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*Czechoslovak Mathematical Journal*, Vol. 44 (1994), No. 4, 603–609

Persistent URL: <http://dml.cz/dmlcz/128494>

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EXISTENCE OF SOLUTIONS OF A FUNCTIONAL-INTEGRAL  
EQUATION IN INFINITE DIMENSIONAL BANACH SPACES<sup>1</sup>

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(Received September 9, 1992)

Let  $\Omega$  be a bounded closed subset of  $\mathbb{R}^n$ . If  $f, k, g$  are functions defined, respectively, in  $\Omega \times E$ ,  $\Omega \times \Omega$ ,  $\Omega \times E$  ( $E$  a Banach space) with values into, respectively,  $E$ ,  $L(F, E)$ ,  $F$  ( $F$  a Banach space,  $L(F, E)$  the space of all linear continuous operators from  $F$  into  $E$ ), we consider the functional-integral equation

$$(1) \quad x(t) = f\left(t, r \int_{\Omega} k(t, s)g(s, x(s)) \, ds\right), \quad t \text{ a.e. in } \Omega$$

and look for solutions of (1) lying in  $L^1(\Omega, E)$ , the usual Bochner function space on  $(\Omega, \mathcal{L}, m)$ , the usual Lebesgue measure space. The equation (1) is quite general, because for  $f(t, x) = x$  we get the Hammerstein integral equation, whereas if  $g(t, x) = x$  we get an equation recently considered in [2] and in [4]. (In particular, we improve the result in [4] because, in the case of  $E = F = \mathbb{R}$  and  $\Omega = [0, 1] \subset \mathbb{R}$ , we are able to dispense with one of the hypotheses used in that paper.) For several applications of the Hammerstein integral equation to partial differential equations we refer to [3] and [8].

The technique we use in the main theorem is the usual one: we construct an operator  $A$  mapping continuously a suitable bounded, closed and convex subset  $Q$  of  $L^1(\Omega, E)$  into itself, and prove that  $A(Q)$  is relatively compact. Hence the Schauder fixed point theorem can be applied. The choice of the set  $Q$  is such that it allows us to avoid the use of certain monotonicity assumptions contained in [2] (see also results and examples in [3] and in [8]) that are not always extendible to the case of functions with values in infinite dimensional Banach spaces; the hypotheses we consider are quite general and "natural" in the sense that they are necessary and sufficient for certain operators to take  $L^1(\Omega, E)$  continuously into itself (see [7]).

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<sup>1</sup> Work performed under the auspices of G.N.A.F.A. of C.N.R. and partially supported by M.U.R.S.T. of Italy (60%; 1990).

One of the main tools we use is the following generalization of the Ascoli-Arzelà theorem to the case of vector-valued continuous functions.

**Theorem 1** (see [1], for instance). *Let  $T$  be a compact metric space and let  $(f_n) \subset C^0(T, E)$  be a sequence of equicontinuous and equibounded functions. If for each  $t \in T$ ,  $\{f_n(t)\}$  is relatively compact in  $E$ , then  $(f_n)$  is relatively compact in  $C^0(T, E)$ . Moreover, the set  $\{f_n(t) : t \in T, n \in \mathbb{N}\}$  is relatively compact in  $E$ , too.*

We even need the following extension of a theorem of Scorza-Dragoni to be found in [9].

**Theorem 2** ([9]). *Let  $T$  be a compact metric space with a Radon measure defined on it,  $E$  a separable metric space,  $F$  a Banach space. If  $f : T \times E \rightarrow F$  is a function verifying the Carathéodory hypotheses, i.e.  $f$  is measurable with respect to  $t \in T$  for all  $x \in E$  and continuous with respect to  $x \in E$  for almost all  $t \in T$ , given  $\varepsilon > 0$  there is a measurable closed subset  $T_\varepsilon$  of  $T$  with  $m(T \setminus T_\varepsilon) < \varepsilon$  and  $f|_{T_\varepsilon \times E}$  continuous.*

We shall make use of the following result concerning compact subsets of separable Banach spaces (see [5]).

**Theorem 3** ([5]). *Let  $M$  be a bounded subset of a separable Banach space  $E$ .  $M$  is relatively compact if and only if for any  $w^*$ -null sequence  $(x_n^*) \subset E^*$  one has  $\limsup_n \sup_{x \in M} |x_n^*(x)| = 0$ .*

We are now ready to give the proof of our result for which we need an easy lemma:

**Lemma 4.** *Let us assume*

(k<sub>1</sub>)  $h_1, h_2 \in L^1(\Omega, \mathbb{R})$ ,  $h_1(t) \geq 0, h_2(t) \geq 0$  a.e. on  $\Omega$ ;

(k<sub>2</sub>)  $\psi : \Omega \times \Omega \rightarrow \mathbb{R}_+$  verifies the Carathéodory hypotheses and the linear operator

$$(\Psi z)(t) = \int_{\Omega} \psi(t, s) z(s) \, ds$$

maps  $L^1(\Omega, \mathbb{R})$  into  $L^1(\Omega, \mathbb{R})$  (in this case  $\Psi$  is continuous ([10]) and  $\|\Psi\|$  denotes its norm);

(k<sub>3</sub>)  $b_1, b_2, r \geq 0$  are such that  $rb_1 b_2 \|\Psi\| < 1$ .

Then there is a nonnegative  $\varphi_0 \in L^1(\Omega, \mathbb{R})$  such that

$$\varphi_0(t) = h_1(t) + rb_1 \int_{\Omega} \psi(t, s) (h_2(s) + b_2 \varphi_0(s)) \, ds, \quad t \text{ a.e. in } \Omega.$$

PROOF. Let us put  $p = \frac{\|h_1\| + \|a\|}{1 - rb_1 b_2 \|\Psi\|}$  where  $a(t) = rb_1 \int_{\Omega} \psi(t, s) h_2(s) ds \in L^1(\Omega, \mathbb{R})$ . It is easy to see that  $Mx \in B_p = \{x: x \in L^1(\Omega, \mathbb{R}), \|x\| \leq p\}$  whenever  $x \in B_p$ , where

$$Mx(t) = h_1(t) + rb_1 \int_{\Omega} \psi(t, s)(h_2(s) + b_2 x(s)) ds, \quad t \in \Omega.$$

It is also clear that  $Mx(t) \geq 0$  a.e. on  $\Omega$  when  $x(t) \geq 0$  a.e. on  $\Omega$  and so  $M(B_p^+) \subset B_p^+$  with  $B_p^+ = B_p \cap \{x: x \in L^1(\Omega, \mathbb{R}), x(t) \geq 0 \text{ a.e. on } \Omega\}$ ; furthermore,  $B_p^+$  is a complete metric space. It is also easy to prove that  $M$  is a contraction when restricted to  $B_p^+$ . Then the Banach-Caccioppoli fixed point theorem applies to give the result. We are done.  $\square$

**Theorem 5.** Let  $E$  be a separable Banach space,  $F$  an arbitrary Banach space and  $\Omega$  a bounded, closed subset of  $\mathbb{R}^n$ . Let us assume

(h<sub>1</sub>)  $f: \Omega \times E \rightarrow E$  verifies the Carathéodory hypotheses and, moreover, there exist  $h_1 \in L^1(\Omega, \mathbb{R})$  and  $b_1 \geq 0$  such that

$$\|f(t, x)\|_E \leq h_1(t) + b_1 \|x\|_E \quad \text{for a.a. } t \in \Omega \text{ and all } x \in E;$$

(h<sub>2</sub>)  $k: \Omega \times \Omega \rightarrow C(F, E)$  (the Banach space of linear compact operators from  $F$  into  $E$  with the usual operator norm) verifies the Carathéodory hypotheses and the linear operator  $K$  defined by

$$(Kz)(t) = \int_{\Omega} \|k(t, s)\|_{C(F, E)} z(s) ds, \quad t \text{ a.e. in } \Omega$$

maps  $L^1(\Omega, \mathbb{R})$  into itself (this last fact implies that  $K$  is continuous, see [10]; let  $\|K\|$  denote its norm);

(h<sub>3</sub>)  $g: \Omega \times E \rightarrow F$  verifies the Carathéodory hypotheses and, moreover, there exist  $h_2 \in L^1(\Omega, \mathbb{R})$  and  $b_2 \geq 0$  such that

$$\|g(t, x)\|_F \leq h_2(t) + b_2 \|x\|_E \quad \text{for a.a. } t \in \Omega \text{ and all } x \in E;$$

(h<sub>4</sub>)  $rb_1 b_2 \|K\| < 1$ .

Then the equation (1) has a solution  $x$  in  $L^1(\Omega, E)$ .

PROOF. Putting  $\psi(t, s) = \|k(t, s)\|_{C(E, F)}$  in Lemma 4, we get that there is a nonnegative  $\varphi_0 \in L^1(\Omega, \mathbb{R})$  such that

$$\varphi_0(t) = h_1(t) + rb_1 \int_{\Omega} \|k(t, s)\|_{C(E, F)} (h_2(s) + b_2 \varphi_0(s)) ds, \quad t \text{ a.e. in } \Omega.$$

First of all, assume  $\varphi_0 = \theta_{L^1(\Omega, R)}$ . In such a case we easily get that  $\theta_{L^1(\Omega, E)}$  is a solution of (1). Indeed, we have

$$\begin{aligned} & \left\| f\left(t, r \int_{\Omega} k(t, s)g(s, \theta_{L^1(\Omega, E)}(s)) \, ds\right) \right\| \\ & \leq h_1(t) + b_1 \left\| r \int_{\Omega} k(t, s)g(s, \theta_{L^1(\Omega, E)}(s)) \, ds \right\| \\ & \leq h_1(t) + b_1 r \int \|k(t, s)\|_{C(E, F)} \|g(s, \theta_{L^1(\Omega, E)}(s))\| \, ds \\ & \leq h_1(t) + b_1 r \int \|k(t, s)\|_{C(E, F)} (h_2(s) + b_2 \|\theta_{L^1(\Omega, E)}(s)\|) \, ds \\ & \leq h_1(t) + b_1 r \int \|k(t, s)\|_{C(E, F)} (h_2(s) + b_2 \theta_{L^1(\Omega, R)}(s)) \, ds \\ & = \theta_{L^1(\Omega, R)}(t), \qquad t \text{ a.e. in } \Omega, \end{aligned}$$

which means that

$$f\left(t, r \int_{\Omega} k(t, s)g(s, \theta_{L^1(\Omega, E)}(s)) \, ds\right) = \theta_{L^1(\Omega, E)}(t), \quad t \text{ a.e. in } \Omega.$$

So let us assume  $\varphi_0 \neq \theta_{L^1(\Omega, R)}$  and consider the following subset of  $L^1(\Omega, E)$ :

$$Q = \{x : x \in L^1(\Omega, E), \|x(t)\|_E \leq \varphi_0(t) \quad \text{a.e. on } \Omega\}.$$

$Q$  is clearly bounded, closed and convex in  $L^1(\Omega, E)$ ; furthermore,  $Q$  is uniformly integrable, i.e.  $\lim_{m(S) \rightarrow 0} \int_S \|x(s)\| \, ds = 0$  uniformly on  $Q$ . We consider the operator

$$(Ax)(t) = f\left(t, r \int_{\Omega} k(t, s)g(s, x(s)) \, ds\right).$$

We shall prove that

- (i)  $A(L^1(\Omega, E)) \subset L^1(\Omega, E)$ ,
- (ii)  $A(Q) \subset Q$ ,
- (iii)  $A|_Q$  is continuous,
- (iv)  $A(Q)$  is relatively compact.

Hence an easy application of the Schauder fixed point theorem will give the existence of a solution of (1). That (i) is true is an easy consequence of our assumptions

$(h_1), (h_2), (h_3)$ . Let us show (ii). If  $x \in Q$  we have, for a.a.  $t \in \Omega$ ,

$$\begin{aligned} \|(Ax)(t)\|_E &= \left\| f\left(t, r \int_{\Omega} k(t, s)g(s, x(s)) \, ds\right) \right\|_E \\ &\leq h_1(t) + b_1 r \left\| \int_{\Omega} k(t, s)g(s, x(s)) \, ds \right\|_E \\ &\leq h_1(t) + b_1 r \int_{\Omega} \|k(t, s)\|_{C(F, E)} \|g(s, x(s))\|_F \, ds \\ &\leq h_1(t) + b_1 r \int_{\Omega} \|k(t, s)\|_{C(F, E)} (h_2(s) + b_2 \|x(s)\|_E) \, ds \\ &\leq h_1(t) + b_1 r \int_{\Omega} \|k(t, s)\|_{C(F, E)} (h_2(s) + b_2 \varphi_0(s)) \, ds = \varphi_0(t) \end{aligned}$$

by virtue of Lemma 4. Let us prove (iii). Let  $(x_n), (x_0) \subset Q$  with  $x_n \rightarrow x_0$ . This means that  $x_n(s) \rightarrow x_0(s)$  almost everywhere on  $\Omega$  (by passing to a subsequence if necessary). Fix  $\bar{t} \in \Omega$ ; we have  $k(\bar{t}, s)g(s, x_n(s)) \rightarrow k(\bar{t}, s)g(s, x_0(s))$  for a.a.  $s \in \Omega$  because of  $(h_3)$  and  $(h_2)$ . Thanks to  $(h_3)$  we also have that, for a.a.  $s \in \Omega$ ,

$$\begin{aligned} &\|k(\bar{t}, s)[g(s, x_n(s)) - g(s, x_0(s))]\|_E \\ &\leq \|k(\bar{t}, s)\|_{C(E, F)} [2h_2(s) + b_2(\|x_n(s)\| + \|x_0(s)\|)] \\ &\leq \|k(\bar{t}, s)\|_{C(E, F)}^2 (h_2(s) + b_2 \varphi_0(s)); \end{aligned}$$

this easily yields

$$\int_{\Omega} k(\bar{t}, s)g(s, x_n(s)) \, ds \rightarrow \int_{\Omega} k(\bar{t}, s)g(s, x_0(s)) \, ds.$$

Hence

$$f\left(\bar{t}, r \int_{\Omega} k(\bar{t}, s)g(s, x_n(s)) \, ds\right) \rightarrow f\left(\bar{t}, r \int_{\Omega} k(\bar{t}, s)g(s, x_0(s)) \, ds\right),$$

i.e.  $Ax_n(\bar{t}) \rightarrow Ax_0(\bar{t})$ , thanks to  $(h_1)$ . But  $\|Ax_n(t) - Ax_0(t)\|_E \leq 2\varphi_0(t)$  and so  $\|Ax_n - Ax_0\|_{L^1(\Omega, E)} \rightarrow 0$ . It remains to show the most difficult (iv). It is clear that we can assume  $Q$  countable; so we do it. First of all, we observe that thanks to Theorem 2, given  $\sigma > 0$  there is  $\Omega_{\sigma} \subset \Omega$ , closed, with  $m(\Omega \setminus \Omega_{\sigma}) < \sigma$ , such that  $f|_{\Omega_{\sigma} \times E}, k|_{\Omega_{\sigma} \times \Omega}$  are continuous. First we shall prove that

$$(j) \quad B = \{Hx|_{\Omega_{\sigma}} : x \in Q\} \subset C^0(\Omega_{\sigma}), \text{ where } (Hx)(t) = \int_{\Omega} k(t, s)g(s, x(s)) \, ds, \\ t \in \Omega,$$

(jj)  $B$  is relatively compact in  $C^0(\Omega_{\sigma})$ .

Let  $t', t'' \in \Omega$ . We have

$$\begin{aligned} \|(Hx)(t') - (Hx)(t'')\|_E &\leq \int_{\Omega} \|k(t', s) - k(t'', s)\|_{C(F,E)} \|g(s, x(s))\| ds \\ &\leq \int_{\Omega} \|k(t', s) - k(t'', s)\|_{C(F,E)} (h_2(s) + b_2\varphi_0(s)) ds. \end{aligned}$$

Since  $k|_{\Omega_{\sigma} \times \Omega}$  is uniformly continuous, we get that  $B$  is an equicontinuous subset of  $C^0(\Omega_{\sigma})$ ; it is also clear that  $B$  is an equibounded subset of  $C^0(\Omega_{\sigma})$ . It remains to show that for all  $t \in \Omega_{\sigma}$ ,  $B(t) = \{Ax|_{\Omega_{\sigma}}(t) : x \in Q\}$  is relatively compact, so that we can apply Theorem 1 to  $B$ . Now we use Theorem 3. Let  $(x_n^*)$  be a  $w^*$ -null sequence in  $E^*$ ; for  $\bar{t} \in \Omega_{\sigma}$  we have

$$(2) \quad x_n^* \int_{\Omega} k(\bar{t}, s)g(s, x(s)) ds = \int_{\Omega} x_n^* k(\bar{t}, s)g(s, x(s)) ds, \quad n \in \mathbb{N}.$$

For almost all  $s \in \Omega$ , the set  $\{g(s, x(s)) : x \in Q\}$  is bounded by virtue of  $(h_3)$  and because of the very definition of  $Q$ ; hence  $\{k(\bar{t}, s)g(s, x(s)) : x \in Q\}$  is compact in  $E$  for a.a.  $s \in \Omega$  and so  $x_n^* k(\bar{t}, s)g(s, x(s)) \rightarrow 0$  uniformly on  $x \in Q$ ; furthermore,  $|x_n^* k(\bar{t}, s)g(s, x(s))| \leq \sup_n \|x_n^*\| \|k(\bar{t}, s)\|_{C(F,E)} (h_2(s) + b_2\varphi_0(s))$ , which implies

$$\sup_{x \in Q} \int_{\Omega} x_n^* k(\bar{t}, s)g(s, x(s)) ds \rightarrow 0.$$

Thanks to (2) we are done:  $B(t)$  is relatively compact for all  $t \in \Omega$ . Hence  $B$  is relatively compact in  $C^0(\Omega_{\sigma})$ . Once we have (j) and (jj) for any  $\sigma > 0$ , we can conclude our proof as follows. Given a sequence  $(x_h) \subset Q$ , it is easy to get a sequence  $(\Omega_n)$  of closed subsets of  $\Omega$  with  $m(\Omega \setminus \Omega_n) \rightarrow 0$  and a subsequence  $(y_h)$  of  $(x_h)$  such that  $(Hy_h)$  is a Cauchy sequence in any  $C^0(\Omega_n)$ . Again thanks to Theorem 1 we have that

$$C_n = \{Hy_h(t) : t \in \Omega_n, h \in \mathbb{N}\}$$

is a relatively compact subset of  $E$  and so  $f|_{\Omega_n \times \bar{C}_n}$  is uniformly continuous. It is then very easy to see (use again Theorem 1) that

$$\{f(\cdot, Hy_h(\cdot)) : \Omega_n \rightarrow E, h \in \mathbb{N}\}$$

is a Cauchy sequence in  $C^0(\Omega_n)$  for all  $n \in \mathbb{N}$ , by passing to a suitable subsequence if necessary. Hence, if  $\varepsilon > 0$ , let  $\sigma > 0$  be such that

$$\sup_{x \in Q} \int_S \|Ax(s)\| ds < \frac{\varepsilon}{4} \quad \text{for all } S \subset \Omega, m(S) < \sigma.$$

Choose  $\bar{n} \in \mathbb{N}$  with  $m(\Omega \setminus \Omega_{\bar{n}}) < \sigma$ . We have

$$\begin{aligned} \int_{\Omega} \|Ay_{h'}(t) - Ay_{h''}(t)\|_E dt &= \int_{\Omega_{\bar{n}}} \|Ay_{h'}(t) - Ay_{h''}(t)\|_E dt \\ &\quad + \int_{\Omega \setminus \Omega_{\bar{n}}} \|Ay_{h'}(t) - Ay_{h''}(t)\|_E dt \\ &\leq \int_{\Omega_{\bar{n}}} \|f(t, Hy_{h'}(t)) - f(t, Hy_{h''}(t))\|_E dt + \frac{\varepsilon}{2} \\ &\leq m(\Omega_{\bar{n}}) \|f(\cdot, Hy_{h'}(\cdot)) - f(\cdot, Hy_{h''}(\cdot))\|_{C^0(\Omega_{\bar{n}})} + \frac{\varepsilon}{2} \end{aligned}$$

$h', h'' \in \mathbb{N}$ .

Since  $(f(\cdot, Hy_h(\cdot)))_{h \in \mathbb{N}}$  is a Cauchy sequence in  $C^0(\Omega_{\bar{n}})$  we are done.  $\square$

*Remark.* If one of the two spaces  $E$  and  $F$  is finite dimensional, then any continuous and linear operator from  $F$  into  $E$  is compact, but this even happens for suitable infinite dimensional Banach spaces; we refer to [6] for a list of such pairs of Banach spaces.

#### References

- [1] *A. Ambrosetti*: Un teorema di esistenza per le equazioni differenziali negli spazi di Banach. *Rend. Sem. Mat. Padova* 39 (1967), 349–360.
- [2] *J. Banas, Z. Knap*: Integrable solutions of a functional-integral equation. *Revista Matematica de la Univ. Complutense de Madrid* 2(1) (1989), 31–38.
- [3] *K. Deimling*: *Nonlinear Functional Analysis*. Springer Verlag, 1985.
- [4] *G. Emmanuele*: About the existence of integrable solutions of a functional integral equation. *Revista Mat. Univ. Complutense Madrid* 4 (1991), 65–69.
- [5] *I. Gel'fand*: Abstrakte Funktionen und lineare Operatoren. *Mat. Sbornik* 4 (1938), 235–284.
- [6] *J. Johnson*: Remarks on Banach spaces of compact operators. *J. Funct. Analysis* 32 (1979), 304–311.
- [7] *M. A. Krasnosel'skii*: *Topological Methods in the Theory of Nonlinear Integral Equations*. Pergamon Press, 1964.
- [8] *R. H. Martin*: *Nonlinear operators and differential equations in Banach spaces*. Wiley e Sons, 1976.
- [9] *B. Ricceri, A. Villani*: Separability and Scorza Dragoni's Property. *Le Matematiche* 37 (1982), 156–161.
- [10] *P. P. Zabreiko, A. I. Koshelev, M. A. Krasnoselskii, S. G. Miklin, L. S. Rakovshchik, V. J. Stecenko*: *Integral Equations*. Noordhoff, Leyden, 1975.

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