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PERRON-TYPE INTEGRATION ON n-DIMENSIONAL INTERVALS AND ITS PROPERTIES

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0. INTRODUCTION

The ϱ-integral of f over an interval I is defined as a limit of a special type of integral sums of the type $\sum_i f(t_i)m(J_i)$; ϱ is a function of the space variable and a real variable and it is used in a regularity condition whose role is to restrict the set of couples (t_i, K_i) which are admissible for the use in the integral sums. The definition of the ϱ-integral and its elementary properties are described in Section 1. The Cousin lemma guarantees the existence of partitions with the desired properties thus making the definition of the ϱ-integral correct; it is formulated in a rather general form (the proof is given in Section 7). If it does not impair readability of the paper the proofs are only indicated.

If pointwise convergence of a sequence of ϱ-integrable functions f_j to a function f is assumed then the proof of a convergence theorem is almost immediate. Since convergence a.e. is more suitable for integration theory, the convergence theorem is adapted so that the convergence of f_j to f a.e. is sufficient.

Section 2 contains results on continuity and differentiability of the primitive of a ϱ-integrable function; the primitive need not be continuous at the boundary of the integration interval and Theorem 2.1 is the best possible continuity result. Moreover, two forms of the regularity condition are shown to lead to the same concept of the ϱ-integral (the technical part of the proof is given in Section 6); in this way the results obtained previously for either of these conditions are unified.

In Section 3 primitives of ϱ-integrable functions are characterized as additive interval functions which are regularly differentiable a.e. and fulfil a supplementary condition on the set of points of nondifferentiability.

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In Section 4 the strong ϱ-integral is introduced by a change in the definition of the ϱ-integral, and analogous results are proved for it as for the ϱ-integral. The strong ϱ-integral will be used in a subsequent paper. Namely, it will be proved that every strongly ϱ-integrable function is the limit of a sequence of step functions in a suitable convergence. Therefore both convergence theorems from Section 1—with pointwise convergence and with convergence a.e.—are modified for the strong ϱ-integration. A ϱ-integrable function which is not strongly ϱ-integrable is constructed in Section 5 (with some rather natural restrictions on ϱ).

1. The ϱ-integral and its elementary properties

Let $n \in \mathbb{N}$, $a_i, b_i \in \mathbb{R}$, $a_i < b_i$ for $i = 1, 2, \ldots, n$, $I = [a_1, b_1] \times \ldots \times [a_n, b_n]$, $\varrho\colon I \times (0, \infty) \to [0, 1]$. For $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ we put $\|t\| = \max\limits_i |t_i|$, for an interval $J = [c_1, d_1] \times \ldots \times [c_n, d_n]$ (with $c_i < d_i$) we put $d(J) = \max\limits_i (d_i - c_i)$, $\operatorname{reg} J = \min\limits_i (d_i - c_i)/d(J)$ and call these quantities the diameter and the regularity of J, respectively. For $t \in \mathbb{R}^n$, $0 < \sigma$ we denote $V(t, \sigma) = \{s \in \mathbb{R}^n;\ \|s - t\| \leqslant \sigma\}$. A finite set $\Delta = \{(t_k, J_k);\ k = 1, 2, \ldots, l\}$ (briefly $\Delta = \{(t, J)\}$) is called *a system* (on I), if $t_k \in J_k \subset I$ and the intervals J_i, J_k are nonoverlapping for $i \neq k$, i, $k = 1$, $2, \ldots, l$. Let K be an interval, $K \subset I$. A system Δ is called *a partition* of K if $\bigcup\limits_k J_k = K$. A function $\delta\colon I \to (0, \infty)$ is called *a gauge*. Let $N \subset I$. A system Δ is called *δ-fine (ϱ-regular, N-tagged)* if $J \subset V\bigl(t, \delta(t)\bigr)$ $\bigl(\operatorname{reg} J > \varrho\bigl(t, d(J)\bigr)$, $t \in N\bigr)$ for $(t, J) \in \Delta$. If $P, Q \subset \mathbb{R}^n$, then $\operatorname{Cl} P$, $\operatorname{Int} P$, $m(P)$, $P \setminus Q$, $P \div Q$ denote the closure of P, interior of P, Lebesgue measure of P, difference of P and Q, symmetrical difference of P and Q, respectively. Throughout the paper it will be assumed that

(1.1) for every gauge δ and every interval $K \subset I$ there exists a δ-fine ϱ-regular partition Δ of K.

The following version of Cousin Lemma introduces a very wide class of ϱ's such that 1.1 holds.

Lemma 1.1. *Assume that*

$$\varrho(t, \sigma) < 1 \quad \text{for } t \in I,\ \sigma > 0. \tag{1.2}$$

Then (1.1) *holds.*

The proof is a modification of the proof of Proposition, [5], Section 2 and is postponed to Section 7.

Definition 1.2. A function $f\colon I \to \mathbb{R}$ is called *ϱ-integrable*, if

(1.3) there exists $\gamma \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a gauge δ such that

$$\Bigl|\gamma - \sum_{\Delta} f(t)m(J)\Bigr| \leqslant \varepsilon$$

for every δ-fine ϱ-regular partition Δ of I.

The number γ is called the *ϱ-integral* of f. It is unique (cf. (1.1)) and will be denoted by $(\varrho)\int_I f$.

Note 1.3. We have defined a ϱ-regular system as a system $\Delta = \{(t, J)\}$ such that the inequality $\operatorname{reg} J > \varrho\bigl(t, d(J)\bigr)$ is satisfied for every $(t, J) \in \Delta$. Let us call a system Δ ϱ-regular* if $\operatorname{reg} J \geqslant \varrho\bigl(t, d(J)\bigr)$ holds for every $(t, J) \in \Delta$, and let f be called ϱ-integrable* if (1.3) holds with "ϱ-regular" replaced by "ϱ-regular*". Evidently, every ϱ-integrable* function is ϱ-integrable as well, and the two integrals coincide. The converse implication also holds, see Proposition 2.2.

Note 1.4. Let $\alpha\colon I \times (0, \infty) \to [0, 1]$, $\alpha(t, \sigma) \leqslant \varrho(t, \sigma)$ for all (t, σ). Obviously any α-integrable function f is ϱ-integrable (and $(\varrho)\int_I f = (\alpha)\int_I f$). The problem to what extent the set of α-integrable functions depends on α was settled in [4] and [5] in the case that α does not depend on t.

Note 1.5. If $n = 1$, then $\operatorname{reg} J = 1$ for any interval J and the ϱ-integral reduces to the Perron integral (cf. [2]).

If $n \geqslant 1$, $\varrho \equiv 0$, then again the Perron integral is obtained (cf. [1], [2]).

It follows that the ϱ-integration is an extension of the Perron (Lebesgue) integration for any ϱ (cf. [2]).

Note 1.6. Let K_1 be an interval, $K_1 \subset I$. Then there are intervals $K_2, \ldots, K_r$ such that $\bigcup_i K_i = I$ and K_i, K_j are nonoverlapping for $i \neq j$.

(i) Let $g\colon K_1 \to \mathbb{R}$. g is called ϱ-integrable if (1.3) is fulfilled with f and "partition Δ of I" being replaced by g and "partition Δ of K_1."

(ii) Let $f\colon I \to \mathbb{R}$ be ϱ-integrable, $\varepsilon > 0$ and let δ correspond to ε by (1.3). Let Θ_1 and Ω_1 be δ-fine ϱ-regular partitions of K_1. Fix δ-fine ϱ-regular partitions Ξ_i of K_i, $i = 2, 3, \ldots, r$ and put

$$\Theta = \Theta_1 \cup \Xi_2 \cup \ldots \cup \Xi_r, \quad \Omega = \Omega_1 \cup \Xi_2 \cup \ldots \cup \Xi_r.$$

Θ and Ω are δ-fine ϱ-regular partitions of I so that

$$\Bigl|\sum_{\Theta} f(t)m(J) - \sum_{\Omega} f(t)m(J)\Bigr| \leqslant 2\varepsilon.$$

Since

$$
\begin{aligned}
&\sum_{\Theta} f(t)m(J) - \sum_{\Omega} f(t)m(J) \\
&= \sum_{\Theta_1} f(t)m(J) - \sum_{\Omega_1} f(t)m(J),
\end{aligned}
$$

it may be concluded that the restriction $f|_{K_1}$ is ϱ-integrable and $(\varrho)\int_{K_1} f|_{K_1}$ (briefly $(\varrho)\int_{K_1} f$) exists for any $K_1 \subset I$.

Put $F(L) = (\varrho)\int_L f$ for any interval $L \subset I$. F is called *the primitive* of f. F is an additive interval function (on I).

In a similar manner the following result can be proved:

Lemma 1.7 (Saks, Henstock). *Let $f\colon I \to \mathbb{R}$ be ϱ-integrable, F being its primitive. Let $\varepsilon > 0$ and let a gauge δ correspond to ε by* (1.3). *Then*

$$
\Big|\sum_{\Delta} \big(f(t)m(J) - F(J)\big)\Big| \leqslant \varepsilon,
$$

$$
\sum_{\Delta} |f(t)m(J) - F(J)| \leqslant 2\varepsilon
$$

for any δ-fine ϱ-regular system $\Delta = \{(t, J)\}$.

Lemma 1.8. *Let $N \subset I$, $m(N) = 0$, $\varphi\colon N \to \mathbb{R}$, $\varepsilon > 0$. Then there is a gauge δ such that $\sum_{\Delta} |\varphi(t)|m(J) \leqslant \varepsilon$ for every δ-fine N-tagged system Δ.*

Sketch of proof. Put $N_1 = \{t \in N;\, |\varphi(t)| < 2\}$, $N_i = \{t \in N;\, 2^{i-1} \leqslant |\varphi(t)| < 2^i\}$, $i = 2, 3, \ldots$. There exist open sets $G_i \subset \mathbb{R}^n$ such that $N_i \subset G_i$, $m(G_i) \leqslant 2^{-2i}$, $i = 1, 2, \ldots$. The assertion of the lemma holds for any gauge δ fulfilling $V\big(t, \delta(t)\big) \subset G_i$ for $t \in N_i$, $i = 1, 2, \ldots$. □

Proposition 1.9. *Let $f\colon I \to \mathbb{R}$ be ϱ-integrable, F being its primitive. The set of $g\colon I \to \mathbb{R}$ such that $g = f$ a.e. is the set of such ϱ-integrable functions the primitive of which is F.*

Proof follows from the following observations:

(i) Let $f\colon I \to \mathbb{R}$, $f = 0$ a.e. Then f is ϱ-integrable and $(\varrho)\int_K f = 0$ for every interval $K \subset I$.

(ii) Let $f\colon I \to \mathbb{R}$ be ϱ-integrable, F being its primitive. Let $F(K) = 0$ for every interval $K \subset I$. Then $f = 0$ a.e.

(i) follows immediately from Lemma 1.7.

If (ii) does not hold, then there exist $\eta > 0$ and $A \subset I$ such that $|f(t)| \geqslant \eta$ for $t \in A$ and $m_e(A) > 0$. Put $\varepsilon = \eta m_e(A)/4$ and let δ be a gauge which corresponds to ε by (1.3). The family $\{V(t,\sigma);\, t \in A, \sigma > 0\}$ is a covering of A in the sense of Vitali. Therefore there is a δ-fine A-tagged system $\Delta = \{(t_k, V(t_k,\sigma_k))\,;\, k = 1,2,\dots,l\}$ such that $\sum_k m(V(t_k,\sigma_k)) \geqslant 2m_e(A)/3$, hence $\sum_k |f(t_k)| m(V(t_k,\sigma_k)) \geqslant 2\eta m_e(A)/3$. Since $V(t,\sigma)$ are cubes, Δ is ϱ-regular. Since $F(K) = 0$ for every interval $K \subset I$, Lemma 1.7 implies that $\sum_k |f(t_k)| m(V(t_k,\sigma_k)) \leqslant 2\varepsilon = \eta m_e(A)/2$, a contradiction which completes the proof of (ii). □

Theorem 1.10. *Let $f\colon I \to \mathbb{R}$ be ϱ-integrable, F being its primitive, $N \subset I$, $m(N) = 0$. Then*

(1.4) *for every $\xi > 0$ there is a gauge ω such that $\sum_\Delta |F(J)| \leqslant \xi$ for every ω-fine ϱ-regular N-tagged system Δ.*

Proof follows by Lemmas 1.7 and 1.8. □

Theorem 1.11. *Let $f_j\colon I \to \mathbb{R}$ be ϱ-integrable, F_j being their primitives, for $j \in \mathbb{N}$, $f\colon I \to \mathbb{R}$. Assume that*

(1.5) *for every $\varepsilon > 0$ there exists a gauge δ such that*

$$\Big|\sum_{\Delta} f_j(t)m(J) - F_j(I)\Big| \leqslant \varepsilon$$

for every δ-fine ϱ-regular partition Δ of I and every $j \in \mathbb{N}$

and

(1.6) *$f_j(t) \to f(t)$ for $j \to \infty$, $t \in I$.*

Then f is ϱ-integrable and

(1.7) *$F_j(K) \to F(K)$ for $j \to \infty$ and any interval $K \subset I$, F being the primitive of f.*

Proof is based on the observation that by (1.5) we have

$$\Big|\sum_{\Delta} f_j(t)m(J) - \sum_{\Theta} f_j(s)m(L)\Big| \leqslant 2\varepsilon \tag{1.8}$$

for any δ-fine ϱ-regular partitions $\Delta = \{(t,J)\}$, $\Theta = \{(s,L)\}$ and (1.6) implies that (1.8) holds with f_j replaced by f. □

If convergence problems are considered, Definition 1.2 corresponds to the pointwise convergence of the sequence of integrands. The characterization of ϱ-integrable functions given in the next theorem corresponds to the convergence a.e.

Theorem 1.12. *Let $f\colon I \to \mathbb{R}$ and let F be an additive interval function on I. The following two conditions are equivalent:*

(1.9) *f is ϱ-integrable and F is its primitive,*

(1.10) *there exists $N \subset I$, $m(N) = 0$ and for every $\xi > 0$ there exists a gauge ϑ such that*

(i) *$\sum_{\Omega} |f(t)m(J) - F(J)| \leqslant \xi$ for every ϑ-fine ϱ-regular $(I \setminus N)$-tagged system $\Omega = \{(t, J)\}$,*

(ii) *$\sum_{\Theta} |F(L)| \leqslant \xi$ for every ϑ-fine ϱ-regular N-tagged system $\Theta = \{(s, L)\}$.*

Proof. (1.10) follows from (1.9) by Lemma 1.7 and Theorem 1.10, on the other hand (1.9) follows from (1.10) by Lemma 1.8. □

Theorem 1.13. *Let $g_j : I \to \mathbb{R}$ be ϱ-integrable, G_j being their primitives, $j \in \mathbb{N}$, $g\colon I \to \mathbb{R}$. Assume that there exists $N \subset I$, $m(N) = 0$ and*

(1.11) *for every $\xi > 0$ there exists a gauge ϑ such that*

(i) *$\sum_{\Omega} |g_j(t)m(J) - G_j(J)| \leqslant \xi$ for every ϑ-fine ϱ-regular $(I \setminus N)$-tagged system $\Omega = \{(t, J)\}$ and $j \in \mathbb{N}$,*

(ii) *$\sum_{\Theta} |G_j(L)| \leqslant \xi$ for every ϑ-fine ϱ-regular N-tagged system $\Theta = \{(s, L)\}$ and $j \in \mathbb{N}$*

and

(1.12) *$g_j(t) \to g(t)$ for $j \to \infty$, $t \in I \setminus N$*

hold. Then g is ϱ-integrable and

(1.13) *$G_j(K) \to G(K)$ for $j \to \infty$ and any interval $K \subset I$, G being the primitive of g.*

Proof. Let (1.11) and (1.12) hold and let $\varepsilon > 0$. Put $\xi = \varepsilon/2$. Let ϑ be a gauge associated to ξ by (1.11). Put $f_j(t) = g_j(t)$, $f(t) = g(t)$ for $t \in I \setminus N$, $j \in \mathbb{N}$ and $f_j(t) = 0$, $f(t) = 0$ for $t \in N$, $j \in \mathbb{N}$. f_j is ϱ-integrable and G_j is its primitive by Proposition 1.9. Let $\Delta = \{(t, J)\}$ be a ϑ-fine ϱ-regular partition of I. Putting $\Omega = \{(t, J) \in \Delta;\, t \in I \setminus N\}$, $\Theta = \{(t, J) \in \Delta;\, t \in N\}$ we obtain from (1.11) that

$$\Big|\sum_{\Delta} f_j(t)m(J) - G_j(I)\Big| \leqslant \varepsilon$$

for $j \in \mathbb{N}$. Moreover, (1.6) holds. Theorem 1.11 implies that f is ϱ-integrable and that $G_j(K) \to G(K)$ for $j \to \infty$ and any interval $K \subset I$, G being the primitive of f. The proof is complete, since $g = f$ a.e. so that g is ϱ-integrable and G is its primitive. □

It will be assumed in this section that

$$\limsup_{\sigma \to 0+} \varrho(t,\sigma) < 1 \quad \text{for } t \in I. \tag{2.1}$$

By (2.1) there exists a gauge $\overline{\delta}$ such that $\varrho(t,\sigma) < 1$ for $t \in I$, $\sigma \leqslant \overline{\delta}(t)$. Therefore it may be assumed without loss of generality that (1.2) holds.

Theorem 2.1. *Let $f\colon I \to \mathbb{R}$ be ϱ-integrable, F being its primitive. Then F is continuous at any interval $L \subset \operatorname{Int} I$ in the following sense: for every $\varepsilon > 0$ there is $\eta > 0$ such that $|F(K) - F(L)| \leqslant \varepsilon$ for every interval $K \subset I$, $m(K \div L) \leqslant \eta$ satisfying the following condition:*

if

$$I = [a_1, b_1] \times \ldots \times [a_n, b_n], L = [c_1, d_1] \times \ldots \times [c_n, d_n], K = [u_1, v_1] \times \ldots \times [u_n, v_n]$$

then

$$\begin{aligned} c_i = a_i &\Rightarrow u_i = a_i, \\ d_i = b_i &\Rightarrow v_i = b_i. \end{aligned}$$

P r o o f is a modification of the proof of Theorem 2.5, [3]; it is given in Section 6. Note that if $L \subset \operatorname{Int} I$ then we have continuity in the current sense.

The next proposition was announced in Note 1.3.

Proposition 2.2. *If a function $\varrho\colon I \times (0,\infty) \to (0,1)$ satisfies (1.2) and*

(2.2) *$\sigma\varrho(t,\sigma)$ is an increasing function of the variable σ for every $t \in I$,*

then every ϱ-integrable function f is ϱ-integrable as well and the two integrals coincide.*

We need

Lemma 2.3. *Let $I = [a_1, b_1] \times \ldots \times [a_n, b_n]$ and let δ be a gauge satisfying for every $t = (t_1, \ldots, t_n) \in I$ the condition*

$$\begin{aligned} &\text{if } a_j < t_j \quad \text{then } a_j < t_j - \delta(t), \\ &\text{if } t_j < b_j \quad \text{then } t_j + \delta(t) < b_j. \end{aligned} \tag{2.3}$$

Let (t, J) be a pair such that $t \in J \subset V(t, \delta(t)) \cap I$ and $\operatorname{reg} J \geqslant \varrho(t, d(J))$. Let f be ϱ-integrable.

Then either $\operatorname{reg} J = 1$ *or for every* $\lambda > 0$ *there is an interval* $K \subset J$ *such that*

(i) $t \in K$,

(ii) $m(J) - m(K) < \lambda$,

(iii) $|F(J) - F(K)| < \lambda$,

(iv) $\operatorname{reg} K > \varrho(t, d(K))$.

Proof. Let $J = [c_1, d_1] \times \ldots \times [c_n, d_n]$, $\operatorname{reg} J < 1$, $\lambda > 0$. By shortening suitably all the intervals $[c_j, d_j]$ with $d_j - c_j = d(J)$ either by increasing c_j or decreasing d_j and leaving the others unchanged, we can obtain an interval K satisfying (i), (ii). By virtue of (2.3) and Theorem 2.1 we can do it in such a way that (iii) holds, too. Since $\operatorname{reg} J < 1$, the shortest edge of J was not changed when passing from J to K; we can assume that it is again the shortest edge of K. Denoting its length by ψ, we have

$$\varrho(t, d(J)) \leqslant \operatorname{reg} J = \frac{\psi}{d(J)} < \frac{\psi}{d(K)} = \operatorname{reg} K,$$

hence

$$\psi \geqslant d(J)\varrho(t, d(J)) > d(K)\varrho(t, d(K))$$

by virtue of (2.2) since $d(K) < d(J)$ and (iv) immediately follows. □

Proof of Proposition 2.2. Given $\varepsilon > 0$, let us find a gauge δ corresponding to ε by the definition of the ϱ-integral of f and satisfying (2.3). Let $\Delta = \{(t, J)\}$ be a δ-fine ϱ-regular* partition, $\mathcal{J} = \{J;\ (t, J) \in \Delta \text{ for some } t \in I\}$. Choose $\lambda = \varepsilon[|\mathcal{J}|(m_f + 1)]^{-1}$ where $m_f = \max\{|f(t)|;\ (t, J) \in \Delta \text{ for some } J\}$ and for every $J \in \mathcal{J}$ find K by Lemma 2.3. Since (iv) from Lemma 2.3 holds, the system $\Theta = \{(t, K)\}$ is ϱ-regular and the Saks-Henstock Lemma 1.7 yields

$$\sum_{\Theta} |f(t)m(K) - F(K)| < 2\varepsilon.$$

Further,

$$\begin{aligned}
\sum_{\Delta} |f(t)m(J) - F(J)| &\leqslant \sum |f(t)m(J) - f(t)m(K)| \\
&\quad + \sum_{\Theta} |f(t)m(K) - F(K)| + \sum |F(K) - F(J)| \\
&\leqslant \lambda \sum |f(t)| + 2\varepsilon + \lambda|\mathcal{J}| < 4\varepsilon,
\end{aligned}$$

which proves the ϱ-integrability* of f. The fact that the two integrals coincide is selfevident. □

Definition 2.4. Let $0 < \alpha < 1$, $s \in I$, $g \in \mathbb{R}$ and let G be an additive interval function in I. G is said to be *α-regularly differentiable* to g at s, if for every $\varepsilon > 0$ there is $\eta > 0$ such that

$$|G(J) - gm(J)| \leqslant \varepsilon m(J)$$

for every interval $J \subset V(s, \eta) \cap I$ with $s \in J$, $\operatorname{reg} J \geqslant \alpha$.

Theorem 2.5. *Let $0 < \beta < \alpha < 1$, $s \in I$, $g \in \mathbb{R}$ and let an additive interval function G be α-regularly differentiable to g at s. Then G is β-differentiable to g at s.*

See [4], Section 2, Theorem 1.

Definition 2.6. Let G be an additive interval function in I, $g \in \mathbb{R}$. G is said to be *regularly differentiable* to g at s, if G is α-regularly differentiable to g at s for some α, $0 < \alpha < 1$. Denote by D_G the set of such $s \in I$ that there is $g \in \mathbb{R}$ such that G is regularly differentiable to g at s; since g is unique, we will write $G'(s)$ instead of g in such a case.

Corollary 2.7. *Let F be an additive interval function in I, $s \in I$, $f(s) \in \mathbb{R}$. Assume that F is not regularly differentiable to $f(s)$ at s. Then there exist $\xi(s) > 0$ and a sequence of intervals $L_k(s)$, $k \in \mathbb{N}$ such that*

$$\begin{aligned} &s \in L_k(s) \quad \text{for } k \in \mathbb{N},\ d\big(L_k(s)\big) \to 0, \\ &\operatorname{reg} L_k(s) \to 1 \quad \text{for } k \to \infty \quad \text{and} \\ &\big|f(s)m\big(L_k(s)\big) - F\big(L_k(s)\big)\big| \geqslant \xi(s)m\big(L_k(s)\big) \\ &\text{for } k \in \mathbb{N}. \end{aligned} \tag{2.4}$$

Theorem 2.8. *Let F be the primitive of a ϱ-integrable $f\colon I \to \mathbb{R}$. Then*

(2.5) *F is regularly differentiable to $f(t)$ at almost every $t \in I$.*

Proof. Let A be the set of $t \in I$ such that F is not regularly differentiable to $f(t)$ at t and assume that $m_e(A) > 0$. Let $\xi(s)$ and $L_k(s)$ have the same meaning as in Corollary 2.5. For $\tau > 0$ put $A(\tau) = \{s \in A;\ \xi(s) \geqslant \tau\}$. Obviously there is $\eta > 0$ such that $m_e\big(A(\eta)\big) > 0$. Put $\varepsilon = \eta m_e\big(A(\eta)\big)/8$ and let δ correspond to ε by (1.3). For $s \in A(\eta)$ there exists $p(s) \in \mathbb{N}$ such that $L_k(s) \subset V\big(s, \delta(s)\big)$, $\operatorname{reg} L_k(s) \geqslant \varrho\big(s, d\big(L_k(s)\big)\big)$, $k = p(s), p(s)+1, p(s)+2, \ldots$. The family $B = \{L_k(s);\ s \in A(\eta), k = p(s), p(s)+1, \ldots\}$ is a covering of $A(\eta)$ in the sense of Vitali. Therefore there exists a system $\Delta\{(t,J)\}$ such that each (t,J) is equal to some $\big(s, L_k(s)\big)$ and $\sum_\Delta m(J) \geqslant m_e\big(A(\eta)\big)/2$. Moreover, Δ is δ-fine and ϱ-regular and

$$\sum_\Delta |f(t)m(J) - F(J)| \geqslant \eta \sum_\Delta m(J) \geqslant 4\varepsilon,$$

which contradicts Lemma 1.6. □

Theorem 2.9. *Let $f\colon I \to \mathbb{R}$ be ϱ-integrable, F being its primitive. Then f is measurable.*

Proof. If $t \in I$, $j \in \mathbb{N}$, $V(t,1/j) \subset I$, put $f_j(t) = F\big(V(t,1/j)\big)(j/2)^n$. Then f_j is continuous and

$$f_j(t) \to j(t) \quad \text{for } j \to \infty \quad \text{a.e.} \tag{2.6}$$

by (2.5). □

Note 2.10. If (2.1) is dropped and if (1.2) and

$$\limsup_{\tau \to \sigma+} \varrho(t,\tau) \leqslant \varrho(t,\sigma) \quad t \in I,\ \sigma > 0 \tag{2.7}$$

are assumed, then Theorem 2.1 holds as well (cf. Note 6.2 and Comment 6.3 bellow). Since (2.6) may be proved directly, it follows that f is measurable.

3. Descriptive characterization of ϱ-integrable functions

It will be assumed in this section that

$$\lambda(t) = \liminf_{\sigma \to 0+} \varrho(t,\sigma) > 0 \quad \text{for } t \in I \tag{3.1}$$

and that (1.1) holds.

Theorem 3.1. *Let G be an additive interval function in I, $m(I \setminus D_G) = 0$ (cf. Definition 2.4). Assume that*

(3.2) *for every $\xi > 0$ there exists a gauge ϑ such that $\sum_\Theta |G(J)| \leqslant \xi$ for any ϑ-fine ϱ-regular $(I \setminus D_G)$-tagged system $\Theta = \{(t,J)\}$.*

Put

(3.3) $g(t) = G'(t)$ *for* $t \in D_G$, $g(t) = 0$ *for* $t \in I \setminus D_G$.

Then g is ϱ-integrable and G is its primitive.

Proof. Let $\varepsilon > 0$. For $t \in D_G$ there exists $\omega(t) > 0$ such that $\varrho(t,\sigma) \geqslant \lambda(t)/2$ for $0 < \sigma \leqslant \omega(t)$ and

$$|G(K) - g(t)m(K)| \leqslant m(K)\varepsilon/2m(I)$$

for any interval $K \subset I$, $t \in K \subset V\bigl(t,\omega(t)\bigr)$, $\operatorname{reg} K \geqslant \lambda(t)/2$. Put $\xi = \varepsilon/2$ and let ϑ correspond to ξ by (3.2). Put $\delta(t) = \omega(t)$ for $t \in D_G$, $\delta(t) = \vartheta(t)$ for $t \in I \setminus D_G$. Let $P \subset I$ be an interval. Let $\Delta = \{(t,J)\}$ be a δ-fine ϱ-regular partition of P. Put $\Omega = \{(t,J) \in \Delta;\, t \in D_G\}$, $\Theta = \{(t,J) \in \Delta;\, t \in P \setminus D_G\}$. Then

$$\begin{aligned}\Bigl|\sum_{\Delta} g(t)m(J) - G(J)\Bigr| &\leqslant \sum_{\Omega} |g(t)m(J) - G(J)| + \sum_{\Theta} |G(J)| \\ &\leqslant \sum_{\Omega} m(K)\varepsilon/2m(I) + \varepsilon/2 \leqslant \varepsilon.\end{aligned}$$

$\square$

Theorem 3.2. *Let* (2.1) *and* (1.2) *hold (in addition to* (3.1)*). Let $f\colon I \to \mathbb{R}$ and let F be an interval function on I. Then the following two conditions are equivalent:*
(A) *f is ϱ-integrable and F is its primitive,*
(B) $m(I \setminus D_F) = 0$, (3.2) *holds and* $f = F'$ a.e.

Proof. (B) follows from (A) by Theorems 2.8 and 1.10. (A) follows from (B) by Theorem 3.1 and Proposition 1.9 (i). $\square$

Corollary 3.3. *Let ϱ fulfil* (1.2), (2.1) *and* (3.1). *Let f be ϱ-integrable, F being its primitive. Let $\omega\colon I \times (0,\infty) \to [0,1]$ fulfil* (1.2), (2.1), (3.1) *and $\omega(t,\sigma) \geqslant \varrho(t,\sigma)$ for $t \in I \setminus D_F$ (no inequality being assumed for $t \in D_F$). Then f is ω-integrable and F is its primitive. Briefly: the values of ϱ are essential only in a neighbourhood of $I \setminus D_F$.*

4. The strong ϱ-integral

It will be proved in a forthcoming paper that every strongly ϱ-integrable function as defined below is the limit of a sequence of step functions in a suitable convergence compatible with strong ϱ-integration (i.e. every limit of a sequence of strongly ϱ-integrable functions is strongly ϱ-integrable). Thus strong ϱ-integration can be viewed as an extension of elementary integration of step functions. It will be assumed throughout this section that (1.1) holds.

Definition 4.1. $f: I \to \mathbb{R}$ is called *strongly ϱ-integrable*, if there exists an additive interval function F such that for every $\varepsilon > 0$ there is a gauge δ such that

$$\sum_{k=1}^{l} |f(t_k)m(M_k) - F(M_k)| \leqslant \varepsilon \tag{4.1}$$

holds provided $\Delta = \{(t_k, J_k);\ k = 1, 2, \ldots, l\}$ is a δ-fine ϱ-regular system and $\mathbb{M} = \{M_k, k = 1, 2, \ldots, l\}$ is a set of intervals such that $M_k \subset J_k$ for $k = 1, 2, \ldots, l$.

Note that $t_k \in M_k$ is not required and no restriction is imposed on $\operatorname{reg} M_k$.

Note 4.2. Every strongly ϱ-integrable function f is ϱ-integrable, since (4.1) implies that

$$\Big|\sum_{\Delta} f(t)m(J) - F(K)\Big| \leqslant \varepsilon$$

for every δ-fine ϱ-regular partition Δ of K, K being an interval, $K \subset I$. Moreover, F is the primitive of f.

Therefore the previous results can be used for strongly ϱ-integrable functions. If (2.1) and (1.2) hold, then F is continuous at any interval $L \subset \operatorname{Int} I$; F is regularly differentiable to $f(t)$ at almost all t and f is measurable (Theorems 2.1, 2.6 and 2.7). On the other hand, Theorems 1.10–1.13, 3.1 and 3.2 will be modified.

An example of a function which is ϱ-integrable but not strongly ϱ-integrable is given in Section 5.

Note 4.3. In the case $n = 1$ every integrable function is strongly integrable.

Let $f\colon [a, b] \to \mathbb{R}$ be integrable, $\varepsilon > 0$ and let the gauge δ correspond to ε by Definition 1.2. Let $\Delta\{(t_k, [c_k, d_k]);\ k = 1, 2, \ldots, l\}$ be a δ-fine system and let $\mathbb{M} = \{[p_k, q_k];\ k = 1, 2, \ldots, l\}$ with $c_k \leqslant p_k < q_k \leqslant d_k$. Put

$$\begin{aligned}
\Delta_1 &= \{(t_k, T_k);\ p_k \leqslant t_k \leqslant q_k, T_k = [p_k, q_k]\},\\
\Delta_2 &= \{(t_k, T_k);\ t_k < p_k, T_k = [t_k, p_k]\},\\
\Delta_3 &= \{(t_k, T_k);\ t_k < p_k, T_k = [t_k, q_k]\},\\
\Delta_4 &= \{(t_k, T_k);\ q_k < t_k, T_k = [q_k, t_k]\},\\
\Delta_5 &= \{(t_k, T_k);\ q_k < t_k, T_k = [p_k, t_k]\}.
\end{aligned}$$

Obviously Δ_i is a δ-fine system for $i = 1, 2, \ldots, 5$ and by Lemma 1.6 we have

$$\begin{aligned}
&\sum_{\Delta} |f(t_k)(q_k - p_k) - F([p_k, q_k])|\\
&\leqslant \sum_{i=1}^{5} \sum_{\Delta_i} |f(t_k)m(T_k) - F(T_k)| \leqslant 10\varepsilon.
\end{aligned}$$

Note 4.4. Usually a brief notation will be used and (4.1) will be written in the form

$$\sum_{\Delta,\mathbb{M}} |F(M) - f(t)m(M)| \leqslant \varepsilon,$$

$\Delta = \{(t, J)\}$, $\mathbb{M} = \{M\}$.

Theorem 4.5. *Let $f\colon I \to \mathbb{R}$ be strongly ϱ-integrable, F being its primitive (cf. Note 4.2), $N \subset I$, $m(N) = 0$. Then*

(4.2) *for every $\xi > 0$ there is a gauge ω such that*

$$\sum_{\Delta,\mathbb{M}} |F(M)| \leqslant \xi$$

for every ω-fine ϱ-regular N-tagged system $\Delta = \{(t, J)\}$ and every set $\mathbb{M} = \{M\}$ such that a one-to-one correspondence between Δ and $\mathbb{M}$ is defined by $M \subset J$.

The proof follows directly from Definition 4.1 and Lemma 1.8.

Theorem 4.6. *Let $f_j\colon I \to \mathbb{R}$ be strongly ϱ-integrable, F_j being their primitives for $j \in \mathbb{N}$, let $f\colon I \to \mathbb{R}$. Assume that*

(4.3) *for every $\varepsilon > 0$ there exists a gauge δ such that*

$$\sum_{\Delta,\mathbb{M}} |f_j(t)m(M) - F_j(M)| \leqslant \varepsilon$$

holds for every δ-fine ϱ-regular system $\Delta = \{(t, J)\}$ and every set $\mathbb{M}$ of intervals M provided a one-to-one correspondence between $\mathbb{M}$ and Δ is defined by $M \subset J$

and (1.6) *hold. Then f is strongly ϱ-integrable and* (1.7) *holds.*

P r o o f. f is ϱ-integrable and (1.8), (1.7) hold by Theorem 1.11. Passing to the limit for $j \to \infty$ in (4.3) we conclude that f is strongly ϱ-integrable. □

Proposition 4.7. *Let $f\colon I \to \mathbb{R}$ be strongly ϱ-integrable, F being its primitive. The set of $g\colon I \to \mathbb{R}$ such that $g = f$ a.e. is the set of all strongly ϱ-integrable functions such that their primitive is F.*

P r o o f. If $g = f$ a.e. then the strong ϱ-integrability follows from Definition 4.1 (which plays the role of the "strong Saks-Henstock Lemma," cf. Lemma 1.7) and Lemma 1.8. If g is strongly ϱ-integrable with the primitive F, then $g = f$ a.e. follows immediately from Proposition 1.9. □

Theorem 4.8. *Let $f\colon I \to \mathbb{R}$ and let F be an additive interval function on I. The following two conditions are equivalent:*

(4.4) *f is strongly ϱ-integrable and F is its primitive;*

(4.5) *there exists $N \subset I$, $m(N) = 0$ and for every $\xi > 0$ there is a gauge ϑ such that*

(i) *$\sum_{\Delta,\mathbb{M}} |f(t)m(M) - F(M)| \leqslant \xi$ for any ϑ-fine ϱ-regular $(I \setminus N)$-tagged system $\Delta = \{(t, J)\}$ and every set $\mathbb{M} = \{M\}$ of intervals M such that the inclusion $M \subset J$ defines a one-to-one correspondence between $\mathbb{M}$ and Δ;*

(ii) *$\sum_{\Delta,\mathbb{M}} |F(M)| \leqslant \xi$ for any ϑ-fine ϱ-regular N-tagged system $\Delta = \{(t, J)\}$ and every set $\mathbb{M} = \{M\}$ of intervals M such that the inclusion $M \subset J$ defines a one-to-one correspondence between $\mathbb{M}$ and Δ.*

Proof. (4.5) follows from (4.4) by Definition 4.1 and Theorem 4.5, while (4.4) follows from (4.5) by Lemma 1.8. □

We can now pass to the analogue of Theorem 1.13, i.e. a convergence theorem for strong ϱ-integrals based on convergence a.e.

Theorem 4.9. *Let g, $g_j\colon I \to \mathbb{R}$, $j \in \mathbb{N}$, let G_j, $j \in \mathbb{N}$, be additive interval functions on I. Assume that there exists $N \subset I$, $m(N) = 0$ and*

(4.6) *for every $\xi > 0$ there exists a gauge ϑ such that*

(i) *$\sum_{\Delta,\mathbb{M}} |g_j(t)m(M) - G_j(M)| \leqslant \xi$ for any ϑ-fine ϱ-regular $(I \setminus N)$-tagged system $\Delta = \{(t, J)\}$, every set $\mathbb{M} = \{M\}$ of intervals M such that the inclusion $M \subset J$ defines a one-to-one correspondence between $\mathbb{M}$ and Δ, and all $j \in \mathbb{N}$;*

(ii) *$\sum_{\Delta,\mathbb{M}} |G_j(M)| \leqslant \xi$ for any ϑ-fine ϱ-regular N-tagged system $\Delta = \{(t, J)\}$, every set $\mathbb{M} = \{M\}$ of intervals M such that the inclusion $M \subset J$ defines a one-to-one correspondence between $\mathbb{M}$ and Δ, and all $j \in \mathbb{N}$;*

(4.7) *$g_j(x) \to g(x)$ for $j \to \infty$, $x \in I \setminus N$.*

Then g is strongly integrable and

(4.8) *$G_j(K) \to G(K)$ for $j \to \infty$ and every interval $K \subset I$, where G is the primitive of g.*

Proof is quite analogous to that of Theorem 1.13. The functions g_j are strongly ϱ-integrable by Theorem 4.8 and we use Proposition 4.7 and Theorem 4.6 instead of Proposition 1.9 and Theorem 1.11. □

Theorem 4.10. *Let* (3.1) *hold (in addition to* (1.1)*). Let G be an additive interval function in I, $m(I \setminus D_G) = 0$ (cf. Definition 2.4). Assume that*

(4.9) *for every $\xi > 0$ there exists a gauge ϑ such that $\sum\limits_{\Theta,\mathbb{M}} |G(M)| \leqslant \xi$ for any ϑ-fine ϱ-regular $(I \setminus D_G)$-tagged system $\Delta = \{(t, J)\}$ and any set $\mathbb{M}$ of intervals M provided a one-to-one correspondence between $\mathbb{M}$ and Δ is defined by $M \subset J$*

holds. Define g by (3.3).

Then g is strongly ϱ-integrable and G is its primitive.

Lemma 4.11. *Let G be an additive interval function in I, $g \in \mathbb{R}$, $t \in \operatorname{Int} I$, $\eta > 0$, $0 < \alpha < 1$. Let G be α-differentiable to g at t. Then there exists $r_1 > 0$ such that*

$$|G(M) - gm(M)| \leqslant \eta(2r_2)^n$$

for $0 < r_2 \leqslant r_1$ and every interval $M \subset V(t, r_2)$.

See [4], Section 2, Corollary 2.

P r o o f o f T h e o r e m 4.10. g is ϱ-integrable and G is its primitive by Theorem 3.1. We have to prove that g is strongly ϱ-integrable. Let $\varepsilon > 0$. By (3.1) there exists a gauge ω such that

$$\varrho(t,\sigma) \geqslant \lambda(t)/2 \quad \text{for } t \in I,\ 0 < \sigma \leqslant \omega(t) \tag{4.10}$$

and

$$|G(M) - g(t)m(M)| \leqslant \varepsilon\big(\lambda(t)\big)^n 2^{-2n-1}(2r_2)^n/m(I) \tag{4.11}$$

for $t \in \operatorname{Int} I$ and any interval $M \subset V(t, r_2)$, $0 < r_2 \leqslant \omega(t)$. Put $\xi = \varepsilon/2$ and let ϑ correspond to ξ by (4.4). Put $\delta(t) = \omega(t)$ for $t \in D_G$, $\delta(t) = \vartheta(t)$ for $t \in I \setminus D_G$. Let $\Delta = \{(t, J)\}$ be a δ-fine ϱ-regular system and let $\mathbb{M}$ be a set of intervals M such that a one-to-one correspondence between $\mathbb{M}$ and Δ is defined by $M \subset J$. Put

$$\begin{aligned}
&\Delta_1 = \{(t,J) \in \Delta;\, t \in D_G\}, \quad \Delta_2 = \{(t,J) \in \Delta;\, t \in I \setminus D_G\},\\
&\mathbb{M}_1 = \{M \in \mathbb{M};\, M \subset J \text{ for some } (t,J) \in \Delta_1\},\\
&\mathbb{M}_2 = \{M \in \mathbb{M};\, M \subset J \text{ for some } (t,J) \in \Delta_2\}.
\end{aligned}$$

Then by (4.11) we have

$$\begin{aligned}
&\sum_{\Delta,\mathbb{M}} |G(M) - g(t)m(M)| \\
&\leqslant \sum_{\Delta_1,\mathbb{M}_1} |G(M) - g(t)m(M)| + \sum_{\Delta_2,\mathbb{M}_2} |G(M) - g(t)m(M)| \\
&\leqslant \sum_{\Delta_1,\mathbb{M}_1} \varepsilon\big(\lambda(t)\big)^n 2^{-2n-1}\big(2r_2(t,J)\big)^n/m(I) + \sum_{\Delta_2,\mathbb{M}_2} |G(t) - g(t)m(M)|,
\end{aligned} \tag{4.12}$$

where $r_2(t,J) = \sup\{\|x-t\|;\ x \in J\}$, $(t,J) \in \Delta_1$. If $(t,J) \in \Delta_1$, then $d(J) \geqslant r_2(t,J)$, J is $\lambda(t)/2$-regular by (4.10) so that any edge of J is greater than or equal to $r_2(t,J)\lambda(t)/2$, $m(J) \geqslant \big(r_2(t,J)\lambda(t)/2\big)^n$ and

$$(4.13) \qquad \sum_{\Delta_1, \mathbb{M}_1} \varepsilon\big(\lambda(t)\big)^n 2^{-2n-1}\big(2r_2(t,J)\big)^n/m(I) \leqslant \sum_{\Delta_1,\mathbb{M}_1} m(J)\varepsilon/2m(I) \leqslant \varepsilon/2.$$

The system Δ_2 is ϑ-fine ϱ-regular $(I \setminus D_G)$-tagged. Since $g(t) = 0$ for $t \in I \setminus D_G$ we get by (4.9)

$$(4.14) \qquad \sum_{\Delta_2,\mathbb{M}_2} |G(M) - g(t)m(M)| \leqslant \xi = \varepsilon/2.$$

The assertion follows from (4.12), (4.13) and (4.14). □

The following descriptive characterization of strongly ϱ-integrable functions is a direct consequence of Theorem 4.10, Lemma 1.7, Theorem 4.5, Note 4.2 and Theorem 2.8.

Theorem 4.12. *Let* (1.2), (2.1) *and* (3.1) *hold. Let* $f\colon I \to \mathbb{R}$ *and let* F *be an interval function on* I. *Then the following two conditions are equivalent:*

(A) *f is strongly ϱ-integrable and F is its primitive,*

(B) *F is additive, $m(I \setminus D_F) = 0$, (4.4) holds and $F' = f$ a.e.*

5. ϱ-INTEGRABILITY DOES NOT IMPLY STRONG ϱ-INTEGRABILITY

In this section we will construct a function $f\colon I \to \mathbb{R}$ which is ϱ-integrable over I but not strongly ϱ-integrable. Actually, we will not do it for quite general ϱ but, nonetheless, for a rather wide class of functions ϱ characterized by conditions (5.1)–(5.4) below. For simplicity of exposition we will assume $n = 2$, i.e. $I \subset \mathbb{R}^2$, and $I = [-1,2] \times [-1,2]$. A modification to $\mathbb{R}^n$ with $n > 2$ and to a general interval is routine. We will modify the idea of the construction introduced in [5, Sec. 3].

Let ϱ be a function defined on $I \times (0,\infty)$ and satisfying the following conditions:

(5.1) $0 < \varrho(t,d) < 1$ for all $(t,d) \in I \times (0,\infty)$;

(5.2) the function $d\varrho(t,d)$ is increasing in d for every $t \in I$;

(5.3) there is a function $\omega\colon (0,\infty) \to (0,1)$ such that

  (i) the function $d\omega(d)$ is increasing;

  (ii) $\omega(d) \leqslant \varrho(t,d)$ for all $(t,d) \in I \times (0,\infty)$;

(5.4) ϱ is continuous on $I \times (0,\infty)$.

Note that evidently

(5.5) $\lim\limits_{d\to 0+} d\varrho(t,d) = 0$ for evey $t \in I$,

(5.6) for every $(t,d) \in I \times (0,\infty)$ there is $\sigma = \sigma(t,d) > 0$ such that

$$\varrho(u,b) > \frac{7}{8}\varrho(t,d)$$

for every $(u,b) \in V\big((t,d),\sigma\big)$.

First we will construct a Cantor discontinuum on $[0,1]$. Choose $t_0 \in (0,1)$, find d_0 such that

$$d_0\varrho(t_0,d_0) < \min\left\{\frac{1}{2}\omega\left(\frac{1}{2}\right), 1 - t_0\right\}$$

(cf. (5.5)) and put

$$t_1 = t_0 + d_0\varrho(t_0,d_0).$$

By (5.4), (5.2), (5.5) and the inequality $\frac{1}{2}\omega\left(\frac{1}{2}\right) \leqslant \frac{1}{2}\varrho\left(t_1,\frac{1}{2}\right)$ there is d_1 such that

$$d_1\varrho(t_1,d_1) = d_0\varrho(t_0,d_0), \qquad d_1 < \frac{1}{2}.$$

Consequently,

$$t_1 = t_0 + d_1\varrho(t_1,d_1).$$

Put

$$c = \max\{d_0,d_1\}, \qquad \tau = \frac{1}{2}(t_0+t_1)$$

and find $\sigma_0^* = \sigma(t_0,c)$, $\sigma_1^* = \sigma(t_1,c)$ from (5.6). Set

$$\sigma_0 = \min\left\{\sigma_0^*, \frac{1}{4}(t_1-t_0)\right\},$$
$$\sigma_1 = \min\left\{\sigma_1^*, \frac{1}{4}(t_1-t_0)\right\}$$

and denote

$$s_0 = \max\{t_0-\sigma_0, 0\}, \qquad s_1 = \min\{t_1+\sigma_1, 1\}.$$

We now repeat the above construction on the intervals $[s_0,t_0]$ and $[t_1,s_1]$. Let us describe the general step.

Let B denote a binary multiindex (i.e. a finite sequence of zeros and ones) and assume we already have numbers t_{B0}, t_{B1}, s_{B0}, s_{B1}, d_{B0}, d_{B1}, c_B, τ_B satisfying

$$0 \leqslant s_{B0} < t_{B0} < t_{B1} < s_{B1} \leqslant 1$$

and

$$t_{B1} - t_{B0} = d_{B0}\varrho(t_{B0},d_{B0}) = d_{B1}\varrho(t_{B1},d_{B1}) < \frac{1}{2^{|B|}}\omega\left(\frac{1}{2^{|B|}}\right), \tag{5.7}$$
$$c_B = \max\{d_{B0},d_{B1}\}, \quad \tau_B = \frac{1}{2}(t_{B0}+t_{B1}).$$

(Here and in the sequel $|B|$ stands for the number of digits of the multiindex B.)

We choose

$$t_{B00} \in (s_{B0}, t_{B0}), \quad t_{B10} \in (t_{B1}, s_{B1}),$$

find d_{B00}, d_{B10} such that

$$\begin{aligned} d_{B00}\varrho(t_{B00}, d_{B00}) &< \min\{2^{-|B|-1}\omega(2^{-|B|-1}), t_{B0} - t_{B00}\}, \\ d_{B10}\varrho(t_{B10}, d_{B10}) &< \min\{2^{-|B|-1}\omega(2^{-|B|-1}), s_{B1} - t_{B10}\} \end{aligned}$$

and put

$$\begin{aligned} t_{B01} &= t_{B00} + d_{B00}\varrho(t_{B00}, d_{B00}), \\ t_{B11} &= t_{B10} + d_{B10}\varrho(t_{B10}, d_{B10}). \end{aligned}$$

Now we find d_{B01}, d_{B11} such that

$$d_{B01}\varrho(t_{B01}, d_{B01}) = d_{B00}\varrho(t_{B00}, d_{B00}),$$

$$d_{B11}\varrho(t_{B11}, d_{B11}) = d_{B10}\varrho(t_{B10}, d_{B10})$$

so that

$$\begin{aligned} t_{B01} &= t_{B00} + d_{B01}\varrho(t_{B01}, d_{B01}), \\ t_{B11} &= t_{B10} + d_{B11}\varrho(t_{B11}, d_{B11}). \end{aligned}$$

We set

$$\begin{aligned} c_{B0} &= \max\{d_{B00}, d_{B01}\}, \quad c_{B1} = \max\{d_{B10}, d_{B11}\}, \\ \tau_{B0} &= \frac{1}{2}(t_{B00} + t_{B01}), \quad \tau_{B1} = \frac{1}{2}(t_{B10} + t_{B11}). \end{aligned}$$

Further, we find from (5.6) the values of

$$\begin{aligned} \sigma^*_{B00} &= \sigma(t_{B00}, c_{B0}), \quad \sigma^*_{B01} = \sigma(t_{B01}, c_{B0}), \\ \sigma^*_{B10} &= \sigma(t_{B10}, c_{B1}), \quad \sigma^*_{B11} = \sigma(t_{B11}, c_{B1}), \end{aligned}$$

put

$$\begin{aligned} \sigma_{B00} &= \min\left\{\sigma^*_{B00}, \frac{1}{4}(t_{B01} - t_{B00})\right\}, \\ \sigma_{B01} &= \min\left\{\sigma^*_{B01}, \frac{1}{4}(t_{B01} - t_{B00})\right\}, \\ \sigma_{B10} &= \min\left\{\sigma^*_{B10}, \frac{1}{4}(t_{B11} - t_{B10})\right\}, \\ \sigma_{B11} &= \min\left\{\sigma^*_{B11}, \frac{1}{4}(t_{B11} - t_{B10})\right\}, \end{aligned}$$

and denote

$$s_{B00} = \max\{t_{B00} - \sigma_{B00}, s_{B0}\}, \quad s_{B01} = \min\{t_{B01} + \sigma_{B01}, t_{B0}\},$$
$$s_{B10} = \max\{t_{B10} - \sigma_{B10}, t_{B1}\}, \quad s_{B11} = \min\{t_{B11} + \sigma_{B11}, s_{B1}\}. \tag{5.8}$$

In this way we construct a sequence of tagged intervals of the form $(\tau_B, [t_{B0}, t_{B1}] \times [0, c_B])$. Any such interval with $|B| = k$ will be called an interval of k-th order; there is one interval of order zero, two of order one, generally 2^k tagged intervals of order k. For brevity, we will denote $[t_{B0}, t_{B1}] = T_B$, $[s_{B0}, t_{B0}] = S_{B0}$, $[t_{B1}, s_{B1}] = S_{B1}$.

Let us notice that the set $D = \bigcap_{k=0}^{\infty} \bigcup_{|B|=k} S_B$ is a Cantor discontinuum.

Now we introduce a function f which we will prove to be ϱ-integrable but not strongly ϱ-integrable. To this end, let us choose sequences of positive numbers ξ_k, η_k, β_k all decreasing to zero and such that the sum of β_k diverges, i.e.

$$\xi_k \searrow 0, \quad \eta_k \searrow 0, \quad \beta_k \searrow 0, \quad \sum_{k=0}^{\infty} \beta_k = \infty. \tag{5.9}$$

(We will subject these numbers to some further conditions later.) For every tagged interval constructed above, let us denote

$$\begin{aligned} Q_B^- &= [\tau_B - \xi_{|B|}, \tau_B] \times [c_B, c_B + \eta_{|B|}], \\ Q_B^+ &= [\tau_B, \tau_B + \xi_{|B|}] \times [c_B, c_B + \eta_{|B|}], \\ Q_B &= Q_B^- \cup Q_B^+, \end{aligned}$$

and set

$$f(x) = \begin{cases} -\beta_{|B|}\left(2^{|B|} m(Q_B^-)\right)^{-1} & \text{for } x \in \operatorname{Int} Q_B^-, \\ \beta_{|B|}\left(2^{|B|} m(Q_B^+)\right)^{-1} & \text{for } x \in \operatorname{Int} Q_B^+, \\ 0 & \text{elsewhere.} \end{cases} \tag{5.10}$$

Note that f is Lebesgue integrable over any compact set H, $H \cap ([0,1] \times \{0\}) = \emptyset$. Therefore we can prove the ϱ-integrability of f via the following proposition.

Proposition 5.1. *Let $I \subset \mathbb{R}^n$ be a compact interval, $f\colon I \to \mathbb{R}$, $S \subset I$ a closed set, $f(x) = 0$ for $x \in S$. Assume that for every closed set $H \subset I$ with $S \cap H = \emptyset$ the integral $\int_H f$ exists in the Lebesgue sense, and let us denote its value by $F(H)$. Let $q \in \mathbb{R}$, $\varrho\colon I \times (0, \infty) \to [0, 1)$. Then the following two assertions are equivalent:*

(a) *the ϱ-integral* $(\varrho)\int_I f$ *exists and is equal to* q*;*

(b) *for every* $\varepsilon > 0$ *there is a gauge* $\delta\colon S \to (0,\infty)$ *such that*

$$\left|F\Big(I \setminus \bigcup_{\Delta} J\Big) - q\right| \leqslant \varepsilon$$

for every δ-fine ϱ-regular system $\Delta = \{(t,J)\}$ *such that* $\operatorname{Int}\bigcup_{\Delta} J \supset S$ *and* $t \in S$ *for all* $(t,J) \in \Delta$.

This result is a modification of Proposition 2 in [4] where it was proved for $\varrho \equiv$ const. It was also used in [5] for the case $\varrho\colon (0,\infty) \to [0,1)$ (i.e. ϱ independent of the tag t). Nonetheless, it is easily verified that the proof from [4] can be applied to the general case of $\varrho\colon I \times (0,\infty) \to [0,1)$ since the only necessary prerequisite is the Saks-Henstock Lemma which is available in the general case (see Lemma 1.7).

By virtue of this proposition, the next lemma yields ϱ-integrability of the function f defined above, with $S = D \times \{0\}$ and $(\varrho)\int_I f = 0$.

Lemma 5.2. *Let* $p \in \mathbb{N}$*. Then there exists a gauge* δ *such that*

$$\left|F\Big(I \setminus \bigcup_{\Delta} J\Big)\right| \leqslant 4\beta_{p+1} \tag{5.11}$$

for every δ-fine ϱ-regular system $\Delta = \{(t,J)\}$ *satisfying*

$$t \in D \times \{0\}, \tag{5.12}$$

$$D \times \{0\} \subset \operatorname{Int}\bigcup_{\Delta} J. \tag{5.13}$$

P r o o f. For $p \in \mathbb{N}$ let us choose a gauge δ such that

$$\delta(t) \leqslant \min\{c_B\,;\ |B| = p\} \tag{5.14}$$

for t satisfying (5.12).

Evidently

$$F\Big(I \setminus \bigcup_{\Delta} J\Big) = \sum_{Q} F\Big(Q \setminus \bigcup_{\Delta} J\Big) \tag{5.15}$$

where the sum is taken over all intervals $Q = Q_B$ such that $F\big(Q \setminus \bigcup_{\Delta} J\big) \neq 0$. (Obviously there is only a finite number of such intervals.) If Q is such an interval, then there is $(t,J) \in \Delta$ such that $F(Q \setminus J) \neq 0$.

It is clear from the construction that for a fixed J there are at most two such intervals Q; moreover, they are of the same order and the corresponding values $F(Q \setminus J)$ have opposite signs. We will estimate the terms that contribute positive values to the sum (5.15); the estimate for those with negative contribution is quite analogous.

Thus, let $F(Q_B \setminus J) > 0$ and let us denote

$$J = [u_1, u_2] \times [v_1, v_2].$$

Then $\tau_B - \xi_{|B|} < u_2 < \tau_B + \xi_{|B|}$, $v_1 < 0$, $v_2 > c_B$.

First, let $u_2 - u_1 < v_2 - v_1$. The ϱ-regularity of J implies

$$\frac{u_2 - u_1}{v_2 - v_1} > \varrho(t, v_2 - v_1).$$

By virtue of (5.2) we have

$$u_2 - u_1 > (v_2 - v_1)\varrho(t, v_2 - v_1) > v_2 \varrho(t, v_2) > c_B \varrho(t, c_B).$$

Assume $u_1 > s_{B0}$. Then (5.6) together with the choice of s_{B0} yields

$$u_2 - u_1 > c_B \varrho(t, c_B) > \frac{7}{8} c_B \varrho(t_{B0}, c_B),$$
$$u_1 < u_2 - \frac{7}{8} c_B \varrho(t_{B0}, c_B) < \frac{1}{2}(t_{B0} + t_{B1}) + \xi_{|B|} - \frac{7}{8} d_{B0} \varrho(t_{B0}, d_{B0}).$$

Subjecting without loss of generality the numbers ξ_k to the condition

$$\xi_k \leqslant \frac{1}{8} \min\{t_{B1} - t_{B0};\ |B| = k\} \tag{5.16}$$

we conclude using (5.7), (5.8)

$$u_1 < \frac{1}{2}(t_{B0} + t_{B1}) + \frac{1}{8}(t_{B1} - t_{B0}) - \frac{7}{8}(t_{B1} - t_{B0}) = t_{B0} - \frac{1}{4}(t_{B1} - t_{B0}) < s_{B0},$$

a contradiction.

If, on the contrary, $u_2 - u_1 \geqslant v_2 - v_1$, then

$$u_1 \leqslant u_2 - (v_2 - v_1) < \tau_B + \xi_{|B|} - c_B$$

and we obtain the inequality $u_1 < s_{B0}$ analogously as above.

In both cases we have arrived to the conclusion that $u_1 < s_{B0}$. This inequality means that the interval $T_{B0} = [t_{B00}, t_{B01}]$ of order $|B| + 1$ immediately next to the interval T_B (to the left) is totally covered by $[u_1, u_2]$, hence for the corresponding Q_{B0} we have $F(Q_{B0} \setminus J) = 0$. Similar situation occurs for all intervals T_{B^*} with $|B^*| > |B|$ lying between the intervals T_{B0}, T_B mentioned above, i.e. with 2^{l-2} intervals of order $|B| + l$, $l \geqslant 2$. Consequently, these intervals contribute nothing to the sum in (5.15).

Let k_l be the number of intervals Q_B of order l on the righthand side of (5.15) for which $F(Q_B \setminus \bigcup_{\Delta} J) > 0$. We have $k_0 = k_1 = \ldots = k_p = 0$ by (5.14), and by (5.13) there exists $m \in \mathbb{N}$ such that $k_{p+m+1} = k_{p+m+2} = \ldots = 0$. Further,

$$
\begin{aligned}
k_{p+1} &\leqslant 2^{p+1}, \\
k_{p+2} &\leqslant 2^{p+2} - k_{p+1}, \\
k_{p+3} &\leqslant 2^{p+3} - k_{p+1} - k_{p+2}, \\
k_{p+4} &\leqslant 2^{p+4} - 2k_{p+1} - k_{p+2} - k_{p+3}, \\
&\ldots\ldots\ldots\ldots\ldots, \\
k_{p+l} &\leqslant 2^{p+l} - 2^{l-3}k_{p+1} - 2^{l-4}k_{p+2} - \ldots - k_{p+l-2} - k_{p+l-1}, \\
&\ldots\ldots\ldots\ldots\ldots, \\
k_{p+m} &\leqslant 2^{p+m} - 2^{m-3}k_{p+1} - 2^{m-4}k_{p+2} - \ldots - k_{p+m-1}.
\end{aligned}
$$

Summing up these inequalities after transferring all but the first terms from the right to the lefthand sides we obtain

$$
\begin{aligned}
&k_{p+1}(1 + 1 + 1 + 2 + \ldots + 2^{m-3}) + k_{p+2}(1 + 1 + 1 + 2 + \ldots + 2^{m-4}) + \ldots \\
&+k_{p+l}(1 + 1 + 1 + 2 + \ldots + 2^{m-l-2}) + \ldots + k_{p+m-2}(1 + 1 + 1) \\
&+k_{p+m-1}(1 + 1) + k_{p+m} \leqslant 2^{p+1}(1 + 2 + \ldots + 2^{m-1}),
\end{aligned}
$$

hence

$$
\begin{aligned}
&k_{p+1}(1 + 2^{m-2}) + k_{p+2}(1 + 2^{m-3}) + \ldots + k_{p+l}(1 + 2^{m-l-1}) + \ldots + \\
&k_{p+m-2}(1 + 2^1) + k_{p+m-1}(1 + 2^0) + k_{p+m} \leqslant 2^{p+1}(2^m - 1)
\end{aligned}
$$

and, a fortiori,

$$
\begin{aligned}
(5.17) \qquad &k_{p+1}2^{m-1} + k_{p+2}2^{m-2} + \ldots + k_{p+m-1}2^1 + k_{p+m}2^0 \leqslant 2^{p+m+2}, \\
&k_{p+1} + k_{p+2}2^{-1} + \ldots + k_{p+m-1}2^{-(m-2)} + k_{p+m}2^{-(m-1)} \leqslant 2^{p+3}.
\end{aligned}
$$

On the other hand, for $|B| = p + l$ the definition of f yields the evident estimate

$$
F\Big(Q_B \setminus \bigcup J\Big) \leqslant \beta_{p+l}2^{-(p+l)},
$$

and summing up these values and using the inequality (5.17) we conclude

$$\sum_{+} F(Q \setminus \bigcup_{\Delta} J) \leqslant k_{p+1}\beta_{p+1}2^{-(p+1)} + k_{p+2}\beta_{p+2}2^{-(p+2)} + \dots$$
$$+ k_{p+m-1}\beta_{p+m-1}2^{-(p+m-1)} + k_{p+m}\beta_{p+m}2^{-(p+m)}$$
$$\leqslant \beta_{p+1}2^{-(p+1)}[k_{p+1} + k_{p+2}2^{-1} + \dots + k_{p+m-1}2^{-m+2} + k_{p+m}2^{-m+1}]$$
$$\leqslant 4\beta_{p+1}.$$

The subscript $+$ at the summation symbol indicates that we add only the positive terms. Since the sum of the negative terms satisfies an analogous estimate, the inequality (5.11) immediately follows. □

We have proved that the function f is ϱ-integrable with vanishing integral. Consequently, if f is strongly ϱ-integrable then its strong ϱ-integral must vanish as well. Thus to prove that f is *not* strongly ϱ-integrable, it suffices to prove the following assertion:

Lemma 5.3. *For every gauge δ on I there exists a ϱ-regular δ-fine partition $\Delta = \{(t, J)\}$ of I and intervals $M \subset J$ (one for each $(t, J) \in \Delta$) such that*

$$\sum_{\Delta} \big(F(M) - f(t)m(M)\big) \geqslant 1.$$

Proof. We will again modify the construction of Δ from [4], proof of Lemma 4. Let $\delta\colon I \to (0, \infty)$ be an arbitrary but fixed gauge. Let us denote

$$W_k = \Big\{w \in D\,;\, \delta(w, 0) > \frac{1}{k}\Big\}, \quad k \in \mathbb{N}.$$

By Baire's theorem on complete spaces there is $p \in \mathbb{N}$, $z \in D$ and $\omega > 0$ such that

$$D \cap [z - \omega, z + \omega] \subset \operatorname{Cl} W_p. \tag{5.18}$$

Since $z \in D = \bigcap_{k=0}^{\infty} \bigcup_{|B|=k} S_B$ there is $q \in \mathbb{N}$ and B^* with $|B^*| = q$ such that

$$S_{B^*} \subset [z - \omega, z + \omega] \tag{5.19}$$

and

$$\max\{c_B\,;\, |B| \geqslant q\} < \frac{1}{2p}. \tag{5.20}$$

Finally, since the sum of β_k diverges, see (5.9), there is $m \in \mathbb{N}$ such that

$$\beta_q + \beta_{q+1} + \ldots + \beta_{q+m} \geqslant 2^{q+1}. \tag{5.21}$$

Now there is an interval T_B of order q such that $T_B \subset S_{B^*}$, two intervals T_B of order $q+1$ contained in S_{B^*}, generally 2^j intervals T_B of order $q+j$ contained in S_{B^*} for $j = 0, 1, \ldots, m$. Since the number of all these intervals is finite ($j \leqslant m$), for each of them there exists $\kappa = \kappa(T_B) > 0$ such that all intervals $\tilde{T}_B = [t_{B0} - \kappa, t_{B1} + \kappa]$ resulting by enlarging the original intervals by κ are pairwise disjoint and contained in $[z - \omega, z + \omega]$, and

$$\frac{t_{B1} - t_{B0} + 2\kappa}{c_B} < 1.$$

(Note that $(t_{B1} - t_{B0})/c_B < 1$ by construction.)

Let e.g. $c_B = d_{B0}$. (If $c_B = d_{B1}$, the procedure is analogous, only we use quantities with indices $B1$ instead of those with $B0$.)

By continuity of the function ϱ there is $\lambda = \lambda_k > 0$ such that

$$\mu\varrho(t, \mu) - d_{B0}\varrho(t_{B0}, d_{B0}) < 2\kappa \tag{5.22}$$

provided $|\mu - d_{B0}| < \lambda$, $|t - t_{B0}| < \lambda$ and $|B| = k$. By virtue of (5.18), (5.19) there is $\zeta_{B0} \in \tilde{T}_B \cap W_p$ such that

$$|\zeta_{B0} - t_{B0}| < \lambda.$$

Without loss of generality we may and will assume that

$$\eta_k < \min\left\{\frac{1}{2p}, \lambda_k\right\} \quad \text{for } k \geqslant q \tag{5.23}$$

and choose $\psi_k > 0$ such that $\eta_k + \psi_k < \min\left\{\frac{1}{2p}, \lambda_k\right\}$. Denote

$$J = \tilde{T}_B \times [-\psi_k, d_{B0} + \eta_{|B|}]. \tag{5.24}$$

Then

$$\operatorname{reg} J = \frac{t_{B1} - t_{B0} + 2\kappa}{d_{B0} + \psi_{|B|} + \eta_{|B|}} = \frac{d_{B0}\varrho(t_{B0}, d_{B0}) + 2\kappa}{d_{B0} + \psi_{|B|} + \eta_{|B|}}.$$

By (5.22) we have $(d_{B0} + \psi_{|B|} + \eta_{|B|})\varrho(\zeta_{B0}, d_{B0} + \psi_{|B|} + \eta_{|B|}) < d_{B0}\varrho(t_{B0}, d_{B0}) + 2\kappa$, hence

$$\operatorname{reg} J > \varrho(\zeta_{B0}, d_{B0} + \psi_{|B|} + \eta_{|B|}) = \varrho\big(\zeta_{B0}, d(J)\big).$$

On the other hand, by (5.24) we have

$$d(J) = d_{B0} + \psi_{|B|} + \eta_{|B|} < \frac{1}{p}.$$

Since $\zeta_{B0} \in W_p$, we have $\delta(\zeta_{B0}, 0) > \frac{1}{p}$ and the pair (ζ_{B0}, J) is both ϱ-regular and δ-fine.

Consequently, all pairs (t, J) constructed above form a δ-fine ϱ-regular system Δ_1. Now we have to complete this system to a partition of I with the required properties.

Let us note that $F(J) = 0$ for every J, $(t, J) \in \Delta_1$. Since $F(I) = 0$, we have $F(I \setminus \bigcup_{\Delta_1} J) = 0$ as well. The set $I - \bigcup_{\Delta_1} J$ is a finite union of intervals, and by the Saks-Henstock Lemma 1.7 there is a δ-fine ϱ-regular system $\Delta_2 = \{(t, K)\}$ covering it for which

$$\sum_{\Delta_2} |F(K) - f(t)m(K)| < 1.$$

Now, $\Delta = \Delta_1 \cup \Delta_2$ is a δ-fine ϱ-regular partition of I. Choose $M = Q_B$ where $Q_B \subset J$ for $(t, J) \in \Delta_1$ (there is a unique such Q_B), $M = K$ for $(t, K) \in \Delta_2$. Then

$$\begin{aligned}\sum_{\Delta} \big(F(M) - f(t)m(M)\big) &\geqslant \sum_{\Delta_1} \big(F(M) - f(t)m(M)\big) - \sum_{\Delta_2} |F(K) - f(t)m(K)| \\ &\geqslant 2^{-q} \sum_{j=0}^{m} \beta_{q+j} - 1 \geqslant 1\end{aligned}$$

by virtue of (5.21). This proves the assertion, and hence nonexistence of the strong ϱ-integral of f. □

Note 5.4. If $n > 2$ the construction is analogous, the intervals Q_B being constructed over $(n-1)$-dimensional intervals that play the role of the Cantor discontinuum. For example, for $n = 3$ we construct the (one-dimensional) intervals T_B as above and form Cartesian products $T_B \times T_B$ (with the same multiindex B). These two-dimensional intervals are then used to construct Q_B. They are located along a diagonal of the square $[0, 1] \times [0, 1]$ and allow similar estimates as in the case $n = 2$.

6. Concerning the proof of Theorem 2.1

Assume that (1.2) and (2.1) hold. The following lemma corresponds to Lemma 2.7 in [3].

Lemma 6.1. *Let $f\colon I \to \mathbb{R}$ be ϱ-integrable, F being its primitive. Let L be an interval, $L \subset I$. Then for every $\varepsilon > 0$ there exists $\eta > 0$ such that*

$$\begin{aligned}&|F(L) - F(K)| \leqslant \varepsilon \quad \text{for every interval } K, \\ &L \subset K \subset I \quad \text{with } m(K \setminus L) \leqslant \eta.\end{aligned} \tag{6.1}$$

P r o o f. Put $\eta(t) = \limsup_{\sigma\to 0+} \varrho(t,\sigma)$, $t \in I$. Since $\eta(t) < 1$ by (2.1), there is a gauge $\overline{\delta}$ such that $\varrho(t,\sigma) \leqslant \frac{1}{2}(1+\eta(t))$ for $0 < \sigma \leqslant \overline{\delta}(t)$. Put $\omega(t,\sigma) = \frac{1}{2}(1+\eta(t))$ for $t \in I$, $\sigma > 0$. Let $L = [c_1, d_1] \times \ldots \times [c_n, d_n]$, $\varepsilon > 0$. Let δ correspond to $\varepsilon/3$ by (1.3). Without loss of generality we may assume that $\delta(t) \leqslant \overline{\delta}(t)$, $\delta(t) < \min_i(d_i - c_i)$ for $t \in I$. By Lemma 1.1 there exists a $\frac{1}{2}\delta$-fine ω-regular partition $\Omega = \{(t, J)\}$ of L.

If $T = [p_1, q_1] \times \ldots \times [p_n, q_n] \subset L$, $0 < q_i - p_i < d_i - c_i$ for $i = 1, 2, \ldots, n$, put

$$\begin{aligned} S_i &= (-\infty, q_i] \text{ provided } p_i = c_i, \\ S_i &= [p_i, q_i] \text{ provided } c_i < p_i < q_i < d_i, \\ S_i &= (p_i, \infty) \text{ provided } q_i = d_i,\ i = 1, 2, \ldots, n, \\ \Lambda(T) &= S_1 \times \ldots \times S_n. \end{aligned}$$

$\Lambda(J_1)$ and $\Lambda(J_2)$ are nonoverlapping provided (t^1, J_1), $(t^2, J_2) \in \Omega$, $J_1 \neq J_2$. Moreover, $\bigcup_{\Omega} \Lambda(J) = \mathbb{R}^n$. For an interval K, $L \subset K \subset I$, and $(t, J) \in \Omega$ put $J' = \Lambda(J) \cap K$. Put $\Omega' = \{(t, J');\ (t, J) \in \Omega\}$. Ω' is a partition of K. There exists $\eta_0 > 0$ such that Ω' is both δ-fine and ϱ-regular if $m(K \setminus L) \leqslant \eta_0$. Moreover, since Ω is fixed, there is $\eta \in (0, \eta_0]$ such that

$$\sum_{\Omega} |f(t)| m(J' \setminus J) \leqslant \varepsilon/3 \text{ provided } m(K \setminus L) \leqslant \eta. \tag{6.2}$$

Assume that $m(K \setminus L) \leqslant \eta$. Since both Ω and Ω' are δ-fine and ϱ-regular, Lemma 1.6 yields

$$\Big|\sum_{\Omega} f(t) m(J) - F(L)\Big| \leqslant \varepsilon/3,$$
$$\Big|\sum_{\Omega'} f(t) m(J') - F(K)\Big| \leqslant \varepsilon/3$$

and (6.1) follows by (6.2). □

Note 6.2. If (2.1) is dropped and if (1.2) and (2.5) are assumed, then Lemma 6.1 holds; it can be proved in the same way provided ω is replaced by ω_1, where $\omega_1(t,\sigma) = \frac{1}{2}(1 + \varrho(t,\sigma))$.

Note 6.3. Making use of Lemma 6.1 we can prove Theorem 2.1 in the same way as Theorem 2.5 in [3] was proved via Lemmas 2.7, 2.6 in [3].

Let

$$J = [c, d] = [c_1, d_1] \times \ldots \times [c_n, d_n] \subset I.$$

First, let us assume that $d_j - c_j$, $j = 1, 2, \ldots, n$ are dyadically rational. Then J is the union of nonoverlapping n-dimensional cubes with the same length of edge equal to 2^{-p}, p a sufficiently large integer. By successively halving the edges of the cubes and applying the standard compactness argument used in the proof of Cousin's lemma we construct a δ-fine partition of J; since all intervals of this partition are cubes, it is ϱ-regular as well since $\varrho(t, \sigma) < 1$ by definition.

Further we proceed by induction, successively decreasing the number of edges of J which are dyadically rational.

Let $k \in \{0, 1, \ldots, n-1\}$ and let the assertion of Lemma 1.1 hold provided the interval J has $k+1$ edges of dyadically rational lengths.

Let

$$K = [g, h] = [g_1, h_1] \times [g_2, h_2] \times \ldots \times [g_n, h_n] \subset I$$

be an interval with k dyadically rational edges with $k \geqslant 1$; without loss of generality we will assume that $h_j - g_j$ is dyadically rational for $j = 1, 2, \ldots, k$. (The case $k = 0$ will be considered later.)

Denote $L = [g_1, h_1] \times \ldots \times [g_k, h_k]$, $\delta_k\big((t_1, t_2, \ldots, t_k)\big) = \delta\big((t_1, \ldots, t_k, g_{k+1}, \ldots, g_n)\big)$ for $t_j \in [g_j, h_j]$, $j = 1, 2, \ldots, k$. By the first part of the proof there exists a $\frac{1}{2}\delta_k$-fine partition Θ of L consisting of k-dimensional cubes with dyadically rational edges, that is, for every $(\tau, Q) \in \Theta$ we have $d(Q) = 2^{-q}$, $\operatorname{reg} Q = 1$. (Here of course $\tau \in \mathbb{R}^k$, $Q \subset \mathbb{R}^k$, q is an integer depending on Q.)

Let $(\tau, Q) \in \Theta$ where $\tau = (t_1, \ldots, t_k)$, $d(Q) = \lambda$. For $j = k+1, k+2, \ldots, n$ find λ_j such that

(i) $h_j - g_j - \lambda_j$ is dyadically rational,

(ii) $\lambda \varrho(t^g, \lambda) < \lambda_j < \lambda$

where $t^g = (t_1, t_2, \ldots, t_k, g_{k+1}, \ldots, g_n)$. Denote

$$M_Q = Q \times [g_{k+1}, g_{k+1} + \lambda_{k+1}] \times \ldots \times [g_n, g_n + \lambda_n].$$

Since $\lambda_j < \lambda < \delta_k\big((t_1, t_2, \ldots, t_k)\big) = \delta(t^g)$, the pair (t^g, M_Q) is δ-fine; since $\lambda = d(M_Q)$ and $\varrho(t^g, \lambda) < \lambda_j/\lambda$, (t^g, M_Q) is ϱ-regular.

Now let us set

$$H_j^0 = [g_j, g_j + \lambda_j], \; H_j^1 = [g_j + \lambda_j, h_j]$$

for $j = k+1, k+2, \ldots, n$. For $\varphi \colon \{k+1, \ldots, n\} \to \{0, 1\}$ denote

$$H^\varphi = H_{k+1}^{\varphi(k+1)} \times H_{k+2}^{\varphi(k+2)} \times \ldots \times H_n^{\varphi(n)}$$

and

$$M_Q^\varphi = Q \times H^\varphi.$$

Two intervals $M_{Q_1}^{\varphi_1}$, $M_{Q_2}^{\varphi_2}$ evidently do not overlap provided $\varphi_1 \neq \varphi_2$ and/or $Q_1 \neq Q_2$. On the other hand, $K = \bigcup\limits_{Q,\varphi} M_Q^\varphi$ where the union is taken over all φ and all Q such that $(\tau, Q) \in \Theta$ for some τ.

If $(\tau, Q) \in \Theta$ and $\varphi \equiv 0$ then $M_Q^\varphi = M_Q$, hence the corresponding pair (t^g, M_Q^φ) is δ-fine and ϱ-regular as shown above.

If $(\tau, Q) \in \Theta$ and $\varphi \not\equiv 0$ then M_Q^φ has k edges of a dyadically rational length λ and at least one more edge of a dyadically rational length $h_j - g_j - \lambda_j$ (with j such that $\varphi(j) = 1$). Hence it has $k + 1$ dyadically rational edges, and by the induction hypothesis there exists a δ-fine ϱ-regular partition of M_Q^φ. The union of all intervals M_Q^0 and all partitions of M_Q^φ for $\varphi \not\equiv 0$ evidently forms the desired partition of K.

If $k = 0$ we have no L, δ_k and Θ. It suffices to start with finding $\lambda_1 < \delta(g)$ satisfying (i) with $j = 1$, and proceed by choosing λ_j, $j = 2, 3, \ldots, n$ satisfying (i), (ii) with $\lambda = \lambda_1$. We continue as above with the obvious modifications; in particular, the role of M_Q^φ is played by the intervals H^φ. They all have at least one dyadically rational edge (cf. (i)) except for H^0, but the pair (g, H^0) is δ-fine and H^0 is ϱ-regular by (ii) and the choice of λ. Consequently, the last induction step is completed by the same argument as above.

References

[1] *J. Kurzweil*: Generalized ordinary differential equations and continuous dependence on a parameter. Czechosl. Math. J. *7* (1957), 418–449.

[2] *J. Kurzweil*: Nichtabsolutkonvergente Integrale, Teubner Texte zur Mathematik, 25. Teubner, Leipzig, 1984.

[3] *J. Kurzweil, J. Jarník*: Equiintegrability and controlled convergence of Perron-type integrable functions. Real. Anal. Exchange *17* (1991–92), 110–139.

[4] *J. Kurzweil, J. Jarník*: Differentiability and integrability in n dimensions with respect to α-regular intervals. Results in Mathematik *21* (1992), 138–151.

[5] *J. Kurzweil, J. Jarník*: Generalized multidimensional Perron integral involving a new regularity condition. Results in Mathematics *23* (1993), 263–273.

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