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ON GIGANTIC DENSITY OF ZEROS OF SOME SIGNALS DEFINED BY PRIME NUMBERS

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1. FORMULATION OF THE RESULTS

We introduce an infinite family of trigonometric sums with prime numbers

$$G(x;t,\varphi)=\sum_{p\leqslant P}\frac{1}{\sqrt{p+x}}\cos\{t\ln(p+x)+\varphi_p\},\quad x\in\langle 0,1\rangle\,, \tag{1}$$

where p and P are prime numbers and

$$t>0,\quad \varphi_p\in\langle -\pi,\pi\rangle\,,\quad \varphi=(\varphi_2,\varphi_3,\varphi_5,\dots,\varphi_p).$$

For $x=0$, $\varphi=0$ we obtain

$$G(0;t,0)=\sum_{p\leqslant P}\frac{1}{\sqrt{p}}\cos(t\ln p), \tag{2}$$

that is, the classical trigonometric sum with prime numbers, connected with the main term of the Riemann-Siegel formula [4, p. 94]. In the opposite direction, (1) results from the amplitude-frequency-phase modulation of (2).

From the family (1) we select an infinite subset corresponding to the horizontal segment $x\in\langle 0,1\rangle$, $t=T$:

$$G(x;T,\varphi)=\sum_{p\leqslant P}\frac{1}{\sqrt{p+x}}\cos\{T\ln(p+x)+\varphi_p\}. \tag{3}$$

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From the viewpoint of theoretical radio-engineering we call the function $G(x;T,\varphi)$, $x \in \langle 0,1\rangle$ with an arbitrary fixed vector φ a signal. Let, for instance, φ_p be independent random quantities distributed uniformly on the interval $\langle -\pi,\pi\rangle$. Then $G(x;t,\varphi)$ is a random process while the signal is its realization.

For signals of the type (3) we have

Theorem. *Let $x \in \langle 0, 1/(9P^2)\rangle$, $P \in \langle 2, T^{1/9}\rangle$. Then for an arbitrary fixed φ the interval*

$$\left(x, x+\Delta\frac{P^6}{T\ln^4 P}\right), \quad T\to\infty$$

with Δ a sufficiently large positive number contains a zero of an odd order of the signal $G(x;t,\varphi)$.

Let $N_0(\langle x_1,x_2\rangle\,;G)$ denote the number of zeros of odd orders of the signal

$$G(x;T,\varphi), \quad x\in\langle x_1,x_2\rangle \subset \left\langle 0,\frac{1}{9P^2}\right\rangle, \quad \Delta\frac{P^6}{T\ln^4 P}\leqslant x_2-x_1. \tag{4}$$

The theorem yields

Corollary.

$$N_0(\langle x_1,x_2\rangle\,;G) > \frac{1}{2\Delta}(x_2-x_1)T\frac{\ln^4 P}{P^6}, \quad T\to\infty.$$

Remark 1. In the case $2\leqslant P\leqslant \ln T$ which is crucial for us the zeros of the signal (4) are distributed directly with gigantic density.

We introduce also an infinite family of complex signals

$$G(z;T,\varphi)=\sum_{p\leqslant P}\frac{1}{\sqrt{p+z}}\cos\{T\ln(p+z)+\varphi_p\},$$

where $z=x+i\tau$, $x\in\langle 0,1\rangle$ and $\ln(p+z)$ denotes the principal value of the logarithm.

Remark 2. It is of interest to consider the problem of existence of zeros of the signal $G(z;T,\varphi)$ which do not lie on the "critical" segment $x\in\langle 0,1\rangle$, $\tau=0$.

The proof of the above theorem is given in Sections 4–9 of the present paper.

2. Oscillation characteristics of the signal $G(x;T,\varphi)$

The signal $G(x;T,\varphi)$ represents the result of interference of oscillators of the type

$$a_p(x)\cdot\cos\{\Phi(x;T,\varphi_p)\},\quad p\leqslant P,$$

where

$$a_p(x)=\frac{1}{\sqrt{p+x}},\quad \Phi(x;T,\varphi_p)=T\ln(p+x)+\varphi_p$$

is respectively the amplitude and the phase,

$$\Omega(p)=\frac{\partial\Phi}{\partial x}=\frac{T}{p+x}$$

is the spectrum of circular frequencies,

$$f(p)=\frac{1}{2\pi}\Omega(p)=\frac{1}{2\pi}\cdot\frac{T}{p+x} \tag{6}$$

is the spectrum of the phases in hertz.

Remark 3. The differences of the adjacent frequencies of the spectrum

$$f(p_{n-1})-f(p_n)=\frac{1}{2\pi}\cdot\frac{T}{(p_{n-1}+x)(p_n+x)}\cdot(p_n-p_{n-1})$$

are very large (p_n denotes the n-th prime number) and for the relative increments of frequencies we have

$$\frac{f(p_{n-1}-f(p_n)}{f(p_{n-1})}=\frac{p_n-p_{n-1}}{p_n+x}.$$

Moreover, the differences p_n-p_{n-1} of the adjacent primes behave in an extremely irregular manner.

3. Definition of the KWN estimate for the number of zeros of the signal

In connection with the Kotelnikov-Whittaker-Neuquist theorem from the theory of information the following result is used in radio-engineering (cf. [1, pp. 81, 86, 96, 97]).

Empirical Rule. If the period of a signal $F(t)$ is U (e.g. $t\in\langle T,T+U\rangle$) and its spectrum is approximately bounded by a frequency W (in hertz) and if $2WU\gg 1$, then the function $F(t)$ is determined "with a high degree of accuracy" by its values at $2WU$ nodes which lie at the distance of $1/2W$.

The quantity $1/2W$ is called the KWN frequency of the signal $F(t)$, and is fundamental for our analysis.

Let $N_0(\langle T, T+U\rangle\,;F)$ denote the number of zeros of odd orders of the signal $F(t)$, $t \in \langle T, T+U\rangle$. As concerns the number of the KWN nodes in the interval $\langle T, T+U\rangle$ we formulate the following conjectures.

Conjecture 1 (strong form).

$$N_0(\langle T, T+U\rangle\,;F) \sim \frac{U}{\frac{1}{2W}} = 2WU.$$

Conjecture 2 (weak form).

$$N_0(\langle T, T+U\rangle\,;F) < (1+\varepsilon)2WU,$$

where ε is an arbitrarily small positive number.

Turning back to our problem we introduce

Definition. A lower estimate of the type

$$N_0(\langle T, T+U\rangle\,;F) > A\cdot 2WU, \quad 0 < A < 1 \tag{7}$$

is called a *KWN estimate.*

Since the KWN frequency corresponding to the signal $F(x;T,\varphi)$, $x \in \langle x_1, x_2\rangle$ is (see (6))

$$2f(2) = \frac{1}{\pi}\cdot\frac{T}{2+x} \leqslant \frac{T}{2\pi} = 2W$$

the estimate (7) assumes the form

$$N_0(\langle x_1, x_2\rangle\,;G) > A\cdot\frac{T}{2\pi}(x_2 - x_1). \tag{8}$$

Consequently, we make

Remark 4. By virtue of (8) the estimate (5) is a KWN estimate only provided $P = O(1)$, $t\to\infty$, that is, provided the number of oscillators generating the signal $G(x;T,\varphi)$ is bounded (e.g. for $P = 5, 17, 257$).

Remark 5. The relation between the KWN Theorem and the Riemann-Siegel formula is dealt with in the papers [2], [3].

4. THE MEAN VALUE OF THE FUNCTION $G^2(x; T, \varphi)$

In what follows, the asymptotic identity

$$\sum_{p \leqslant P} \frac{1}{p} = \ln\ln P + O(1), \quad P \to \infty \tag{9}$$

due to Mertens and the estimates

$$\sum_{p \leqslant P} 1 = \pi(P) = O\Big(\frac{P}{\ln P}\Big), \quad \sum_{p \leqslant P} \frac{1}{\sqrt{P}} = O\Big(\frac{\sqrt{P}}{\ln P}\Big), \tag{10}$$
$$\sum_{p \leqslant P} p^{\frac{k}{2}} = O\Big(\frac{P^{\frac{k}{2}+1}}{\ln P}\Big), \quad k = 1, 2, \dots .$$

will be useful. We have

Lemma 1. *For all sufficiently large T, $\Delta > 0$ and for arbitrary x_1, x_2 satisfying*

$$\langle x_1, x_2 \rangle \subset \langle 0, 1 \rangle, \quad \Delta \frac{P^3}{T \ln^2 P} \leqslant x_2 - x_1 \tag{11}$$

we have

$$\alpha(1) + O\Big(\frac{1}{\Delta}\Big) < \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} G^2(x; T, \varphi)\, \mathrm{d}x < \alpha(0) + O\Big(\frac{1}{\Delta}\Big), \tag{12}$$
$$\alpha(x) = \frac{1}{2} \sum_{p \leqslant P} \frac{1}{p + x}, \quad 2 \leqslant P \leqslant K,$$

where K is a sufficiently large number. Further,

$$\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} G^2(x; T, \varphi)\, \mathrm{d}x \sim \frac{1}{2} \ln\ln P, \quad P \to \infty. \tag{13}$$

The relations (12), (13) *are valid uniformly for x_1, x_2 and φ.*

Proof. We have $(p, q \leqslant P)$

$$G^2(x; T, \varphi) = \frac{1}{2} \sum_{p} \frac{1}{p + x} + \sum_{p > q}\sum \frac{\cos \omega_3}{\sqrt{(p + x)(q + x)}} \tag{14}$$
$$+ \frac{1}{2} \sum_{p,q}\sum \frac{\cos \omega_3}{\sqrt{(p + x)(q + x)}} = S_1 + S_2 + S_3,$$

where

$$\omega_2 = T\ln\frac{p+x}{q+x} + \varphi_p - \varphi_q, \quad \omega_3 = T\ln[(p+x)(q+x)] + \varphi_p + \varphi_q.$$

Since

$$\int_{x_1}^{x_2} S_2\,\mathrm{d}x = \sum_{p>q}\sum I_2,$$

where

$$I_2 = \int_{x_1}^{x_2} \frac{\cos\omega_2}{\sqrt{(p+x)(q+x)}}\,\mathrm{d}x = \int_{x_1}^{x_2} \frac{\mathrm{d}(\sin\omega_2)}{\omega_2'\sqrt{(p+x)(q+x)}}$$

$$= \frac{1}{T(q-p)}\Big\{\big[\sqrt{(p+x)(q+x)}\sin\omega_2\big]_{x_1}^{x_2} - \frac{1}{2}\int_{x_1}^{x_2} \frac{p+q+2x}{\sqrt{(p+x)(q+x)}}\sin\omega_2\,\mathrm{d}x\Big\}$$

$$= O\Big(\frac{1}{2}\sqrt{pq}\Big) + O\Big(\frac{x_2-x_1}{T}\sqrt{\frac{p}{q}}\Big) = O\Big(\frac{1}{2}\sqrt{pq}\Big),$$

we have (see (10))

$$\int_{x_1}^{x_2} S_2\,\mathrm{d}x = O\Big(\frac{P^3}{T\ln^2 P}\Big) \tag{15}$$

uniformly for x_1, x_2 and φ. Quite analogously we obtain the estimate

$$\int_{x_1}^{x_2} S_3\,\mathrm{d}x = O\Big(\frac{P^2}{T\ln^2 P}\Big). \tag{16}$$

Since

$$\alpha(1)\cdot(x_2-x_1) + O\Big(\frac{P^3}{T\ln^2 P}\Big) < \int_{x_1}^{x_2} G^2\,\mathrm{d}x < \alpha(0)\cdot(x_2-x_1) + O\Big(\frac{P^3}{T\ln^2 P}\Big)$$

provided $2 \leqslant P \leqslant K$ (see (14)–(16)), by virtue of (11) we obtain (12). If $P \to \infty$ then (see(9))

$$S_1 = \frac{1}{2}\sum_{p\leqslant P}\frac{1}{p} + O\Big(\sum_p \frac{1}{P^2}\Big) = \frac{1}{2}\ln\ln P + O(1)$$

and consequently

$$\int_{x_1}^{x_2} G^2\,\mathrm{d}x = \frac{1}{2}(x_2-x_1)\ln\ln P + O(x_2-x_1) + O\Big(\frac{P^3}{T\ln^2 P}\Big).$$

By virtue of (11) this implies (13). □

5. Asymptotic formula for the double sum

We have

Lemma 2. *The identity*

$$(17)\qquad \sum\sum_{\substack{p_1,p_2\leqslant P\\ p_1\neq p_1}}\int\limits_{x_1}^{x_2}\frac{dx}{(p_1+x)(p_2+x)}=(x_2-x_1)(\ln\ln P)^2+O\{(x_2-x_1)\ln\ln P\}.$$

holds for all sufficiently large P and $\langle x_1,x_2\rangle\subset\langle 0,1\rangle$.

P r o o f. First of all, we have

$$(18)\qquad \begin{aligned}\frac{1}{(p_1+x)(p_2+x)}&=\frac{1}{p_2-p_1}\sum_{k=0}^{\infty}(-1)^k\Big(\frac{1}{p_1^{k+1}}-\frac{1}{p_2^{k+1}}\Big)x^k\\&=\frac{1}{p_1p_2}+\frac{1}{p_2-p_1}\sum_{k=1}^{\infty}(-1)^k\Big(\frac{1}{p_1^{k+1}}-\frac{1}{p_2^{k+1}}\Big)x^k\\&=\frac{1}{p_1p_2}+R.\end{aligned}$$

Since ($p_1<p_2$)

$$\begin{aligned}|R|&<\frac{2}{p_2-p_1}\sum_{k=1}^{\infty}\Big(\frac{1}{p_1^{k+1}}-\frac{1}{p_2^{k+1}}\Big)x^k=2\frac{p_2+p_1-1}{p_1^2p_2^2\big(1-\frac{1}{p_1}\big)\big(1-\frac{1}{p_2}\big)}\\&<8\frac{p_1+p_2}{p_1^2p_2^2}=8\Big(\frac{1}{p_1p_2^2}+\frac{1}{p_1^2p_2}\Big)\end{aligned}$$

and also (see (9))

$$\begin{aligned}\sum\sum_{p_1\neq p_2}\frac{1}{p_1p_2}&=\Big(\sum\frac{1}{p_1}\Big)^2-\sum\frac{1}{p_1^2}=(\ln\ln P)^2+O(\ln\ln P),\\ \sum\sum_{p_1\neq p_2}R&=O\Big(\sum\frac{1}{p_1}\sum\frac{1}{p_2^2}\Big)=O(\ln\ln P),\end{aligned}$$

we obtain (17) from (18). □

We have

Lemma 3. *Let*

$$V(x_1,x_2,T,P;\varphi)=\sum_{\substack{p_1\neq p_4,p_2\neq p_4,p_3\neq p_4\\ p_1\neq p_2,p_1\neq p_3,p_2\neq p_3}}\sum\sum\sum I, \tag{19}$$

where

$$I=\int_{x_1}^{x_2}\frac{\cos\omega}{\sqrt{(p_1+x)\dots(p_4+x)}}\,\mathrm{d}x,\quad x_1,x_2\in\left\langle 0,\frac{1}{8P^2}\right\rangle, \tag{20}$$
$$\omega=T\ln\frac{(p_1+x)(p_2+x)(p_3+x)}{p_4+x}+\varphi_{p_1}+\varphi_{p_2}+\varphi_{p_3}-\varphi_{p_4}.$$

Then for all sufficiently large $T>0$ the estimate

$$V=O\Big(\frac{1}{T}\cdot\frac{P^6}{\ln^4 P}\Big) \tag{21}$$

holds uniformly for x_1, x_2 and φ.

Proof. We have

$$I=\frac{1}{T}\int_{x_1}^{x_2}\frac{1}{M(x)}\sqrt{(p_1+x)\dots(p_4+x)}\,\mathrm{d}(\sin\omega),$$
$$M(x)=(p_1+p_2)p_3p_4+(p_4-p_3)p_1p_2+2p_4(p_1+p_2+p_3)x$$
$$+(p_1+p_2+p_3+3p_4)x^2+2x^3=a+2bx+cx^2+2x^3.$$

If $a=0$ then

$$(p_1+p_2)p_3p_4=(p_3-p_4)p_1p_2. \tag{22}$$

However, in our case

$$p_3\nmid(p_3-p_4),\ p_3\nmid p_1,\ p_3\nmid p_2,$$
$$p_4\nmid(p_3-p_4),\ p_4\nmid p_1,\ p_4\nmid p_2,$$
$$(p_1-p_2)\nmid p_1,\ (p_1+p_2)\nmid p_2,\ (p_1+p_2)\nmid(p_3-p_4);$$

consequently, if $(p_1+p_2)\mid(p_3-p_4)$ then $p_3-p_4=k(p_1+p_2)$ and (see (22)) $p_3p_4=kp_1p_2$, a contradiction. Hence $|a|\geqslant 1$. Further (see (20), $P\geqslant 2$),

$$2bx+cx^2+2x^3\leqslant\frac{3}{4}+2^{-6}+2^{-12}<0.77.$$

Consequently,

$$\begin{aligned}&|M(x)|\geqslant 0.23,\\&M'(x)=2b+2cx+6x^2<A(p_1+p_2+p_3)p_4\end{aligned}$$

for $x\in\langle x_1,x_2\rangle$. Now

$$\begin{aligned}I&=O\Big(\frac{1}{T}\sqrt{p_1p_2p_3p_4}\Big)+\frac{1}{T}\int_{x_1}^{x_2}\Big\{\frac{1}{2M(x)}\Big(\sqrt{\frac{(p_1+x)(p_2+x)(p_2+x)}{p_4+x}}+\dots\Big)\\&\quad-\sqrt{(p_1+x)\dots(p_4+x)}\cdot\frac{M'(x)}{M^2(x)}\Big\}\sin\omega\,\mathrm{d}x\\&=O\Big(\frac{1}{T}\sqrt{p_1p_2p_3p_4}\Big)+\Big\{\frac{x_2-x_1}{T}\Big(\sqrt{\frac{p_2p_3p_4}{p_1}}+\dots\Big)\Big\}\\&\quad+O\Big\{\frac{x_2-x_1}{T}p_4^{3/2}\sqrt{p_1p_2p_3}(p_1+p_2+p_3)\Big\}\end{aligned}$$

and finally (see (10), (19), (20))

$$\begin{aligned}V&=O\Big\{\frac{1}{T}\Big(\sum\sqrt{p_1}\Big)^4\Big\}+O\Big\{\frac{x_2-x_1}{T}\sum\frac{1}{p_1}\Big(\sum\sqrt{p_2}\Big)^3\Big\}\\&\quad+O\Big\{\frac{x_2-x_1}{T}\Big(\sum p_1^{3/2}\Big)^2\Big(\sum\sqrt{p_2}\Big)^2\Big\}\\&=O\Big(\frac{1}{T}\cdot\frac{P^6}{\ln^4 P}\Big)=O\Big(\frac{x_2-x_1}{T}\cdot\frac{P^8}{\ln^4 P}\Big)\\&=O\Big(\frac{1}{T}\cdot P^6\ln^4 P\Big)\end{aligned}$$

uniformly for x_1, x_2 and φ. □

7. Mean value of the function $G^4(x;T,\varphi)$

We have

Lemma 4. *For all sufficiently large T, $\Delta > 0$ and for arbitrary x_1, x_2 satisfying the conditions*

$$\langle x_1, x_2\rangle \subset \left\langle 0, \frac{1}{8P^2}\right\rangle, \quad \frac{P^6}{T\ln^4 P} \leqslant x_2 - x_1; \quad 2 \leqslant P \leqslant T^{1/9}, \tag{23}$$

we have

$$\beta(1) + O\Big(\frac{1}{\Delta}\Big) < \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} G^4(x;T,\varphi)\,\mathrm{d}x < \beta(0) + O\Big(\frac{1}{\Delta}\Big), \tag{24}$$

$$\beta(x) = \frac{3}{4}\sum_{p_1\neq p_2}\sum \frac{1}{(p_1+x)(p_2+x)} + \frac{3}{8}\sum_{p_1}\frac{1}{(p_1+x)^2}, \quad 2 \leqslant P \leqslant K,$$

$$\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} G^4(x;T,\varphi)\,\mathrm{d}x \sim \frac{3}{4}(\ln\ln P)^2, \quad P \to \infty. \tag{25}$$

The relations (24), (25) are valid uniformly for x_1, x_2 and φ.

Proof. We have

$$G^4(x;T,\varphi) = S_4 + S_5 + S_6 + S_7 + S_8, \tag{26}$$

where $(p_1, \dots, p_4 \leqslant P)$

$$S_4 = \beta(x), \tag{27}$$

$$\begin{aligned} S_5 = S_{51} + S_{52} + S_{53} &= \frac{1}{2}\sum\sum_{p_1,p_2,p_3}\sum \frac{\cos\omega_{51}}{(p_1+x)\sqrt{(p_2+x)(p_3+x)}} \\ &+ \frac{1}{2}\sum\sum_{p_1\neq p_2,p_3}\sum \frac{\cos\omega_{52}}{(p_1+x)\sqrt{(p_2+x)(p_3+x)}} \\ &+ \frac{1}{2}\sum\sum_{p_1\neq p_2,p_1\neq p_3}\sum \frac{\cos\omega_{53}}{(p_1+x)\sqrt{(p_2+x)(p_3+x)}}, \end{aligned} \tag{28}$$

$$\omega_{51} = \omega_{52} = \omega_{53} = T\ln[(p_2+x)(p_3+x)] + \varphi_{p_2} + \varphi_{p_3},$$

$$(29)\qquad S_6 = S_{61} + S_{62} + S_{63} = \frac{3}{4}\sum\sum\sum_{p_1,p_2\neq p_3} \frac{\cos\omega_{61}}{(p_1+x)\sqrt{(p_2+x)(p_3+x)}}$$
$$+ \frac{3}{4}\sum\sum\sum_{p_1\neq p_2\neq p_3\neq p_1} \frac{\cos\omega_{62}}{(p_1+x)\sqrt{(p_2+x)(p_3+x)}}$$
$$+ \frac{3}{8}\sum\sum_{p_1\neq p_2} \frac{\cos\omega_{63}}{(p_1+x)(p_2+x)},$$
$$\omega_{61} = \omega_{62} = T\ln\frac{p_2+x}{p_3+x} + \varphi_{p_2} - \varphi_{p_3},$$
$$\omega_{63} = 2T\ln\frac{p_1+x}{p_2+x} + 2\varphi_{p_1} - 2\varphi_{p_2},$$

$$(30)\qquad S_7 = S_{71} + S_{72} + S_{73} = \frac{3}{4}\sum\sum\sum_{\substack{p_1\neq p_2, p_1\neq p_3\\ p_2\neq p_3}} \frac{\cos\omega_{71}}{(p_1+x)\sqrt{(p_2+x)(p_3+x)}}$$
$$+ \frac{3}{8}\sum\sum\sum\sum_{\substack{p_1\neq p_3, p_1\neq p_4, p_1\neq p_2\\ p_2\neq p_3, p_2\neq p_4, p_3\neq p_4}} \frac{\cos\omega_{72}}{\sqrt{(p_1+x)\dots(p_4+x)}}$$
$$+ \frac{1}{8}\sum\sum\sum\sum_{p_1,p_2,p_3,p_4} \frac{\cos\omega_{73}}{\sqrt{(p_1+x)\dots(p_4+x)}},$$
$$\omega_{71} = T\ln\frac{(p_1+x)^2}{(p_2+x)(p_3+x)} + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3},$$
$$\omega_{72} = T\ln\frac{(p_1+x)(p_2+x)}{(p_3+x)(p_4+x)} + \varphi_{p_1} + \varphi_{p_2} - \varphi_{p_3} - \varphi_{p_4},$$
$$\omega_{73} = T\ln[(p_1+x)\dots(p_4+x)] + \varphi_{p_1} + \varphi_{p_2} + \varphi_{p_3} + \varphi_{p_4},$$

$$(31)\qquad S_8 = S_{81} + S_{82} + S_{83} = \frac{1}{2}\sum\sum_{p_1\neq p_2} \frac{\cos\omega_{81}}{(p_1+x)^{3/2}\sqrt{p_2+x}}$$
$$+ \frac{3}{2}\sum\sum\sum_{\substack{p_1\neq p_3, p_2\neq p_3\\ p_1\neq p_2}} \frac{\cos\omega_{82}}{(p_1+x)\sqrt{(p_2+x)(p_3+x)}}$$
$$+ \frac{1}{2}\sum\sum\sum\sum_{\substack{p_1\neq p_4, p_2\neq p_4, p_3\neq p_4\\ p_1\neq p_2, p_1\neq p_3, p_2\neq p_3}} \frac{\cos\omega_{83}}{\sqrt{(p_1+x)\dots(p_4+x)}},$$
$$\omega_{81} = T\ln\frac{(p_1+x)^3}{p_2+x} + 3\varphi_{p_1} - \varphi_{p_2},$$

$$\omega_{82} = T\ln\frac{(p_1+x)^2(p_2+x)}{p_3+x} + 2\varphi_{p_1} + \varphi_{p_2} - \varphi_{p_3},$$
$$\omega_{83} = T\ln\frac{(p_1+x)(p_2+x)(p_3+x)}{p_4+x} + \varphi_{p_1} + \varphi_{p_2} + \varphi_{p_3} - \varphi_{p_4}.$$

□

7.1. A typical and relatively the most difficult is the case of an estimate of the integral of the sum S_{83}. In that case we have the estimate (see (19)–(21), (31))

$$\int_{x_1}^{x_2} S_{83}\,\mathrm{d}x = V = O\Big(\frac{1}{T}\cdot\frac{P^6}{\ln^4 P}\Big)$$

uniformly for x_1, x_2 and φ. Analogously we establish the estimates

$$\int_{x_1}^{x_2} S_{5k}\,\mathrm{d}x = O\Big(\frac{1}{T}\cdot\frac{P^2\ln\ln P}{\ln^2 P}\Big),\quad k=1,2,3,$$
$$\int_{x_1}^{x_2} S_{6l}\,\mathrm{d}xD = O\Big(\frac{1}{T}\cdot\frac{P^3\ln\ln P}{\ln^2 P}\Big),\quad l=1,2,$$
$$\int_{x_1}^{x_2} S_{63}\,\mathrm{d}x = O\Big(\frac{1}{T}\cdot\frac{P^2}{\ln^2 P}\Big),\quad \int_{x_1}^{x_2} S_{71}\,\mathrm{d}x = O\Big(\frac{1}{T}\cdot\frac{P^4}{\ln^3 P}\Big),$$
$$\int_{x_1}^{x_2} S_{72}\,\mathrm{d}x = O\Big(\frac{1}{T}\cdot\frac{P^6}{\ln^4 P}\Big),\quad \int_{x_1}^{x_2} S_{73}\,\mathrm{d}x = O\Big(\frac{1}{T}\cdot\frac{P^3}{\ln^3 P}\Big),$$
$$\int_{x_1}^{x_2} S_{81}\,\mathrm{d}x = O\Big(\frac{1}{T}\cdot\frac{P^2}{\ln^2 P}\Big),\quad \int_{x_1}^{x_2} S_{82}\,\mathrm{d}x = O\Big(\frac{1}{T}\cdot\frac{P^4}{\ln^3 P}\Big),$$

uniformly for x_1, x_2 and φ. Consequently, we conclude (see (26), (28)–(31)) that the estimate

$$\int_{x_1}^{x_2} (S_5+S_6+S_7+S_8)\,\mathrm{d}x = O\Big(\frac{1}{T}\cdot\frac{P^6}{\ln^4 P}\Big) \tag{32}$$

holds uniformly for x_1, x_2 and φ.

7.2. Now, in the case $2 \leqslant P \leqslant K$ we have (see (24), (26), (27), (32))

$$\beta(1)\cdot(x_2-x_1) + O\Big(\frac{P^6}{T\ln^4 P}\Big) < \int_{x_1}^{x_2} G^4\,\mathrm{d}x < \beta(0)\cdot(x_2-x_1) + O\Big(\frac{P^6}{T\ln^4 P}\Big)$$

and by virtue of (23) this implies (24). Since

$$\sum_{p_1}\int_{x_1}^{x_2}\frac{dx}{(p_1+x)^2}=O(x_2-x_1),$$

we have

$$\int_{x_1}^{x_2} G^4\,dx=\frac{3}{4}(x_2-x_1)(\ln\ln P)^2+O\{(x_2-x_1)\ln\ln P\}+O\Big(\frac{P^6}{T\ln^4 P}\Big)$$

for $P\to\infty$ (see (17), (24), (27), (32)), and by virtue of (23) this yields (25).

8. Integral order of the function $|G(x;T,\varphi)|$

First of all, using (12), (13), (24), (25) we establish estimates

$$\int_{x_1}^{x_2} G^2(x;T,\varphi)\,dx>\begin{cases}(1-\varepsilon)\alpha(1)\cdot(x_2-x_1), & 2\leqslant P\leqslant K,\\ \frac{1-\varepsilon}{2}(x_2-x_1)\ln\ln P, & P\to\infty,\end{cases}$$

$$\int_{x_1}^{x_2} G^4(x;T,\varphi)\,dx<\begin{cases}(1+\varepsilon)\beta(0)\cdot(x_2-x_1), & 2\leqslant P\leqslant K,\\ \frac{3}{4}(1+\varepsilon)(x_2-x_1)(\ln\ln P), & P\to\infty.\end{cases}$$

Further, applying (12), (13) and the Cauchy-Bunyakovskii inequality we obtain estimates

$$\int_{x_1}^{x_2}|G(x;T,\varphi)|\,dx<\begin{cases}(1+\varepsilon)\cdot(x_2-x_1)\sqrt{\alpha(0)}, & 2\leqslant P\leqslant K,\\ \frac{1+\varepsilon}{\sqrt{2}}(x_2-x_1)\sqrt{\ln\ln P}, & P\to\infty.\end{cases}$$

The well known inequality

$$\int_a^b|g(x)|\,dx\geqslant\Big\{\int_a^b g^2(x)\,dx\Big\}^{3/2}\cdot\Big\{\int_a^b g^4(x)\,dx\Big\}^{-1/2}.$$

(use the Hoelder inequality with $g^2=|g|^{\frac{2}{3}}\cdot|g|^{\frac{4}{3}}$, $p=\frac{3}{2}$, $q=3$) implies

Lemma 5. *For all sufficiently large T, $\Delta>0$ and for arbitrary x_1, x_2 satisfying the conditions* (23) *we have*

$$(33)\qquad (1-3\varepsilon)\sqrt{\frac{\alpha^3(1)}{\beta(0)}}<\frac{1}{x_2-x_1}\int_{x_1}^{x_2}|G(x;T,\varphi)|\,dx<(1+\varepsilon)\sqrt{\alpha(0)}$$

provided $2 \leqslant P \leqslant K$ *and*

$$(34)\qquad \frac{1-3\varepsilon}{\sqrt{6}}\sqrt{\ln\ln P} < \frac{1}{x_2-x_1}\int_{x_1}^{x_2}|G(x;T,\varphi)|\,\mathrm{d}x < \frac{1+\varepsilon}{\sqrt{2}}\sqrt{\ln\ln P}$$

provided $P \to \infty$. *Moreover, both* (33), (34) *are valid uniformly for* x_1, x_2 *and* φ.

Remark 6. We have thus determined the order of the area of the curvilinear trapezoid corresponding to the graph of an arbitrary signal. In this connection a question arises of the proof of an asymptotic identity improving the estimate (34).

9. Proof of the theorem—conclusion

Since

$$\int_{x_1}^{x_2} G(x;T,\varphi)\,\mathrm{d}x = \sum_{p\leqslant P} \overset{*}{I}, \quad \overset{*}{\omega} = T$$

where

$$\begin{aligned}\overset{*}{I} &= \int_{x_1}^{x_2} \frac{\cos\overset{*}{\omega}}{\sqrt{p+x}}\,\mathrm{d}x = \frac{1}{T}\int_{x_1}^{x_2}\sqrt{p+x}\,\mathrm{d}(\sin\overset{*}{\omega}) \\ &= \frac{1}{T}\Big[\sqrt{p+x}\sin\overset{*}{\omega}\Big]_{x_1}^{x_2} + \frac{1}{2T^2}\Big[\sqrt{p+x}\cos\overset{*}{\omega}\Big]_{x_1}^{x_2} - \frac{1}{4T^2}\overset{*}{I},\end{aligned}$$

we have

$$\overset{*}{I} = O\Big(\frac{\sqrt{p}}{T}\Big).$$

Consequently,

$$(35)\qquad \int_{x_1}^{x_2} G(x;T,\varphi)\,\mathrm{d}x = O\Big(\frac{P^{3/2}}{T\ln P}\Big)$$

uniformly for x_1, x_2 and φ.

If under the conditions (23) the function $G(x;T,\varphi)$, $x \in \langle x_1, x_2\rangle$ does not change sign, then (35) implies the inequality

$$(36)\qquad \int_{x_1}^{x_2} |G(x;T,\varphi)|\,\mathrm{d}x < A\frac{P^{3/2}}{T\ln P}.$$

Further, for sufficiently large T, $\Delta > 0$ and an arbitrary fixed vector φ we have estimates (see (33), (34), $\varepsilon = 1/6$)

$$\int_{x_1}^{x_2} |G(x;T,\varphi)|\, dx > \begin{cases} A \cdot (x_2 - x_1), & 2 \leqslant P \leqslant K, \\ A \cdot (x_2 - x_1)\sqrt{\ln\ln P}, & P \to \infty. \end{cases} \tag{37}$$

However, under the conditions

$$\Delta \frac{P^6}{T \ln^4 P} = x_2 - x_1, \quad 0 \leqslant x_1 \leqslant \frac{1}{9P^2}$$

(see (23) and the assumptions of the theorem) the relations (36), (37) are contradictory. Consequently, the interval

$$\left(x_1, x_1 + \Delta \frac{P^6}{T \ln^4 P}\right), \quad 0 \leqslant x_1 \leqslant \frac{1}{9P^2}$$

contains a zero of an odd order of the function $G(x;T,\varphi)$ (of course, the condition $x_1 \in \langle 0, 1/9P^2 \rangle$ implies that $\langle x_1, x_2 \rangle \subset \langle 0, 1/(8P^2) \rangle$ for sufficiently large T).

10. Concluding remarks

The method of the proof of the theorem makes it possible to obtain analogous results even for families of signals obtained for example in the following way:

(A) by differentiation:

$$\frac{\partial G}{\partial x}, \frac{\partial G}{\partial t}, \ldots,$$

(B) by replacing the summation in (3) by a summation

(a) by an arbitrary choice of primes not greater than P,

(b) by numbers of the type

$$q_1 = p_1 p_2, \; q_2 = p_3 p_4, \; q_3 = p_5 p_6, \; \ldots \leqslant P,$$

(say, by products of twin primes),

(c) or, generally, by numbers

$$n_1 < n_2 < \ldots < n_k \leqslant P, \quad (n_i, n_j) = 1, \quad i \neq j, \quad i,j \leqslant k,$$

(C) by a substitution

$$\sqrt{p+x} \to (p+x)^\sigma, \quad \sigma \in \langle -L, L \rangle.$$

In particular, let us mention the family of signals (C) $\sigma = 0$, i.e.

$$G_0(x;T,\varphi) = \sum_{p \leqslant P} \cos\{T \ln(p+x) + \varphi_p\}, \quad x \in \langle 0,1 \rangle$$

with random excitation of phases.

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