\cdot)$, defined so far as a set function on open sets, extends naturally to all Borel sets $M\subset\mathbb{R}^m$ by

(41)
$$\mu_\omega^q(M) = \int\limits_{M\cap\widehat{\partial G}} q|n^G\cdot\omega|\,\mathrm{d}\lambda_{m-1}.$$

Proposition 2. *Suppose that $S_j\subset\mathbb{R}^m\setminus\partial A$ are connected sets such that $\eta\in\operatorname{cl}S_j\cap\partial A$, $\operatorname{contg}(S_j,\eta) = H(\eta,\theta_j)$ $(j=1,\dots,m)$ (cf. (3)), where $\theta_1,\dots,\theta_m\in\partial B(0,1)$ are linearly independent. If* (30) *holds for $S=\bigcup\limits_{j=1}^m S_j$ and $\mu^q(\cdot)$ is defined by* (11), *then* (12) *is true.*

P r o o f. Our assumptions on S_j $(j=1,\dots,m)$ guarantee the existence of positive constants δ, c such that

$$\left(n\in\partial B(0,1),\ z^j\in S_j\cap B(\eta,\delta)\ (j=1,\dots,m)\right) \implies \sum_{j=1}^m\left|n\cdot\frac{z^j-\eta}{|z^j-\eta|}\right| \geqslant c.$$

We may suppose that $\delta>0$ has been fixed sufficiently small so that, for every $r\in\,]0,\delta[$ and $j=1,\dots,m$, $\partial B(\eta,r)\cap S_j\neq\emptyset$, which implies the existence of an $\omega_j\in\partial B(0,1)$ with $\eta+r\omega_j\equiv z^j\in S_j$. In view of (11), (14), $\mu^q(\cdot)$ can be defined by

(42)
$$\mu^q(M) = \int_{M\cap\widehat{\partial G}} q\,\mathrm{d}\lambda_{m-1}$$

for any Borel set $M\subset\mathbb{R}^m$. By Lemma 6 we have for $0<r<\delta$

$$\begin{aligned} v^q(\eta) &+ r^{1-m}\mu^q(B(\eta,r))\\ &\leqslant v^q(\eta) + \frac{1}{c}r^{1-m}\sum_{j=1}^m \mu_{\omega_j}^q(B(\eta,r))\\ &\leqslant v^q(\eta) + \frac{m}{c}\sup\left\{r^{1-m}\mu_\omega^q(B(\eta,r)); r>0,\ \omega\in\partial B(0,1),\ \eta+r\omega\in S\right\}\\ &<\infty. \end{aligned}$$

Obviously, for $r \geqslant \delta$ we have $r^{1-m}\mu^q\big(B(\eta,r)\big) \leqslant \delta^{1-m}\mu^q(\mathbb{R}^m) < \infty$ and (12) is verified. $\square$

Proposition 3. *Let $m = 2$ and let $S \subset \mathbb{R}^2 \setminus \partial A$ be a connected set with $\eta \in \operatorname{cl} S \cap \partial A$. Denote by*

$$\check{S} = \{2\eta - x\,;\ x \in S\}$$

its reflection at η and suppose that

$$\operatorname{contg}(\widehat{\partial G},\eta) \cap \operatorname{contg}(S,\eta) = \emptyset = \operatorname{contg}(\widehat{\partial G},\eta) \cap \operatorname{contg}(\check{S},\eta).$$

Then (30) *implies* (12).

Proof. There are constants $\delta \in]0,\pi/4[$, $\varrho \in]0,\infty[$ such that, for any $y \in \widehat{\partial G} \cap \big[B(\eta,\varrho) \setminus \{\eta\}\big]$ and $z \in S \cap B(\eta,\varrho)$, the angle enclosed by the vectors

$$\frac{y-\eta}{|y-\eta|},\quad \frac{z-\eta}{|z-\eta|}$$

exceeds 2δ and the same holds for the vectors

$$\frac{y-\eta}{|y-\eta|},\quad -\frac{z-\eta}{|z-\eta|}.$$

In view of (30) we may assume that $\varrho > 0$ has been fixed small enough so that

$$0 < r < \varrho \implies S \cap \partial B(\eta,r) \neq \emptyset,$$
$$\sup\{v^q(z)\,;\ z \in S \cap B(\eta,\varrho)\} \equiv c < \infty.$$

It follows from Corollary 1 that also

$$v^q(\eta) \leqslant c.$$

Let now $0 < r < \varrho$, $y \in \widehat{\partial G} \cap B(\eta,r)$, $z \in S \cap \partial B(\eta,r)$. If $n^G(y)$ encloses with one of the vectors

$$\frac{z-\eta}{|z-\eta|},\quad -\frac{z-\eta}{|z-\eta|} \tag{43}$$

the angle not exceeding $\frac{1}{2}\pi - \delta$, then

$$|n^G(y)\cdot(y-\eta)| + |n^G(y)\cdot(y-z)| \geqslant |n^G(y)\cdot(z-\eta)| \geqslant r\cos(\tfrac{1}{2}\pi - \delta).$$

If both vectors (43) enclose with $n^G(y)$ angles exceeding $\frac{1}{2}\pi - \delta$, then at least one of the vectors

$$\frac{y-\eta}{|y-\eta|}, \quad -\frac{y-\eta}{|y-\eta|}$$

encloses with $n^G(y)$ an angle which is less than $\frac{1}{2}\pi - 2\delta + \delta = \frac{1}{2}\pi - \delta$, whence

$$|n^G(y)\cdot(y-\eta)| \geqslant |y-\eta|\cos(\tfrac{1}{2}\pi - \delta).$$

In any case (note that $|y-z| \leqslant |y-\eta| + |z-\eta| \leqslant 2r$) we have

$$\frac{|n^G(y)\cdot(y-\eta)|}{|y-\eta|^2} + \frac{|n^G(y)\cdot(y-z)|}{|y-z|^2} \geqslant \frac{1}{4}r^{-1}\cos(\tfrac{1}{2}\pi - \delta),$$

whence by (42), (27) we get

$$\begin{aligned}
&r^{-1}\mu^q(B(\eta,r)) \\
&\quad\leqslant 4\cos^{-1}(\tfrac{1}{2}\pi-\delta)\int_{\widehat{\partial G}\cap B(\eta,r)}\left[\frac{|n^G(y)\cdot(y-\eta)|}{|y-\eta|^2} + \frac{|n^G(y)\cdot(y-z)|}{|y-z|^2}\right] q(y)\,\mathrm{d}\lambda_1(y) \\
&\quad\leqslant 8\pi\cos^{-1}(\tfrac{1}{2}\pi-\delta)[v^q(\eta)+v^q(z)] \leqslant 16\pi c\cos^{-1}(\tfrac{1}{2}\pi-\delta).
\end{aligned}$$

If $r \geqslant \varrho$, then $r^{-1}\mu^q(B(\eta,r)) \leqslant \varrho^{-1}\mu^q(\mathbb{R}^2) < \infty$ and (12) (where now $m=2$) is verified. □

Lemma 7. *Let* $S \subset \mathbb{R}^m \setminus \partial A$, $\eta \in \partial A \cap \operatorname{cl} S$, $v^q(\eta) < \infty$, $\operatorname{contg}(\widehat{\partial G},\eta) \cap \operatorname{contg}(S,\eta) = \emptyset$. *If*

$$\text{(44)} \qquad \sup\left\{t^{1-m}\mu^q_\theta(B(\eta,t))\,;\,\theta = \frac{z-\eta}{|z-\eta|},\ z\in S\cap B(\eta,\delta),\ t>|z-\eta|\right\} < \infty$$

for some $\delta > 0$, *then* (30) *holds.*

Proof. It follows from

$$\operatorname{contg}(\widehat{\partial G},\eta)\cap\operatorname{contg}(S,\eta) = \emptyset$$

that there exist constants $a > 0$, $\varepsilon > 0$ such that

$$z \in S\cap B(\eta,\varepsilon) \implies \operatorname{dist}(z,\widehat{\partial G}) \geqslant a|z-\eta|$$

(cf. [3]). Clearly we may suppose that $\varepsilon \leqslant \delta$, where δ occurs in the assumption (44). We shall first show that there is a $c \in\,]0,\infty[$ such that, for any $z \in S\cap B(\eta,\varepsilon)$ and any Borel measure $\nu \geqslant 0$ concentrated on $\widehat{\partial G}$,

$$\text{(45)} \qquad \sup\{r^{1-m}\nu(B(z,r))\,;\,r>0\} \leqslant c\sup\{r^{1-m}\nu(B(\eta,r))\,;\,r>|z-\eta|\}.$$

Setting $r_1 = |z-\eta|$ and $r = r_1 b$ we get for $b \in]0,a]$

$$B(z,r) \cap \widehat{\partial G} = \emptyset \ (\Longrightarrow \ \nu(B(z,r)) = 0),$$

while for $b > a$

$$\begin{aligned}
&r^{1-m}\nu(B(z,r)) \\
&\leqslant r^{1-m}\nu(B(\eta, r+r_1)) = [r_1(1+b)]^{1-m}\nu(B(\eta,(1+b)r_1)) \left(\frac{1+b}{b}\right)^{m-1} \\
&\leqslant \left(\frac{1+a}{a}\right)^{m-1} \sup\{t^{1-m}\nu(B(\eta,t))\,;\, t > r_1\}.
\end{aligned}$$

We see that (45) holds with $c = a^{1-m}(1+a)^{m-1}$.

Now fix any $z \in S \cap B(\eta, \varepsilon)$ and put $r = |z-\eta|$, $\theta = (z-\eta)/|z-\eta|$. Defining the Borel measures μ^q and μ^q_θ by (42) and (41) we get by (27)

$$\begin{aligned}
v^q(z) &\leqslant \frac{1}{\sigma_m}\int_{\widehat{\partial G}} \frac{|n^G(y)\cdot(y-\eta)|}{|y-\eta|^m}\,\mathrm{d}\mu^q(y) \\
&\quad + \frac{1}{\sigma_m}\int_{\widehat{\partial G}} \left|\frac{n^G(y)\cdot(y-z)}{|y-z|^m} - \frac{n^G(y)\cdot(y-\eta)}{|y-\eta|^m}\right| \mathrm{d}\mu^q(y) \\
&= v^q(\eta) + \frac{1}{\sigma_m}\int_{\widehat{\partial G}} \left|\frac{|y-\eta|^m - |y-z|^m}{|y-\eta|^m\cdot|y-z|^m} n^G(y)\cdot(y-\eta)\right. \\
&\qquad\qquad \left. - \frac{rn^G(y)\cdot\theta}{|y-z|^m}\right| \mathrm{d}\mu^q(y) \\
&\leqslant 2v^q(\eta) + \frac{1}{\sigma_m}\int_{\widehat{\partial G}} \frac{|n^G(y)\cdot(y-\eta)|}{|y-\eta|^m}\cdot\left(\frac{|y-\eta|}{|y-z|}\right)^m \mathrm{d}\mu^q(y) \\
&\quad + \frac{r}{\sigma_m}\int_{\widehat{\partial G}} \frac{\mathrm{d}\mu^q_\theta(y)}{|y-z|^m}.
\end{aligned}$$

In order to get an estimate of the last integral recall that, for any non-negative Baire function f on $\mathbb{R}^m$,

$$\int_{\mathbb{R}^m} f\,\mathrm{d}\mu^q_\theta = \int_0^\infty \mu^q_\theta(\{x\in\mathbb{R}^m\,;\, f(x) > t\})\,\mathrm{d}t.$$

Hence we get by (45), denoting by k the supremum occurring in (44),

$$\begin{aligned}
r\int_{\widehat{\partial G}} |y-z|^{-m}\,\mathrm{d}\mu^q_\theta(y) &= r\int_0^{(ar)^{-m}} \mu^q_\theta(B(z, t^{-\frac{1}{m}}))\,\mathrm{d}t \\
&= rm\int_{ar}^\infty \mu^q_\theta(B(z,s))s^{-1-m}\,\mathrm{d}s \\
&\leqslant rmck\int_{ar}^\infty s^{-2}\,\mathrm{d}s = mck/a.
\end{aligned}$$

If $y \in \widehat{\partial G}$, then $|y-z| \geqslant a|z-\eta|$, whence $|y-\eta| \leqslant |y-z| + |z-\eta| \leqslant (1+\frac{1}{a})|y-z|$. Hence

$$\frac{1}{\sigma_m} \int_{\widehat{\partial G}} \frac{|n^G(y)\cdot(y-\eta)|}{|y-\eta|^m} \cdot \left(\frac{|y-\eta|}{|y-z|}\right)^m \mathrm{d}\mu^q(y) \leqslant \left(\frac{a+1}{a}\right)^m v^q(\eta).$$

Finally, we arrive at the estimate

$$v^q(z) \leqslant \left[2 + \left(\frac{a+1}{a}\right)^m\right] v^q(\eta) + \frac{cmk}{\sigma_m a}.$$

□

The following assertion is an immediate consequence of Lemma 7.

Corollary 2. *Let* $S \subset \mathbb{R}^m \setminus \partial A$, $\eta \in \partial A \cap \operatorname{cl} S$, $\operatorname{contg}(\widehat{\partial G}, \eta) \cap \operatorname{contg}(S, \eta) = \emptyset$. *Then* (12) *implies* (30).

Lemma 8. *Let* $S \subset \mathbb{R}^m$ *be connected,* $\eta \in (\operatorname{cl} S \setminus S) \cap \partial A$ *and suppose that, for suitable* $k \in]0, \infty[$ *and* $\theta \in \partial B(0,1)$, *the following implication holds:*

$$z \in S \implies \left|\theta - \frac{z-\eta}{|z-\eta|}\right| \leqslant k|z-\eta|^{m-1}. \tag{46}$$

Then the following conditions (i)–(iii) *are mutually equivalent:*

(i) $\sup\limits_{r>0} r^{1-m} \mu^q_\theta\big(B(\eta,r)\big) < \infty$;

(ii) $\sup \big\{ r^{1-m} \mu^q_\omega\big(B(\eta,r)\big)\,;\ \eta + r\omega \in S,\ \omega \in \partial B(0,1) \big\} < \infty$;

(iii) $\sup \big\{ t^{1-m} \mu^q_\omega\big(B(\eta,t)\big)\,;\ \eta + r\omega \in S,\ \omega \in \partial B(0,1),\ t > r \big\} < \infty$.

Proof. We shall first verify the equivalence (i) $\iff$ (ii). Choose $\delta > 0$ small enough to have $S \cap \partial B(\eta, r) \neq \emptyset$ whenever $0 < r < \delta$. Having fixed such an r we choose $\omega \in \partial B(0,1)$ with $\eta + r\omega \in S$ and get from (46)

$$\begin{aligned} r^{1-m}\big|\mu^q_\theta\big(B(\eta,r)\big) - \mu^q_\omega\big(B(\eta,r)\big)\big| &\leqslant r^{1-m} \int_{\widehat{\partial G}\cap B(\eta,r)} q(y)|n^G(y)\cdot(\theta-\omega)|\,\mathrm{d}\lambda_{m-1}(y) \\ &\leqslant k\mu^q(\mathbb{R}^m). \end{aligned}$$

For $r \geqslant \delta$ and any $\omega \in \partial B(0,1)$ the inequality $r^{1-m}\mu^q_\omega\big(B(\eta,r)\big) \leqslant \delta^{1-m}\mu^q(\mathbb{R}^m)$ holds, whence the equivalence (i) $\iff$ (ii) follows at once. It remains to verify (ii) $\iff$ (iii).

Let $\omega \in \partial B(0,1)$, $\eta + r\omega \in S$, $t > r$. If $t \geqslant \delta$, then $t^{1-m}\mu^q_\omega(B(\eta,t)) \leqslant \delta^{1-m}\mu^q(\mathbb{R}^m) < \infty$. If $t \in]0,\delta[$, then we can choose $\omega_t \in \partial B(0,1)$ with $\eta + t\omega_t \in S$. We then have

$$t^{1-m}\left|\mu^q_\omega(B(\eta,t)) - \mu^q_{\omega_t}(B(\eta,t))\right| \leqslant t^{1-m}\Bigg[\int\limits_{\widehat{\partial G}\cap B(\eta,t)} q(y)|n^G(y)\cdot(\omega-\theta)|\,\mathrm{d}\lambda_{m-1}(y) + \int\limits_{\widehat{\partial G}\cap B(\eta,t)} q(y)|n^G(y)\cdot(\theta-\omega_t)|\,\mathrm{d}\lambda_{m-1}(y)\Bigg] \leqslant 2k\mu^q(\mathbb{R}^m),$$

which shows that (ii) $\Longleftrightarrow$ (iii). □

Now we are in position to prove Theorems 1, 2 announced in the introduction.

Proof of Theorem 1. If the limit (9) exists for every $f \in \mathscr{C}(\partial A, q)$, then (30) holds by Lemma 4 and (39) follows by Lemma 6. The implication (ii) $\Longrightarrow$ (i) from Lemma 8 yields (10). Conversely, (10) and Lemmas 8, 7 imply (30) and Lemma 4 guarantees the existence of the limit (9) for every $f \in \mathscr{C}(\partial A, q)$. □

Remark 3. The above proof shows that for Theorem 1 to be valid, the assumption concerning $\operatorname{contg}(S,\eta)$ can be weakened to

$$\operatorname{contg}(S,\eta) \cap \operatorname{contg}(\widehat{\partial A},\eta) = \emptyset.$$

Proof of Theorem 2. If the limits (13) exist for each $f \in \mathscr{C}(\partial A, q)$, then (30) holds for $S = \bigcup\limits_{j=1}^{m} S_j$ by Lemma 4 and (12) follows by Proposition 2. Conversely, assume (12) and suppose that

$$\operatorname{contg}(S_j,\eta) \cap \operatorname{contg}(\widehat{\partial A},\eta) = \emptyset, \quad 1 \leqslant j \leqslant m. \tag{47}$$

Since $\mu^q(\cdot) \geqslant \mu^q_\theta(\cdot)$ for any $\theta \in \partial B(0,1)$ we get by Lemma 7 that (30) holds for $S = \bigcup\limits_{j=1}^{m} S_j$ and Lemma 4 implies the existence of the limits (13) for any $f \in \mathscr{C}(\partial A, q)$. □

Remark 4. As we have observed in the above proof, the weaker assumption (47) concerning the contingents of S_j suffices for validity of Theorem 2. In the plane case $m = 2$ we obtain the following result concerning angular limits of double layer potentials.

Theorem 3. *Let $m = 2$ and let $S \subset \mathbb{R}^2 \setminus A$ be a connected set with $\eta \in \operatorname{cl} S \cap \partial A$. Denote by*

$$\check{S} = \{2\eta - x; x \in S\}$$

the reflection of S at η and suppose that

$$\operatorname{contg}(\widehat{\partial G}, \eta) \cap \operatorname{contg}(S, \eta) = \emptyset = \operatorname{contg}(\widehat{\partial G}, \eta) \cap \operatorname{contg}(\check{S}, \eta).$$

Then the limit (9) *exists and is finite for every $f \in \mathscr{C}(\partial A, q)$ iff*

$$v^q(\eta) + \sup_{r>0} \mu^q\big(B(\eta, r)\big)/r < \infty. \tag{48}$$

Proof. If the limit (9) exists for every $f \in \mathscr{C}(\partial A, q)$, then (30) holds by Lemma 4, and (12) (with $m = 2$, which is just (48)) follows by Proposition 3. Conversely, (48) together with the inequality $\mu^q(\cdot) \geqslant \mu^q_\theta(\cdot)$ valid for each $\theta \in \partial B(0,1)$ yields (30) by Lemma 7, and Lemma 4 guarantees the existence of (9) whenever $f \in \mathscr{C}(\partial A, q)$. □

Remark 5. Our main results concerning angular limits of the double layer potentials were based on obtaining geometric conditions on the boundary guaranteeing the validity of the relation (30) occurring in Lemma 4. The same relation serves as a basis for obtaining geometric conditions on ∂A guaranteeing the existence of ordinary limits (along $\operatorname{int} A$ or $\operatorname{int} G$) of the double layer potentials at a boundary point η. The following result from [10] may serve as an illustration.

Theorem 4. *Suppose that either $S = \operatorname{int} A$ or $S = \operatorname{int} G$, $\eta \in \partial S$. Then the limit* (9) *exists and is finite for every $f \in \mathscr{C}(\partial A, q)$ iff*

$$\limsup_{\substack{y \to \eta \\ y \in \partial S}} v^q(y) < \infty. \tag{49}$$

Proof. We know from Lemma 4 that (30) is necessary and sufficient for the existence of (9) for each $f \in \mathscr{C}(\partial A, q)$. It follows from the lower-semicontinuity of $v^q(\cdot)$ (cf. Corollary 1) that (30) implies (49). The converse implication (49) $\Longrightarrow$ (30) has been proved in [10], p. 32. □

Remark 6. There is a vast literature concerning boundary behaviour of double layer potentials. Various results on angular limits which are close in spirit to those occurring in the present paper may be found in [3], [6], [8], [12], [13], [14], [17], [18].

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