OSCILLATION OF THE EULER DIFFERENTIAL EQUATION WITH DELAY

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1. INTRODUCTION

Consider the equation

$$y''(t) + \frac{k^2}{t^2} y(ct) = 0, \quad t \geqslant t_0 > 0, \tag{1}$$

where $0 < c < 1$ and $k \neq 0$. This equation is a natural generalization of the familiar Euler differential equation

$$y''(t) + \frac{k^2}{t^2} y(t) = 0 \tag{2}$$

and will be called in the sequel the Euler differential equation with delay. It is well known, see [4], that

$$(2) \text{ oscillates} \iff k^2 > \frac{1}{4}.$$

Mahfoud in [5] attempted to extend this result to Eq. (1) that is, to get a necessary and sufficient condition in terms of k and c, for all solutions of (1) to oscillate. As a partial answer to this problem he obtained the following result:

$$k^2 > \frac{1}{4c} \implies (1) \text{ is oscillatory,}$$

$$k^2 \leqslant \frac{1}{4} \implies (1) \text{ is nonoscillatory.}$$

However, his results do not apply to the remaining case $\frac{1}{4} < k^2 \leqslant \frac{1}{4c}$. In this paper we will fill this gap and get an efficient necessary and sufficient condition for oscillation of all solutions of Eq. (1). Our main result is

Theorem 1. *Every solution of Eq.* (1) *oscillates if and only if*

$$k^2 > \frac{\sqrt{r^2+4}-2}{r^2}\cdot \mathrm{e}^{\frac{r-2+\sqrt{r^2+4}}{2}}, \quad r = -\ln c. \tag{3}$$

By a solution of Eq. (1) we mean a function defined on a set $[ct_0, t_0]$ which satisfies (1) for every $t \geqslant t_0 > 0$. A solution $y(t)$ of Eq. (1) is said to be oscillatory if it has zeros for arbitrarily large t. Otherwise it is called nonoscillatory.

Main results

First we will transform Eq. (1) to the form of a delay differential equation with constant coefficients and constant delay.

Set

$$t = \exp(s), \quad y(t) = x(s) \quad \text{and} \quad c = \exp(-r) \tag{4}$$

with $r > 0$. Then as usual

$$t\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}s}, \quad t^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x}{\mathrm{d}s^2} - \frac{\mathrm{d}x}{\mathrm{d}s}$$

and (1) becomes

$$\frac{\mathrm{d}^2x}{\mathrm{d}s^2} - \frac{\mathrm{d}x}{\mathrm{d}s} + k^2x(s-r) = 0. \tag{5}$$

Obviously Eq. (1) oscillates if and only if Eq. (5) does.

Now, we will prove Theorem 1.

Proof of Theorem 1. With Eq. (5) we associate the characteristic equation

$$\lambda^2 - \lambda + k^2\exp(-\lambda r) \equiv \lambda^2 - \lambda + k^2c^\lambda = 0 \tag{6}$$

which is obtained by setting $x(s) = \exp(\lambda s)$ in (5). It is known, see [1], that (5) oscillates if and only if (6) has no real roots. Now we will show that (6) has no real roots if and only if (3) is satisfied.

Set

$$f(\lambda) = (\lambda - \lambda^2)\exp(r\lambda).$$

Then Eq. (6) can be written as $f(\lambda) = k^2$. First we find the maximum $f_{\max}$ of $f(\lambda)$ for $\lambda \geqslant 0$:

$$f'(\lambda) = [1 + (r-2)\lambda - r\lambda^2]\exp(r\lambda)$$

and

$$f'(\lambda) = 0 \Longleftrightarrow \lambda = \frac{r - 2 \pm \sqrt{r^2 + 4}}{2r}.$$

Since $\frac{r-2-\sqrt{r^2+4}}{2r} < 0$, we conclude that

$$f_{\max}(r) = f\Big(\frac{r - 2 + \sqrt{r^2 + 4}}{2}\Big) = \frac{\sqrt{r^2 + 4} - 2}{r^2} \cdot \mathrm{e}^{\frac{r-2+\sqrt{r^2+4}}{2}}.$$

Obviously, the equation $f(\lambda) = k^2$ does not have real root if and only if $k^2 > f_{\max}(r)$, that is, if and only if (3) is satisfied. $\square$

Remark 1. Obviously

$$\lim_{r \to 0} f_{\max}(r) = \frac{1}{4}$$

and so we set $f_{\max}(0) = \frac{1}{4}$. Thus in the limiting case $r = 0$ we get the familiar condition $k^2 > \frac{1}{4}$ for all solutions of Eq. (2) to oscillate.

Using the same technique one can extend the above result to the more general equation

$$y''(t) + \frac{a}{t}y'(t) + \frac{k^2}{t^2}y(ct) = 0, \quad 0 < c < 1. \tag{7}$$

In this case the necessary and sufficient condition has the form

$$k^2 > \frac{-2 + \sqrt{(a-1)^2 r^2 + 4}}{r} \exp\Big(\frac{r - ra - 2 + \sqrt{(a-1)^2 r^2 + 4}}{2}\Big).$$

Remark 2. It should be mentioned that in the case of the first order Euler differential equation

$$y'(t) + \frac{a}{t}y(ct) = 0, \quad t \geqslant t_0 > 0 \tag{8}$$

where $0 < c < 1$, every solution oscillates if $ar > \frac{1}{\mathrm{e}}$. In fact, performing the transformation (4) we get

$$x'(s) + ax(s - r) = 0 \tag{9}$$

which oscillates if and only if its characteristic equation

$$\lambda + a\exp(-\lambda r) = 0$$

has no real roots, see [4], that is, if and only if $ar > \frac{1}{\mathrm{e}}$.

Further, this result can be extended to the Euler differential equation with several delays

$$y'(t) + \frac{1}{t}\sum_{i=1}^{n} a_i y(c_i t) = 0$$

where $a_i > 0$ and $0 < c_i < 1$ for $i = 1, \ldots, n$. Using the transformation (4) we get

$$x'(s) + \sum_{i=1}^{n} a_i x(s - r_i) = 0,$$

where $r_i = -\ln c_i$. As is well known, see [4], the last equation oscillates if and only if corresponding characteristic equation

$$\lambda + \sum_{i=1}^{n} a_i \exp(-\lambda r_i) \equiv \lambda + \sum_{i=1}^{n} a_i c_i^{\lambda} = 0$$

has no real roots.

Remark 3. The above result can be extended to the case of the general n-th order Euler differential equation with several delays

$$y^{(n)}(t) + \frac{a_{n-1}}{t} y^{(n-1)}(t) + \ldots + \frac{a_1}{t^{n-1}} y'(t) + \frac{1}{t^n}\sum_{i=1}^{p} a_{0i} x(c_i t) = 0, \tag{10}$$

where $0 < c_i < 1$ and $a_{0i} > 0$ for $i = 1, \ldots, p$. Using the transformation (4) we get a linear delay differential equation with constant coefficients and constant delays whose characteristic equation is

$$\lambda(\lambda - 1)\cdots(\lambda - n + 1) + a_{n-1}\lambda(\lambda - 1)\cdots(\lambda - n + 2) + \ldots + a_1\lambda + \sum_{i=1}^{p} a_{0i} c_i^{\lambda} = 0. \tag{11}$$

Using [1], we conclude that (10) oscillates if and only if this characteristic equation has no real roots.

Remark 4. Our result can be extended to the system of Euler differential equations

$$y'(t) + \frac{1}{t} A y(ct) = 0, \quad t \geqslant t_0 > 0$$

where $0 < c < 1$, A is an $n \times n$ matrix and $y(t) \in \mathbb{R}^n$ for every t. Using (4) again, we get

$$x'(s) + Ax(s - r) = 0$$

which is, in view of [1], oscillatory if and only if the corresponding characteristic equation

$$\det(\lambda I + \exp(-\lambda r)A) \equiv \det(\lambda I + c^{\lambda} A) = 0$$

where I is the $n \times n$ unit matrix, has no real roots.

Remark 5. In the case of the Euler differential equation with constant delay of the form $t-r$, that is

$$y''(t) + \frac{k^2}{t^2} y(t-r) = 0, \quad t \geqslant t_0 > r$$

where $r > 0$, it is well known, see [6], that the delay does not effect the oscillation. In other words, this equation oscillates if and only if the corresponding equation without delay (2) oscillates, that is if and only if

$$k^2 > \frac{1}{4}.$$

Remark 6. A related transformation can be applied to the equation

$$y'(t) + (t \ln t)^{-1} a y(t^{\sigma}) = 0, \quad t \geqslant t_0 > 1, \tag{12}$$

where $a > 0$, $0 < \sigma \leqslant 1$ are constants. In this case the corresponding transformation is

$$t = \mathrm{e}^{\mathrm{e}^{s}}, \quad y(t) = x(s) \quad \text{and} \quad \sigma = \mathrm{e}^{-r}$$

with $r > 0$. Then it is easy to see that

$$t \ln t \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}s} \quad \text{and} \quad y(t^{\sigma}) = x(s-r)$$

and (12) takes the form (9), which is discussed in Remark 2.

In a similar way one can treat the more general case of equations with several delays

$$y'(t) + (t \ln t)^{-1} \sum_{i=1}^{p} a_i y(t^{\sigma_i}) = 0, \quad t \geqslant t_0 > 1,$$

where $a_i > 0$, $0 < \sigma_i \leqslant 1$, $i = 1, \ldots, p$ are constants and systems of equations

$$y'(t) + (t \ln t)^{-1} A y(t^{\sigma}) = 0, \quad t \geqslant t_0 > 1$$

where A is an $n \times n$ constant matrix and $y(t) \in \mathbb{R}^n$ for every t.

These results together with the result of Remark 2 extend the results of Jaroš [2] and [3].

References

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