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LIE DERIVATIVES ON REAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

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0. INTRODUCTION

Let $P_n(\mathbb{C})$ be an n-dimensional complex projective space with Fubini-Study metric G of constant holomorphic sectional curvature 4, and let M be a real hypersurface of $P_n(\mathbb{C})$. Then M has an almost contact metric structure (φ, ξ, η, g) (cf. x 1) induced from the complex structure J of $P_n(\mathbb{C})$. Many differential geometers have studied M by using the structure (φ, ξ, η, g). Typical examples of real hypersurfaces in $P_n(\mathbb{C})$ are homogeneous ones. Takagi ([12]) classified homogeneous real hypersurfaces of $P_n(\mathbb{C})$. By virtue of his work, we find that a homogeneous real hypersurface of $P_n(\mathbb{C})$ is locally congruent to one of the six model spaces type A_1, A_2, B, C, D and E (for details, see Theorem A).

In differential geometry of real hypersurfaces of $P_n(\mathbb{C})$, it is very interesting to give a characterization of homogeneous real hypersurfaces. In particular, many geometers characterized homogeneous ones of type A_1 and A_2, because two examples have a lot of beautiful geometric properties. We here recall the work of Okumura ([11]). He showed that a real hypersurface M of $P_n(\mathbb{C})$ is locally congruent to one of homogeneous ones of type A_1 and A_2 if and only if the structure vector ξ is an infinitesimal isometry, that is $L_\xi g = 0$, where L is the Lie derivative. Motivated by this result, Udagawa and the second author established Theorem D and Ki, Kim and Lee ([2]) proved the fact that "M is of type A_1 or type A_2" is equivalent to "$L_\xi A = 0$, where A is the shape operator of M". In this paper we investigate real hypersurfaces M of $P_n(\mathbb{C})$ by using the Lie derivatives on M (cf. Theorems 1, 2 and Proposition 4).

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1. Preliminaries

Let M be an orientable real hypersurface of $P_n(\mathbb{C})$ and let N be a unit normal vector field on M. The Riemannian connections $\widetilde{\nabla}$ in $P_n(\mathbb{C})$ and ∇ in M are related by the following formulae for any vector fields X and Y on M:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{1.1}$$

$$\widetilde{\nabla}_X N = -AX, \tag{1.2}$$

where g denotes the Riemannian metric of M induced from the Fubini-Study metric G of $P_n(\mathbb{C})$ and A is the shape operator of M in $P_n(\mathbb{C})$. An eigenvector X of the shape operator A is called a *principal curvature vector.* Also an eigenvalue λ of A is called a *principal curvature.* In what follows, we denote by V_λ the eigenspace of A associated with eigenvalue λ. It is known that M has an almost contact metric structure induced from the complex structure J on $P_n(\mathbb{C})$, that is, we define a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η on M by $g(\varphi X, Y) = G(JX, Y)$ and $g(\xi, X) = \eta(X) = G(JX, Y)$. Then we have

$$\varphi^2 X = -X + \eta(X)\xi, \quad g(\xi,\xi) = 1, \quad \varphi\xi = 0. \tag{1.3}$$

It follows from (1.1) that

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{1.4}$$

$$\nabla_X \xi = \varphi AX. \tag{1.5}$$

Let $\widetilde{R}$ and R be the curvature tensors of $P_n(\mathbb{C})$ and M, respectively. Since the curvature tensor $\widetilde{R}$ has a nice form, we have the following Gauss and Codazzi equations:

$$\begin{aligned} g(R(X,Y)Z,W) &= g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &+g(\varphi Y,Z)g(\varphi X,Z) - g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W) \\ &+g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W), \end{aligned} \tag{1.6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi. \tag{1.7}$$

From (1.3), (1.5), (1.6) and (1.7) we get

$$SX = (2n+1)X - 3\eta(X)\xi + hAX - A^2X, \tag{1.8}$$

$$\begin{aligned} (\nabla_X S)Y = &- 3\{g(\varphi AX, Y)\xi + \eta(Y)\varphi AX\} + (Xh)AY \\ &+ (hI - A)(\nabla_X A)Y - (\nabla_X A)AY, \end{aligned} \tag{1.9}$$

where $h = \operatorname{tr} A$, S is the Ricci tensor of type $(1,1)$ on M and I is the identity map.

In the following, we use the same terminology and notations as above unless otherwise stated. Now we prepare without proof the following in order to prove our results:

Theorem A ([12]). *Let M be a homogeneous real hypersurface of $P_n(\mathbb{C})$. Then M lies on a tube of radius r over one of the following Kaehler submanifolds:*

(A_1) *hyperplane $P_{n-1}(\mathbb{C})$, where $0 < r < \frac{\pi}{2}$,*
(A_2) *totally geodesic $P_k(\mathbb{C})$ $(1 \leqslant k \leqslant n-2)$, where $0 < r < \frac{\pi}{2}$,*
(B) *complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,*
(C) *$P_1(\mathbb{C}) \times P_{(n-1)/2}(\mathbb{C})$, where $0 < r < \frac{\pi}{4}$ and n $(\geqslant 5)$ is odd,*
(D) *complex Grassmann $G_{2,5}(\mathbb{C})$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*
(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

Theorem B ([3]). *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

Theorem C ([10]). *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then the following are equivalent:*

(i) $\varphi A = A\varphi$,
(ii) *M is locally congruent to one of homogeneous real hypersurfaces of type* A_1 *and* A_2.

Theorem D ([8]). *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then the following are equivalent:*

(i) *$L_\xi \varphi = 0$, where L is the Lie derivative on M. Namely, ξ is an infinitesimal automorphism of φ.*
(ii) *M is locally congruent to one of homogeneous real hypersurfaces of type* A_1 *and* A_2.

Proposition A ([9]). *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

Proposition B ([9]). *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX = \lambda X$ for $X \perp \xi$, then we have $A\varphi X = ((\alpha\lambda + 2)/(2\lambda - \alpha))\varphi X$.*

Proposition C ([1]). *Let M be a connected orientable real hypersurface (with unit normal vector N) in $P_n(\mathbb{C})$ on which ξ is a principal curvature vector with principal curvature $\alpha = 2\cot 2r$ and the focal map φ_r has constant rank on M. Then the following hold:*

(i) *M lies on a tube (in the direction of $\eta = \gamma'(r)$, where $\gamma(r) = \exp_x(rN)$ and x is a base point of the normal vector N) of radius r over a certain Kaehler submanifold $\widetilde{N}$ in $P_n(\mathbb{C})$.*

(ii) *Let $\cot\theta$ be a principal curvature of the shape operator A_η at $y = \gamma(r)$ of the Kaehler submanifold $\widetilde{N}$. Then the real hypersurface M has a principal curvature $\cot(\theta - r)$ at $x = \gamma(0)$.*

Proposition D ([7]). *Let M be a real hypersurface with constant mean curvature in $P_n(\mathbb{C})$. Suppose that ξ is a principal curvature vector and the corresponding principal curvature is non-zero. If $\nabla_\xi S = 0$, then M is a tube of radius r over one of the following Kaehler submanifolds:*

(A_1) *hyperplane $P_{n-1}(\mathbb{C})$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$,*

(A_2) *totally geodesic $P_k(\mathbb{C})$ $(1 \leqslant k \leqslant n-2)$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$,*

(B) *complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$ and $\cot^2 2r = n-2$,*

(C) *$P_1(\mathbb{C}) \times P_{(n-1)/2}(\mathbb{C})$, where $0 < r < \frac{\pi}{4}$, $\cot^2 2r = 1/(n-2)$ and n $(\geqslant 5)$ is odd,*

(D) *complex Grassmann $G_{2,5}(\mathbb{C})$, where $0 < r < \frac{\pi}{4}$, $\cot^2 2r = 3/5$ and $n = 9$,*

(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$, $\cot^2 2r = 5/9$ and $n = 15$.*

2. Results

We denote by S the Ricci tensor of type $(1,1)$ on a real hypersurface M of $P_n(\mathbb{C})$. We investigate M by using the condition "$L_\xi S = 0$", where L is the Lie derivative of M. We have

Theorem 1. *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then M satisfies $L_\xi S = 0$ if and only if ξ is a principal curvature vector, in addition except for the null set on which the focal map φ_r degenerates, M lies on a tube of radius r over one of the following Kaehler submanifolds:*

(a) *totally geodesic $P_k(\mathbb{C})$ $(1 \leqslant k \leqslant n-1)$, where $0 < r < \frac{\pi}{2}$,*

(b) *complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$ and $\cot^2 2r = n-2$,*

(c) *$P_1(\mathbb{C}) \times P_{(n-1)/2}(\mathbb{C})$, where $0 < r < \frac{\pi}{4}$, $\cot^2 2r = 1/(n-2)$ and n $(\geqslant 5)$ is odd,*

(d) *complex Grassmann* $G_{2,5}(\mathbb{C})$, *where* $0 < r < \frac{\pi}{4}$, $\cot^2 2r = 3/5$ *and* $n = 9$,

(e) *Hermitian symmetric space* $SO(10)/U(5)$, *where* $0 < r < \frac{\pi}{4}$, $\cot^2 2r = 5/9$ *and* $n = 15$,

(f) k*-dimensional Kaehler submanifold* $\widetilde{N}$ *on which the rank of each shape operator is not greater than 2 with nonzero principal curvatures not equal to* $\pm\sqrt{(2k-1)/(2n-2k-1)}$ *and* $\cot^2 r = (2k-1)/(2n-2k-1)$, *where* $k = 1, \dots, n-1$.

Proof. From (1.5), for any $X \in TM$ we see that

$$\begin{aligned}(L_\xi S)X &= [\xi, SX] - S[\xi, X] \\ &= (\nabla_\xi S)X - \varphi ASX + S\varphi AX.\end{aligned}$$

And hence "$L_\xi S = 0$" is equivalent to

$$\nabla_\xi S = \varphi AS - S\varphi A. \tag{2.1}$$

Since $\nabla_\xi S$ is symmetric, (2.1) shows that

$$(\varphi A - A\varphi)S = S(\varphi A - A\varphi). \tag{2.2}$$

On the other hand, (1.8) yields that

$$(\varphi S - S\varphi) = h(\varphi A - A\varphi) - (\varphi A^2 - A^2\varphi),$$

which implies that

$$\operatorname{tr}(\varphi S - S\varphi)^2 = h \cdot \operatorname{tr}(\varphi A - A\varphi)(\varphi S - S\varphi) - \operatorname{tr}(\varphi A^2 - A^2\varphi)(\varphi S - S\varphi). \tag{2.3}$$

In general, we get

$$\operatorname{tr}(\varphi A - A\varphi)(\varphi S - S\varphi) = 2\operatorname{tr}\varphi A\varphi S - \operatorname{tr}A\varphi^2 S - \operatorname{tr}\varphi AS\varphi. \tag{2.4}$$

It follows from (2.2) that

$$\operatorname{tr}\varphi^2 AS - 2\operatorname{tr}\varphi S\varphi A + \operatorname{tr}\varphi^2 SA = 0. \tag{2.5}$$

So, from (2.4) and (2.5) we obtain

$$\operatorname{tr}(\varphi A - A\varphi)(\varphi S - S\varphi) = 0. \tag{2.6}$$

Now we find that

$$\text{(2.7)} \qquad \operatorname{tr}(\varphi A^2 - A^2\varphi)(\varphi S - S\varphi) = 2\operatorname{tr}\varphi A^2\varphi S - \operatorname{tr} A^2\varphi^2 S - \operatorname{tr}\varphi A^2 S\varphi.$$

It follows from (2.2) that

$$\varphi A(\varphi AS - S\varphi A + SA\varphi - A\varphi S) = 0,$$

so that

$$\text{(2.8)} \qquad \operatorname{tr}\varphi ASA\varphi = \operatorname{tr}\varphi A^2\varphi S.$$

Hence from (2.7) and (2.8) we find

$$\text{(2.9)} \qquad \operatorname{tr}(\varphi A^2 - A^2\varphi)(\varphi S - S\varphi) = 2\operatorname{tr}\varphi^2 ASA - \operatorname{tr}\varphi^2 SA^2 - \operatorname{tr}\varphi^2 A^2 S.$$

Then from (1.3), (1.8), (2.3), (2.6) and (2.9) we can see that

$$\operatorname{tr}(\varphi S - S\varphi)^2 = -6\|\varphi A\xi\|^2,$$

which, together with the fact that $\varphi S - S\varphi$ is symmetric, shows that $\operatorname{tr}(\varphi S - S\varphi)^2 = 0$ and ξ is a principal curvature vector. Here note that "$\operatorname{tr}(\varphi S - S\varphi)^2 = 0$" implies that "$\varphi S = S\varphi$", because $\varphi S - S\varphi$ is symmetric.

We shall classify real hypersurfaces M satisfying $\varphi S = S\varphi$ and ξ is a principal curvature vector. The following discussion is indebted to Kimura ([4, 5]): Let X be a principal curvature (unit) vector orthogonal to ξ with principal curvature λ. Since $\varphi SX = S\varphi X$, from (1.3), (1.8) and Proposition B we get the following equation

$$\text{(2.10)} \qquad \left\{\lambda - \frac{\alpha\lambda + 2}{2\lambda - \alpha}\right\} \cdot \left\{h - \lambda - \frac{\alpha\lambda + 2}{2\lambda - \alpha}\right\} = 0.$$

Since ξ is a principal curvature vector, except for the null set on which the focal map φ_r degenerates, our manifold M is a tube (of radius r) over a certain (k-dimensional) Kaehler submanifold $\widetilde{N}$ in $P_n(\mathbb{C})$. So we may put $\alpha = 2\cot 2r (= \cot r - \tan r)$ (cf. Proposition C). Hence, solving the equation $\lambda - (\alpha\lambda + 2)/(2\lambda - \alpha) = 0$, we find that $\lambda = \cot r, -\tan r$. We here denote by λ_1, λ_2 ($\neq \cot r, -\tan r$) the solutions for the quadratic equation $h - \lambda - (\alpha\lambda + 2)/(2\lambda - \alpha) = 0$. Note that (cf. Proposition B)

$$\text{(2.11)} \qquad \varphi V_{\cot r} = V_{\cot r}, \quad \varphi V_{-\tan r} = V_{-\tan r}, \text{ and } \varphi V_{\lambda_1} = V_{\lambda_2}.$$

Then M has at most five distinct principal curvatures $2\cot 2r$ (with multiplicity 1), $\cot r$ (with multiplicity $2n-2k-2$), $-\tan r$ (with multiplicity $2k-2m$), λ_1 (with multiplicity $m \geqslant 0$) and λ_2 (with multiplicity $m \geqslant 0$). Hence

$$h = (2n-2k-1)\cot r - (2k-2m+1)\tan r + m(\lambda_1+\lambda_2). \tag{2.12}$$

On the other hand

$$h = \lambda_1 + \lambda_2. \tag{2.13}$$

It follows from (2.12), (2.13) and $\lambda_2 = (\alpha\lambda_1 + 2)/(2\lambda_1 - \alpha)$ that

$$(2n-2k-1)\cot r - (2k-2m+1)\tan r + (m-1)\left\{\lambda_1 + \frac{\alpha\lambda_1+2}{2\lambda_1-\alpha}\right\} = 0. \tag{2.14}$$

In the following, our discussion is divided into three cases (I) $m=0$, (II) $m=1$ and (III) $m \geqslant 2$.

Case (I): In the case of $k=n-1$, M has two distinct constant principal curvatures $2\cot 2r$ (with multiplicity 1) and $-\tan r$ (with multiplicity $2n-2$), so that M is locally congruent to a homogeneous one of type A_1. In case that $1 \leqslant k \leqslant n-2$, M has three distinct constant principal curvatures $2\cot 2r$ (with multiplicity 1), $\cot r$ (with multiplicity $2n-2k-2$) and $-\tan r$ (with multiplicity $2k$), so that M is locally congruent to a homogeneous one of type A_2 (cf. [9]). Hence M is of case (a) in Theorem 1. As matter of course, our manifold M satisfies $\varphi S = S\varphi$ (see Theorem C).

Case (II): Our non-homogeneous real hypersurface M has at most five distinct principal curvatures $2\cot 2r$ (with multiplicity 1), $\cot r$ (with multiplicity $2n-2k-2$), $-\tan r$ (with multiplicity $2k-2$), λ_1 (with multiplicity 1) and λ_2 (with multiplicity 1). Here note that both λ_1 and λ_2 are not constant. (Moreover, Proposition C asserts that λ_1 and λ_2 can expressed as: $\lambda_1 = \cot(r-\theta)$ and $\lambda_2 = \cot(r+\theta)$, where $\cot\theta$ is a principal curvature of the Kaehler submanifold $\widetilde{N}$). In addition, equation (2.14) shows that

$$\cot^2 r = \frac{2k-1}{2n-2k-1}. \tag{2.15}$$

Hence we find that M is of case (f) in Theorem 1.

Case (III): It follows from (2.14) and Proposition A that λ_1 is constant. Therefore we can see that our manifold M is homogeneous (cf. Theorem B). Now we shall check $\varphi S = S\varphi$ one by one for homogeneous real hypersurfaces of type B, C, D and E. Since ξ is a principal curvature vector, $\varphi S\xi = 0 = S\varphi\xi$ holds. So, we have only to consider the condition that $\varphi SX = S\varphi X$ for any $X(\perp\xi)$.

Let M be of type B (which is a tube of radius r). Let $x = \cot r$. Then M has three distinct constant principal curvatures $r_1 = (1+x)/(1-x)$ with multiplicity $n-1$, $r_2 = (x+1)/(x-1)$ with multiplicity $n-1$ and $\alpha = (x^2-1)/x$ with multiplicity 1. Since $\varphi V_{r_1} = V_{r_2}$, $\varphi S = S\varphi$ is equivalent to $h - r_1 - r_2 = 0$. Then we have the following algebraic equation $x^4 - 2(2n-3)x^2 + 1 = 0$. Hence we find $x^2 = 2n - 3 \pm 2\sqrt{(n-1)(n-2)}$ so that $x = \sqrt{n-1} + \sqrt{n-2}$, since $x > 1$. So, M is of case (b) in Theorem 1. Now let M be of type C (which is a tube of radius r). Let $x = \cot r$. Then M has five distinct constant principal curvatures $r_1 = (1+x)/(1-x)$ with multiplicity 2, $r_2 = (x+1)/(x-1)$ with multiplicity 2, $r_3 = x$ with multiplicity $n-3$, $r_4 = -1/x$ with multiplicity $n-3$ and $\alpha = (x^2-1)/x$ with multiplicity 1. In case that $X \in V_{r_3}$ or V_{r_4}, $\varphi SX = S\varphi X$ for any radius r. So, $\varphi S = S\varphi$ is equivalent to $h - r_1 - r_2 = 0$. Then we have the following equation $(n-2)x^4 - 2nx^2 + n - 2 = 0$. And hence we find $x^2 = (n \pm 2\sqrt{n-1})/(n-2)$ so that $x = (\sqrt{n-1}+1)/\sqrt{n-2}$, since $x > 1$. Hence, M is of case (c) in Theorem 1. Let M be of type D (which is a tube of radius r). Let $x = \cot r$. Then M has five distinct constant principal curvatures $r_1 = (1+x)/(1-x)$ with multiplicity 4, $r_2 = (x+1)/(x-1)$ with multiplicity 4, $r_3 = x$ with multiplicity 4, $r_4 = -1/x$ with multiplicity 4 and $\alpha = (x^2-1)/x$ with multiplicity 1. By virtue of the computation in case of type C we have only to solve the equation $h - r_1 - r_2 = 0$. Namely we get the following $5x^2 - 22x^2 + 5 = 0$ so that $x = (\sqrt{8} + \sqrt{3})/\sqrt{5}$. Hence, M is of type (d) in Theorem 1. Let M be of type E (which is a tube of radius r). Let $x = \cot r$. Then M has five distinct constant principal curvatures $r_1 = (1+x)/(1-x)$ with multiplicity 6, $r_2 = (x+1)/(x-1)$ with multiplicity 6, $r_3 = x$ with multiplicity 8, $r_4 = -1/x$ with multiplicity 8 and $\alpha = (x^2-1)/x$ with multiplicity 1. Considering the equation $h - r_1 - r_2 = 0$, we have the following $9x^4 - 38x^2 + 9 = 0$ so that $x = (\sqrt{5} + \sqrt{14})/3$. Hence, M is of case (e) in Theorem 1.

The rest of the proof is to check $L_\xi S = 0$ for examples (a), (b), (c), (d), (e) and (f) in Theorem 1. Since our all examples satisfy $\varphi S = S\varphi$ and $SA = AS$ (that is, ξ is a principal curvature vector), (2.1) tells us that "$L_\xi S = 0$" is equivalent to "$\nabla_\xi S = 0$." So we shall verify $\nabla_\xi S = 0$ one by one for the six model spaces of case (a), (b), (c), (d), (e) and (f) in Theorem 1:

Let M be of case (a). Then we see that $\nabla_\xi A = 0$ (cf [9]). Moreover, ξ is a principal curvature vector and $\xi(\mathrm{tr}\, A) = 0$. And hence (1.9) yields that $\nabla_\xi S = 0$. Next, let M be of case (b), (c), (d) or (e). Obvious (cf. Proposition D). Finally let M be of case (f). The manifold M (which is a tube of radius r) has at most five distinct principal curvatures $2\cot 2r$ (with multiplicity 1), $\cot r$ (with multiplicity $2n-2k-2$), $-\tan r$ (with multiplicity $2k-2$), $\lambda_1 = \cot(r-\theta)$ (with multiplicity 1) and $\lambda_2 = \cot(r+\theta)$ (with multiplicity 1) and M satisfies (2.11), (2.13) and (2.15). First we shall compute

$\nabla_\xi A$. Since ξ is a principal curvature vector, we easily see that

$$(\nabla_\xi A)\xi = 0. \tag{2.16}$$

From (1.5), (1.7) and (2.11) we obtain the following:

$$(\nabla_\xi A)X = 0 \quad \text{for any } X \in V_{\cot r}, \tag{2.17}$$

$$(\nabla_\xi A)Y = 0 \quad \text{for any } Y \in V_{-\tan r}, \tag{2.18}$$

$$(\nabla_\xi A)Z = a\varphi Z \quad \text{for any } Z \in V_{\lambda_1}, \tag{2.19}$$

$$(\nabla_\xi A)W = -a\varphi W \quad \text{for any } W \in V_{\lambda_2}, \tag{2.20}$$

where $a = (1-\cot^4 r)\cot\theta/\{\cot r(\cot r + \cot\theta)(\cot r - \cot\theta)\}$. And hence the equation (2.16) $\sim$ (2.20) imply that

$$\xi(\operatorname{tr} A) = \operatorname{tr}(\nabla_\xi A) = 0. \tag{2.21}$$

Then, from (1.9), (2.11), (2.16), (2.17), (2.18) and (2.21) we get

$$(\nabla_\xi S)\xi = (\nabla_\xi S)X = (\nabla_\xi S)Y = 0 \quad \text{for any } X \in V_{\cot r} \text{ and for any } Y \in V_{-\tan r}. \tag{2.22}$$

It follows from (1.9), (2.11), (2.19) and (2.21) that

$$(\nabla_\xi S)Z = a(h - \lambda_1 - \lambda_2)\varphi Z \quad \text{for any } Z \in V_{\lambda_1},$$

which, combined with (2.13), shows that

$$(\nabla_\xi S)Z = 0 \quad \text{for any } Z \in V_{\lambda_1}. \tag{2.23}$$

Similarly, from (1.9), (2.11), (2.13), (2.20) and (2.21) we see that

$$(\nabla_\xi S)W = 0 \quad \text{for any } W \in V_{\lambda_2}. \tag{2.24}$$

Thus, from (2.22), (2.23) and (2.24) we find that the manifold M satisfies $\nabla_\xi S = 0$. $\square$

Remark 1. (1) In case (f), condition "the Kaehler submanifold $\widetilde{N}$ does not have principal curvatures $\pm\sqrt{(2k-1)/(2n-2k-1)}$" is necessary. In general, Statement (ii) in Proposition C shows that the point x $(= \gamma(0))$ is a singular point of M (that is, M is not smooth at the point x) in the case of $r = \theta$.

(2) When $k = 1$ in case (f), M is a tube of radius r with $\cot^2 r = 1/(2n-3)$ over a complex curve (in $P_n(\mathbb{C})$) with nonzero principal curvatures $\neq \pm 1/\sqrt{2n-3}$.

(3) In general, "$\nabla_\xi A = 0$" implies "$\nabla_\xi S = 0$" (cf. Proposition 1). Of course the inverse is not true. All examples (a), (b), (c), (d), (e) and (f) satisfy $\nabla_\xi S = 0$. But $\nabla_\xi A \neq 0$ for examples (b), (c), (d), (e) and (f) in the case of $n \neq 2k$. The classification problem of real hypersurfaces M (in $P_n(\mathbb{C})$) satisfying $\nabla_\xi A = 0$ was solved by the present authors. Note that "$\nabla_\xi A = 0$" is equivalent to "$A\xi = 0$" for non-homogeneous real hypersurfaces M (for details, see [7]). However, the classification problem of real hypersurfaces M (in $P_n(\mathbb{C})$) satisfying $\nabla_\xi S = 0$ is an open problem.

Proposition 1. *$\nabla_\xi A = 0$ always implies $\nabla_\xi S = 0$.*

Proof. We remark that "$\nabla_\xi A = 0$" implies that "ξ is a principal curvature vector" (see, Proposition 7 in [7]) and $\xi(\operatorname{tr} A) = \operatorname{tr}(\nabla_\xi A) = 0$. So, equation (1.9) asserts that $\nabla_\xi S = 0$. □

Now, in relation to Theorem D we establish the following

Theorem 2. *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then the following are equivalent:*

(i) *ξ is a principal curvature vector and $(L_\xi \varphi)^2 = -c^2\varphi^2$, where c is locally constant.*

(ii) *M is locally congruent to one of homogeneous real hypersurfaces of type* A_1, A_2 *and* B.

Proof. For any $X \in TM$ we see that

$$\begin{aligned}(L_\xi \varphi)X &= [\xi, \varphi X] - \varphi[\xi, X] \\ &= (\nabla_\xi \varphi)X - \nabla_{\varphi X}\xi + \varphi \nabla_X \xi.\end{aligned}$$

This, together with (1.3), (1.4) and (1.5), yields

$$(L_\xi \varphi)X = \eta(X)A\xi - AX - \varphi A \varphi X. \tag{2.25}$$

Then from (1.3) and (2.25), for any $X \in TM$ we get

$$\begin{aligned}(L_\xi \varphi)^2 X = {} & (\varphi A - A\varphi)^2 X - g(\varphi A\xi, X)\varphi A\xi \\ & + \eta(X)\{\eta(A\xi)A\xi - A^2\xi - \varphi A \varphi A \xi\}.\end{aligned} \tag{2.26}$$

In the following we study M satisfying condition (i) in Theorem 2. Since ξ is a principal curvature vector, from (2.26) we see that $(L_\xi\varphi)^2 = -c^2\varphi^2$ is equivalent to

$$(\varphi A - A\varphi)^2 X = -c^2\varphi^2 X \quad \text{for any } X \in TM. \tag{2.27}$$

Let X be a principal curvature (unit) vector orthogonal to ξ with principal curvature λ. From (1.3), (2.27) and Proposition B we obtain the following

$$4(\lambda^2 - \alpha\lambda - 1)^2 = c^2(2\lambda - \alpha)^2, \tag{2.28}$$

which, combined with Proposition A, yields that λ is constant. Therefore by virtue of Proposition A and Theorem B we find that M is homogeneous.

First let M be of type A_1 or type A_2. Then Theorem C guarantees (2.27). Now we consider M of type B (which is a tube of radius r). Let $x = \cot r$. Then M has three distinct constant principal curvatures r_1, r_2 and α (see, Case (III) in the proof of Theorem 1). Then by a direct calculation we know that (2.27) holds, when $c = 2(x^2+1)/(x^2-1)$. Next let M be of type C, type D or type E (which is a tube of radius r). Then M has five distinct constant principal curvatures r_1, r_2, r_3, r_4 and α (cf. Case (III)). Suppose that (2.27) holds. Then (2.28) tells us that $c = 0$, when $\lambda = r_3, r_4$. On the other hand, (2.28) shows that $c = 2(x^2+1)/(x^2-1)$ $(\neq 0)$, when $\lambda = r_1, r_2$. These statements contradict each other. □

Remark 2. As an immediate consequence of Theorem 2, we find

Corollary 1. *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then the following are equivalent:*

(i) *ξ is a principal curvature vector and $(L_\xi\varphi)^2 = -c^2\varphi^2$, where c is nonzero locally constant.*
(ii) *M is locally congruent to a homogeneous real hypersurface of type* B.

If we omit the hypothesis that ξ is a principal curvature vector, Theorem 2 is not true (cf. Proposition 2). The first author of the present paper constructed a class of non-homogeneous real hypersurfaces M (in $P_n(\mathbb{C})$) which are called ruled real hypersurfaces (cf. [6]). We say that M is a *ruled* real hypersurface if there is a foliation of M by complex hyperplanes $P_{n-1}(\mathbb{C})$. More precisely, let T^0M be the distribution defined by $T^0_xM = \{X \in X_xM \mid X \perp \xi\}$ for $x \in M$. Then M is ruled if and only if T^0M is integrable and its integral manifold is a totally geodesic submanifold $P_{n-1}(\mathbb{C})$.

Now, for later use we shall write down the shape operator A of a ruled real hypersurface M:

$$
\begin{cases} A\xi = \mu\xi + \nu U \quad (\nu \neq 0), \\ AU = \nu\xi, \\ AX = 0 \quad (\text{for any } X \perp \xi,\ U), \end{cases} \tag{2.29}
$$

where U is a unit vector orthogonal to ξ, μ and ν are differentiable functions on M.

So, (2.29) implies that the vector ξ of any ruled real hypersurface M in $P_n(\mathbb{C})$ is not principal. Moreover we have

Proposition 2. *Any ruled real hypersurface M in $P_n(\mathbb{C})$ satisfies $(L_\xi\varphi)^2 = 0$.*

Proof. It follows from (1.3), (2.25) and (2.29) that $(L_\xi\varphi)\xi = (L_\xi\varphi)X = 0$ for any $X(\perp\xi, U)$. In addition, from (2.25)and (2.29) we find that $(L_\xi\varphi)U = -\nu\xi$. Hence $(L_\xi\varphi)^2U = -\nu(L_\xi\varphi)\xi = 0$. Therefore we get our conclusion. □

Now we shall a provide a characterization of a ruled real hypersurface M in $P_n(\mathbb{C})$. First we prepare the following

Proposition 3. *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then the following are equivalent:*

(i) $(L_\xi\varphi)^2 = 0$,

(ii) $(\varphi A - A\varphi)X = \eta(X)\varphi A\xi + g(\varphi A\xi, X)\xi$ *for any* $X \in TM$,

(iii) $g((\varphi A - A\varphi)X, Y) = 0$ *for any* $X, Y \perp \xi$.

Proof. Equation (2.26) tells us that $(L_\xi\varphi)^2 = 0$ is equivalent to

$$
\begin{aligned} (\varphi A - A\varphi)^2 X = {} & g(\varphi A\xi, X)\varphi A\xi \\ & + \eta(X)\{\varphi A\varphi A\xi + A^2\xi - \eta(A\xi)A\xi\} \\ & \text{for any } X \in TM. \end{aligned} \tag{2.30}
$$

(i) $\Rightarrow$ (ii): Since $(\varphi A - A\varphi)^2$ is symmetric, (2.30) yields that $(\varphi A\varphi A\xi + A^2\xi - \eta(A\xi)A\xi)$ is proportional to ξ. Then (1.3) shows that

$$
\varphi A\varphi A\xi + A^2\xi - \eta(A\xi)A\xi = \|\varphi A\xi\|^2\xi. \tag{2.31}
$$

Since $\varphi A - A\varphi$ is symmetric, the equations (2.30) and (2.31) provide us with

$$
(\varphi A - A\varphi)X = 0 \text{ for any } X \perp \xi, \varphi A\xi. \tag{2.32}
$$

From (1.3) and (2.31) we obtain the following

$$(\varphi A - A\varphi)\xi = \varphi A\xi, \tag{2.33}$$

$$(\varphi A - A\varphi)\varphi A\xi = \|\varphi A\xi\|^2\xi. \tag{2.34}$$

It follows from (2.32), (2.33) and (2.34) that condition (ii) holds.

(ii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (ii): Condition (iii) implies that

$$g((\varphi A - A\varphi)\varphi X, \varphi Y) = 0 \text{ for any } X, Y \in TM. \tag{2.35}$$

From (1.3) and (2.35) we get condition (ii). □

We are now in a position to prove the following

Proposition 4. *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then $(L_\xi\varphi)^2 = 0$ and the holomorphic distribution T^0M $(= \{X \in TM \mid \eta(X) = 0\})$ is integrable if and only if M is locally congruent to a ruled real hypersurface of $P_n(\mathbb{C})$.*

Proof. ($\Rightarrow$) It is known that "T^0M is integrable" is equivalent to the following (see, Proposition 5 in [6]):

$$g((\varphi A + A\varphi)X, Y) = 0 \text{ for any } X, Y \in T^0M. \tag{2.36}$$

It follows from condition (iii) in Proposition 3 and (2.36) that $g(AX, Y) = 0$ for any $X, Y \in T^0M$. This implies that our manifold M is locally congruent to a ruled real hypersurface.

($\Leftarrow$) See, Proposition 2. □

Remark 3. The classification problem of real hypersurfaces M in $P_n(\mathbb{C})$ satisfying $(L_\xi\varphi)^2 = 0$ is still open.

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