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ON EXTENDED CYCLIC ORDERS

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The notion of cyclically ordered set will be applied in the same sense as in the papers [5] and [6].

Let G be a nonempty set and let C be a cyclic order on G. We define a ternary relation C_0 on G by putting, for any $x, y, z \in G$,

$$(x, y, z) \in C_0 \text{ iff either } (x, y, z) \in C \text{ or } x = y = z.$$

The relation C_0 will be said to be an extended cyclic order (corresponding to the cyclic order C).

It is clear that C and C_0 are uniquely determined by each other. Hence every result on C_0 can be considered in a certain sense as a result on C.

The pair (G, C_0) will be said to be an ec-set. If, moreover, G is a group such that the group operation is compatible with the relation C_0, then $(G, +, C_0)$ will be called an ec-group.

The present paper deals with subdirect product decompositions of ec-sets and direct product decompositions of ec-groups.

1. PRELIMINARIES

For the sake of completeness we recall here the basic definitions on cyclic orders.

A ternary relation C on a set $G \neq \emptyset$ is called a cyclic order whenever the following conditions are satisfied:

(I) If $(x, y, z) \in C$, then $(z, y, x) \bar{\in} C$.
(II) If $(x, y, z) \in C$, then $(z, x, y) \in C$.
(III) If $(x, y, z) \in G$ and $(x, z, u) \in C$, then $(x, y, u) \in C$.

Under the above conditions, the pair $\mathbf{G} = (G, C)$ is said to be a cyclically ordered set. $\mathbf{G}$ is called a cycle if, moreover, for each $(x, y, z) \in G^3$ such that the elements x, y and z are distinct we have either $(x, y, z) \in C$ or $(z, y, x) \in C$.

We denote by $\mathscr{C}$ the class of all cyclically ordered sets. If $\mathbf{G} \in \mathscr{C}$, then (I) and (II) imply that whenever $(x, y, z) \in C$, then $\operatorname{card}\{x, y, z\} = 3$.

For $\mathbf{G} = (G, C) \in \mathscr{C}$ let C_0 be as above. The pair $\mathbf{G}_0 = (G, C_0)$ will be said to be an ec-set. The class of all ec-sets will be denoted by $\mathscr{C}_0$. Next, we denote by $\mathscr{C}^1$ the class of all cycles; let $\mathscr{C}_0^1$ be the class of all $(G, C_0) \in \mathscr{C}_0$ such that $(G, C) \in \mathscr{C}^1$.

Isomorphisms between cyclically ordered sets (or ec-sets) are defined in an obvious way. If two cyclically ordered sets $\mathbf{G}$ and $\mathbf{H}$ are isomorphic, then we express this fact by writing $\mathbf{G} \cong \mathbf{H}$; a similar notation will be applied for elements of $\mathscr{C}_0$.

Let $\mathbf{G} = (G; C_0) \in \mathscr{C}_0$. An element $a \in G$ will be said to be isolated (in $\mathbf{G}$) if there are no elements b and c in G with $b \neq a \neq c$ such that $(a, b, c) \in C_0$.

Let $(G, \leqslant)$ be a partially ordered set. We define a ternary relation $C_{\leqslant}$ on G as follows. For $x, y, z \in G$ we put $(x, y, z) \in C_{\leqslant}$ iff some of the following condition is valid:

$$x < y < z; \quad y < z < x; \quad z < x < y.$$

It is easy to verify that $(G, C_{\leqslant})$ belongs to $\mathscr{C}$.

Again, let $\mathbf{G} \in \mathscr{C}$ and let G_1 be a nonempty subset of G. Put $C^1 = C \cap G_1^3$. Then $\mathbf{G}_1 = (G_1, C^1)$ belongs to $\mathscr{C}$; it will be called a subsystem of $\mathbf{G}$. Analogously, $\mathbf{G}_{10} = (G_1, C_0^1)$ is said to be a subsystem of $\mathbf{G}_0$.

2. Direct and subdirect products

In this section the notions of direct and subdirect product of ec-sets will be defined and it will be proved that each ec-set (G, C_0) with $\operatorname{card} G > 1$ can be represented as a subdirect product of ec-sets (G_i, C_{i0}) $(i \in I)$ such that for each $i \in I$ either $\operatorname{card} G_i = 2$ or $\operatorname{card} G_i = 3$ is valid.

Assume that I is a nonempty set and that $\mathbf{G}_i = (G_i, C_i) \in \mathscr{C}_0$ for each $i \in I$. Put $G = \prod\limits_{i \in I} G_i$ and let C be a ternary relation on G such that for $x, y, z \in G$ we have $(x, y, z) \in C$ iff $(x(i), y(i), z(i)) \in C_i$ for each $i \in I$. Then $\mathbf{G} = (G, C) \in \mathscr{C}_0$ and we denote $\mathbf{G} = \prod\limits_{i \in I} \mathbf{G}_i$; we also say that $\mathbf{G}$ is the direct product of ec-sets G_i.

Let us apply the above notation and let G^1 be a nonempty subset of G. Then we can construct the corresponding subsystem $\mathbf{G}^1$ of $\mathbf{G}$ as above. For $i \in I$ we put $G^1(i) = \{t \in G_i\colon$ there is $g^1 \in G^1$ with $g^1(i) = t\}$. If $G^1(i) = G_i$ for each $i \in I$, then $\mathbf{G}^1$ will be said to be the subdirect product of e-cyclically ordered sets $\mathbf{G}_i$ $(i \in I)$.

The direct products of cyclically ordered sets and of ordered sets are defined analogously (cf. [1] and [6]). Also, the notion of the subdirect product for these cases can be introduced in the same way as in the case of e-cyclically ordered sets above.

An ec-set $\mathbf{G} = (G, C_0)$ will be said to be elementary if either (i) $\operatorname{card} G = 2$, or (ii) $\operatorname{card} G = 3$ and $C \neq \emptyset$. It is obvious that whenever $\mathbf{G}_i = (G_i, C_{0i})$ $(i = 1, 2)$ are elementary ec-sets such that $\operatorname{card} G_1 = \operatorname{card} G_2$, then $\mathbf{G}_1$ and $\mathbf{G}_2$ are isomorphic. Hence there are, up to isomorphism, only two elementary ec-sets.

We define $\mathbf{A}_2$ and $\mathbf{A}_3$ in $\mathscr{C}$ as follows. We put $\mathbf{A}_2 = (A_2, C_2)$, where $A_2 = \{0, 1\}$ and C_2 is the diagonal of A_2^3. Next, let $\mathbf{A}_3 = (A_3, C_3)$, where $A_3 = \{0, 1, 2\}$ and $C_3 = (C_{\leqslant})_0$, where $\leqslant$ is the natural linear order on A_3.

2.1. Theorem. *Let $\mathbf{G} = (G, C_0) \in \mathscr{C}_0$, $\operatorname{card} G \geqslant 2$. Then $\mathbf{G}$ is isomorphic to a subdirect product of ec-sets $\mathbf{G}_i$ $(i \in I)$ where I is a nonempty set and for each $i \in I$ either $\mathbf{G}_i = \mathbf{A}_2$ or $\mathbf{G}_i = \mathbf{A}_3$.*

Proof. Let I_1 be a set having the property that there exists a one-to-one mapping φ_1 of C onto I_1. Next, let G^0 be the set of all isolated elements of G and let φ_2 be a one-to-one mapping of G^0 onto a set I_2, where $I_1 \cap I_2 = \emptyset$. Put $I = I_1 \cup I_2$. Let us remark that I_1 can be empty, and similarly for I_2.

We set $\mathbf{G}_{i(1)} = \mathbf{A}_3$ and $\mathbf{G}_{i(2)} = \mathbf{A}_2$ for each $i(1) \in I_1$ and each $i(2) \in I_2$. Now we construct the direct product $\prod\limits_{i \in I} \mathbf{G}_i$ which will be denoted by $\mathbf{H} = (H, C')$.

Let us define a mapping $f\colon G \to H$ as follows. For $a \in G$ and $i \in I$ we have to define $f(a)(i)$.

First let $i \in I_1$. There are distinct elements x, y and z in G such that $(x, y, z) \in C$ and $\varphi_1^{-1}(i) = (x, y, z)$. We put

$$f(a)(i) = 0 \quad \text{if either} \quad x = a \quad \text{or} \quad a \overline{\in} \{x, y, z\},$$

$$f(a)(i) = 1 \quad \text{if} \quad y = a,$$

$$f(a)(i) = 2 \quad \text{if} \quad z = a.$$

Next, let $i \in I_2$. There is $x \in G^0$ with $\varphi_2^{-1}(i) = x$. We put $f(a)(i) = 1$ if $x = a$, and $f(a)(i) = 0$ otherwise.

Put $f(G) = H'$ and let C'' be the extended cyclic order on H' which is inherited from C'. Let a and a' be distinct elements of G. If a is isolated, then for $i = \varphi_2(a)$ we have $f(a)(i) \neq f(a')(i)$, thus $f(a) \neq f(a')$. Next, assume that a fails to be isolated. Then there are $b, c \in G$ with $b \neq a \neq c$ such that $(b, a, c) \in C$. Denote $\varphi_1((b, a, c)) = i$. Thus $f(a)(i) = 1$ and $f(a')(i) \neq 1$. Therefore f is injective.

Let $a, b, c \in G$ and assume that $(a, b, c) \in C_0$. If $a = b = c$, then we have clearly $(f(a), f(b), f(c)) \in C''$. Suppose that a, b and c are distinct. Hence $(a, b, c) \in C$ and there is $i \in I_1$ with $i = \varphi_1(a, b, c)$). Thus $(f(a)(i), f(b)(i), f(c)(i) \in C_3$. Moreover, for each $j \in I$ with $j \neq i$ the relation $f(a)(j) = f(b)(j) = f(c)(j) = 0$ is valid. Therefore we have again $(f(a), f(b), f(c)) \in C''$.

Now let us assume that a, b and c are elements of G such that $(a, b, c) \bar{\in} C_0$. Hence at least two of the elements a, b and c are distinct. If a is isolated and $\varphi_2(a) = i$, then $(f(a)(i), f(b)(i), f(c)(i)) \bar{\in} C_2$, whence $(f(a), f(b), f(c)) \bar{\in} C''$. Suppose that the element a is not isolated. Thus there are b' and c' in G with $b' \neq a \neq c'$ such that $(a, b', c') \in C$. Put $i = \varphi_1(a, b', c')$. Hence $f(a)(i) = 0$. If $b' = b$, then $c' \neq c$ and thus $f(c)(i) \neq 2$ implying that $(f(a)(i), f(b)(i), f(c)(i)) \bar{\in} C_3$. If $b' \neq b$, then $f(b)(i) \neq 1$ and hence in this case we also have $(f(a)(i), f(b)(i), f(c)(i)) \bar{\in} C_3$. Therefore $(f(a), f(b), f(c)) \bar{\in} C''$.

Thus we have proved that f is an isomorphism of $\mathbf{G}$ onto (H', C''). It remains to verify that (H', C'') is a subdirect product of ec-sets $\mathbf{G}_i$ $(i \in I)$.

Let $i \in I_1$ and $t \in \{0, 1, 2\}$. There is $(a, b, c) \in C$ such that $\varphi_1((a, b, c)) = i$. Hence there is $x \in \{a, b, c\}$ such that $f(x)(i) = t$.

Next, let $i \in I_2$ and $t \in \{0, 1\}$. Hence there is $a \in G^0$ such that $\varphi_2(a) = i$. Then $f(a)(i) = 1$. Since $\operatorname{card} G > 1$ there is $a' \in G$ with $a' \neq a$. Hence $f(a')(i) = 0$.

Summarizing, we conclude that (H', C'') is a subdirect product of the system $\{\mathbf{G}_i\}_{i \in I}$; the proof is complete. □

2.2. Corollary. *Let $\mathbf{G} = (G, C_0) \in \mathscr{C}_0$, $\operatorname{card} G \geqslant 2$. Assume that $\mathbf{G}$ has no isolated element. Then $\mathbf{G}$ is isomorphic to a subdirect product of ec-sets $\mathbf{G}_i$ $(i \in I)$ where I is a nonempty set and $\mathbf{G}_i = \mathbf{A}_3$ for each $i \in I$.*

2.3. Remark. The above result 2.1 can be considered to be a representation theorem for ec-sets $\mathbf{G} = (G, C_0)$ with $\operatorname{card} G \geqslant 2$ (i.e., it gives an embedding of $\mathbf{G}$ into a direct product of "standard" ec-sets $\mathbf{G}_i$; the "standardness" of $\mathbf{G}_i$ means that all $\mathbf{G}_i$ are elementary ec-sets). A representation theorem for cyclically ordered sets was proved in [6]; in the corresponding theorem of [6] all direct factors under consideration are isomorphic, but a subdirect product representation is obtained.

2.4. Remark. If I is as in 2.1 and if we put $\mathbf{H}_i = \mathbf{A}_2 \times \mathbf{A}_3$ for each $i \in I$, then by an obvious modification of the proof of 2.1 we obtain an embedding f' of $\mathbf{G}$ into the direct product $\prod_{i \in I} \mathbf{H}_i$; but in such a case $f'(G)$ fails to be a subdirect product of the system $\{\mathbf{H}_i\}_{i \in I}$.

By a direct product decomposition of a cyclically ordered set $\mathbf{G}$ we understand a triple $(\mathbf{G}, \prod_{i \in I} \mathbf{G}_i, f)$, where all $\mathbf{G}_i$ are cyclically ordered sets and f is an isomorphism of $\mathbf{G}$ onto $\prod_{i \in I} \mathbf{G}_i$.

In an analogous manner we can define direct product decompositions of ec-sets and of partially ordered sets.

The natural question arises whether the relations between different types of direct product decompositions are "good".

For example: let $\mathbf{G} = (G, C) \in \mathscr{C}$ and let $(\mathbf{G}, \prod_{i \in I} G_i, f)$ be a direct product decomposition of $\mathbf{G}$. Put $\mathbf{G}_0 = (G, C_0)$; we can ask whether $(\mathbf{G}_0, \prod_{i \in I} \mathbf{G}_{i0}, f)$ must be a direct decomposition of $\mathbf{G}_0$ (where $\mathbf{G}_i = (G_i, C_i)$ and $\mathbf{G}_{i0} = (G_i, C_{i0})$).

The answers to this question and to some other analogous questions are negative in general (cf. 2.5–2.7; the proofs are routine and so they will be omitted).

Let us remark that there exist positive results for an analogous situation in the theory of directed groups (cf. [3]).

In 2.5 and 2.6 we apply the above introduced notation.

2.5. Proposition. *Let $\mathbf{G} \in \mathscr{C}$ and let $(\mathbf{G}, \prod_{i \in I} \mathbf{G}_i, f)$ be a direct product decomposition of $\mathbf{G}$. Assume that there is $a \in G$ such that a fails to be isolated. Then $(\mathbf{G}_0, \prod_{i \in I} \mathbf{G}_{i0}, f)$ is not a direct decomposition of $\mathbf{G}_0$.*

2.6. Proposition. *Let $\mathbf{G} \in \mathscr{C}$ and let $(\mathbf{G}_0, \prod_{i \in I} \mathbf{G}_{i0}, f)$ be a direct product decomposition of $\mathbf{G}_0$. Assume that there is $a \in G$ such that a fails to be isolated. Then $(\mathbf{G}, \prod_{i \in I} \mathbf{G}_i, f)$ is not a direct product decomposition of $\mathbf{G}$.*

2.7. Proposition. *Let $(G, \leqslant)$ be a partially ordered set, $C = C_{\leqslant}$. Let $((G, \leqslant), \prod_{i \in I} (G_i, \leqslant), f)$ be a direct product decomposition of $(\mathbf{G}, \leqslant)$. Assume that there are $i(1), i(2) \in I$, $i(1) \neq i(2)$ and elements $a_1, a_2 \in G_{i(1)}$, $b_1, b_2 \in G_{i(2)}$ such that $a_1 < a_2$ and $b_1 < b_2$. Let C_i be the cyclic order defined by means of the relation $\leqslant$ on G_i. Then $((G, C), \prod_{i \in I} (G_i, C_i), f)$ is not a direct product decomposition of (G, C); similarly, $((G, C_0), \prod_{i \in I} (G_i, C_{i0}), f)$ is not a direct product decomposition of (G, C_0).*

3. Groups endowed with an extended cyclic order

In the present section we will investigate direct product decompositions of ec-groups. A sufficient condition for cancellability of direct factors will be found. This result will be applied in the next section for studying a particular type of direct product decompositions.

Cyclically ordered groups $(G,+,C)$ such that (G,C) is a cycle were investigated by several authors; cf., e.g., the citations in [4]; the more general case where (G,C) is any cyclically ordered set was dealt with in [8], [9] and [10].

We will apply the folloving definition.

Let $(G,+)$ be a group and let (G,C) be a cyclically ordered set such that whenever $(a,b,c) \in C$ and $x,y \in G$, then

$$(x+a+y, x+b+y, x+c+y) \in C \quad \text{and} \quad (-c,-b,-a) \in C.$$

Under these assumption $(G,+,C)$ is said to be a cyclically ordered group. The class of all cyclically ordered groups will be denoted by $\mathscr{G}_c$.

Next, we denote by $\mathscr{G}_c^0$ the class of all structures $(G,+,C_0)$, where $(G,+,C) \in \mathscr{G}_c$. The elements of $\mathscr{G}_c^0$ will be called ec-groups. If $(G,+,C_0) \in \mathscr{G}_c^0, \operatorname{card} G > 2$ and $(G,C_0) \in \mathscr{C}_0^1$ (cf. Section 1), then $(G,+,C_0)$ is said to be an ℓc-group.

Let I be a nonempty set and for each $i \in I$ let $\mathbf{G}_i = (G_i,+,C_{i0}) \in \mathscr{G}_c^0$ and $\mathbf{G} \in \mathscr{G}_c^0$. The direct product $\prod\limits_{i\in I} \mathbf{G}_i$ is defined in an obvious way. The meaning of the notation $((\mathbf{G}, \prod\limits_{i\in I} \mathbf{G}_i, f)$ is analogous to that introduced in Section 2.

Under the above notation let $i(1) \in I$. Put

$$H_{i(1)} = \{x \in G \colon f(x)(i) = 0 \quad \text{for each} \quad i \in I \setminus \{i(1)\}\}.$$

The corresponding ec-group (with the extended cyclic order and the group operation inherited from $\mathbf{G}$) will be denoted by $\mathbf{H}_{i(1)}$. We will call $\mathbf{H}_{i(1)}$ a direct factor of $\mathbf{G}$.

Let $F(\mathbf{G})$ be the system of all direct factors of $\mathbf{G}$; this system is partially ordered by inclusion. Then $\mathbf{G}$ and $\mathbf{O} = \{\{0\},+,(0,0,0)\}$ are the greatest and the least elements of $F(\mathbf{G})$, respectively.

In an analogous way we can define the system $S(G)$ of direct factors of a directed group G. It is well-known that $S(G)$ is a Boolean algebra.

Returning to $F(\mathbf{G})$ let us remark that the question whether $F(\mathbf{G})$ is a lattice remains open. Some results concerning $F(\mathbf{G})$ will be proved below.

Let $\mathbf{G} = (G,+,C_0) \in \mathscr{G}_c^0$ and let H be a subgroup of G. The ec-group $\mathbf{H}$ is defined by the inherited extended cyclic order; this will be denoted by $C_0(H)$. (Analogous notation are applied below.)

3.1. Lemma. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be as above. Then the following conditions are equivalent:*

(i) $\mathbf{H} \in F(\mathbf{G})$.

(ii) *There exists a subgroup H' of G such that the group G is a direct product of H and H' and for each triple $(x,y,z) \in G^3$ the relation $(x,y,z) \in C_0$ is valid iff $(x(H),y(H),z(H)) \in C_0(H)$ and $(x(H'),y(H'),z(H')) \in C_0(H')$ (where $x(H')$ is the component of x in H or in H' with respect to the direct product decomposition $G = H \times H'$, and similarly for y and z).*

Proof. This is an immediate consequence of the definition of $F(\mathbf{G})$. □

If the condition (ii) from 3.1 is satisfied then we write $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$.

More generally, let $A_1, A_2, \ldots, A_n$ be subgroups of G. The corresponding ec-groups will be denoted by $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$. We will write $\mathbf{G} = \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$ if

(i) the group G is a direct product of its subgroups $A_1, A_2, \ldots, A_n$;

(ii) if $g^i \in G, g^i = a_1^i + a_2^i + \ldots + a_n^i$ with $i = 1,2,3$, and $a_j^i \in A_j^i$ for $j = 1,2,\ldots,n$ and $i = 1,2,3$, then the relation $(g^1,g^2,g^3) \in C$ is valid iff $(a_j^1,a_j^2,a_j^3) \in C(A_j)$ holds for each $j \in \{1,2,\ldots,n\}$.

It is clear that each $\mathbf{A}_j$ belongs to $F(\mathbf{G})$. The above definition yields

3.2. Lemma. *Let $\mathbf{G} = \mathbf{A} \times \mathbf{B}$, $\mathbf{A} = \mathbf{C} \times \mathbf{D}$. Then $\mathbf{G} = \mathbf{C} \times \mathbf{D} \times \mathbf{B}$.*

3.3. Lemma. *Let $\mathbf{H}$ and $\mathbf{H}_1$ be elements of $F(\mathbf{G})$ such that $H_1 \subseteq H$. Let $\mathbf{H}'$ and $\mathbf{H}_1'$ be defined analogously as in 3.1. Then $\mathbf{H} = \mathbf{H}_1 \times \mathbf{H}_2$, where $H_2 = H \cap H_1'$.*

Proof. The validity of the relation $H = H_1 \times H_2$ in the group theoretical sense is obvious. The remaining part of the proof concerning the extended cyclic order on H is a consequence of the relation $\mathbf{G} = \mathbf{H}_1 \times \mathbf{H}_1'$. □

3.4. Corollary. *Let $\mathbf{H}, \mathbf{K}$ and $\mathbf{H}_1$ be elements of $F(\mathbf{G})$ such that $H_1 \subseteq H, H_1 \subseteq K$ and $H \subset K$. Then $H \cap H_1' \subset K \cap H_1'$.*

3.5. Lemma. *Let $\mathbf{H}, \mathbf{K}$ and $\mathbf{H}_1$ be elements of $F(\mathbf{G})$ such that $H_1 \subseteq H, H_1 \subseteq K$ and $H \not\subseteq K$. Then $H \cap H_1' \not\subseteq K \cap H_1'$ (where H_1' is as in 3.3).*

Proof. By way of contradiction; if $H \cap H_1' \subseteq K \cap H_1'$, then in view of 3.3 we would have $H \subseteq K$. □

Now, 3.1, 3.4 and 3.5 yield

3.6. Proposition. *Let $\mathbf{G} = \mathbf{H} \times \mathbf{H}' \in \mathcal{G}_c^0$. Put $L = \{\mathbf{X} \in F(\mathbf{G})\colon H \subseteq X\}$ and $\varphi(\mathbf{X}) = \mathbf{Y}$, where $Y = H' \cap X$ for each $\mathbf{X} \in L$. Let L be partially ordered by inclusion. Then φ is an isomorphism of L onto $F(\mathbf{H}')$.*

Next, from 3.4 we infer

3.7. Lemma. *Let* $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$. *Put* $L_1 = \{X \in F(\mathbf{G}) : X \subseteq H\}$. *Then* $L_1 = F(\mathbf{H})$.

3.8. Corollary. *Let* $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$. *Assume that* $F(\mathbf{G})$ *is a lattice. Then* $F(\mathbf{H})$ *is a lattice as well.*

An ec-group $\mathbf{G} = (G, C)$ will be said to be a dc-group if it satisfies the following condition:

(i) Whenever a and b are distinct elements of G, then there exists $c \in G$ such that either $(a, b, c) \in C$ or $(b, a, c) \in C$.

The condition (i) is obviously equivalent to the condition

(ii) Whenever $a \in G$ and $a \neq 0$, then there exists $b \in G$ such that either $(0, a, b) \in C$ or $(a, 0, b) \in C$.

3.9. Theorem. *Let* $\mathbf{G}$ *be an ec-group,* $\mathbf{G} = \mathbf{A} \times \mathbf{B}$ *and* $\mathbf{G} = \mathbf{A} \times \mathbf{D}$. *Assume that* $\mathbf{D}$ *is a dc-group. Then* $\mathbf{B} = \mathbf{D}$.

Proof. Since the extended cyclic orders on B and D are inherited from the cyclic order on G it suffices to verify that $B = D$. By way of contradiction, assume that $B \neq D$.

First, suppose that $D \subset B$. Hence in view of 3.3 there exists a direct decomposition $\mathbf{B} = \mathbf{D} \times \mathbf{B}_1$ with $B_1 \neq \{0\}$. Thus according to 3.2 we have

(1) $\mathbf{G} = \mathbf{A} \times \mathbf{D} \times \mathbf{B}_1$.

There exists $b_1 \in B_1$ with $b_1 \neq 0$. The relation (1) yields that b_1 does not belong to $A + D$. But from $\mathbf{G} = \mathbf{A} \times \mathbf{D}$ we infer that b_1 is an element of $A + D$, which is a contradiction.

Next, suppose that D fails to be a subset of B. Hence there is $g \in D \setminus B$. Since D is a dc-group there exists $h \in D$ such that either $(0, g, h) \in C$ or $(0, h, g) \in C$.

Let $(0, g, h) \in C$ (the case $(0, h, g) \in C$ is analogous). Then $g \neq 0 \neq h$ and thus $g \,\overline{\in}\, A$, $h \,\overline{\in}\, A$. Next, the relation $h \in B$ would imply that $g \in B$; therefore h does not belong to B.

There are uniquely determined elements $g_1, h_1 \in A$ and $g_2, h_2 \in B$ such that $g = g_1 + g_2$ and $h = h_1 + h_2$. From the above mentioned relation we infer that the elements $0, g_1, h_1$ are distinct; similarly, the elements 0, g_2, h_2 are distinct. Hence

(2) $(0, g_1, h_1) \in C$ and

(3) $(0, g_2, h_2) \in C$.

Next, from the relations

$$g_2 = -g_1 + g, \qquad h_2 = -h_1 + h$$

and from $\mathbf{G} = \mathbf{A} \times \mathbf{D}$ we obtain (by applying (3)) that $(0, -g_1, -h_1)$ holds, which contradicts (2). □

Let $(G, \leqslant, +)$ be a partially ordered group. Put $C = C_{\leqslant}$. Then $(G, C_0, +)$ is an ec-group; it will be said to be generated by the partially ordered group $(G, \leqslant, +)$.

An ec-group $\mathbf{G}$ is said to be directly indecomposable if, whenever $G = \mathbf{G}_1 \times \mathbf{G}_2$, then either $\operatorname{card} G_1 = 1$ or $\operatorname{card} G_2 = 1$.

3.10. Theorem. *Let $\mathbf{G}$ be an ec-group which is generated by a nonzero directed group $(G, \leqslant, +)$. Then $\mathbf{G}$ is directly indecomposable.*

Proof. By way of contradiction, let us suppose that $\mathbf{G} = \mathbf{A} \times \mathbf{B}$, $\operatorname{card} A > 1$, $\operatorname{card} B > 1$.

First suppose that there exists $a \in A$ such that the element a is isolated in $\mathbf{A}$. Then all elements are isolated in $\mathbf{A}$. Since $\operatorname{card} A > 1$, there is $a_1 \in A$ with $a_1 \neq 0$. Because G is directed, there are elements x and y in G such that $0 < x < y$ and $a_1 < x < y$. Hence we have

(1) $(0, x, y) \in C$,

(2) $(a_1, x, y) \in C$.

There are uniquely determined elements $a_3, a_4 \in A$ and $b_1, b_2 \in B$ such that $x = a_3 + b_1$ and $y = a_4 + b_2$.

If $a_3 \neq a_1$, then (2) implies that (a_1, a_3, a_4) belongs to C, which is a contradiction. Therefore $a_3 = a_1$. Hence according to (1) the triple $(0, a_1, a_4)$ belongs to C, which is impossible. Therefore there is no $a \in A$ which is isolated in $\mathbf{A}$.

Hence there are a_1, a_2 and a_3 in A such that $(a_1, a_2, a_3) \in C$. By a routine calculation we obtain that there are a_1' and a_2' in A with $0 < a_1' < a_2'$. Similarly, there are b_1' and b_2' in B such that $0 < b_1' < b_2'$.

Hence

(3) $0 < a_1' < a_1' + b_1'$.

Thus $(0, a_1', a_1' + b_1') \in C$. From this and from the relation $\mathbf{G} = \mathbf{A} \times \mathbf{B}$ we infer that $(0, a_1', a_1') \in C$, which is impossible. □

4. Direct products of ℓc-groups

In this section it will be proved that if an ec-group $\mathbf{G}$ possesses a direct product decomposition such that all direct factors in this decomposition are ℓc-groups, then the partially ordered set $F(\mathbf{G})$ is an atomic Boolean algebra.

In what follows we assume that $\mathbf{G}$ is an ec-group.

4.1. Lemma. *Each ℓc-group is directly indecomposable.*

Proof. Let $\mathbf{G}$ be an ℓc-group and suppose that $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$, where $\operatorname{card} G_1 > 1$ and $\operatorname{card} G_2 > 1$. Thus there are elements $g_1 \in G_1$ and $g_2 \in G_2$ such that

$g_1 \neq 0 \neq g_2$. Obviously $g_1 \neq g_2$. Then neither $(0, g_2, g_1) \in C$ nor $(0, g_2, g_1)$ is valid, which is a contradiction. □

4.2. Lemma. *Let* $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$ *and let* $\mathbf{D}$ *be a direct factor of* $\mathbf{G}$ *such that* $\mathbf{D}$ *is an* ℓc*-group. Then either* $H \cap D = \{0\}$ *or* $D \subseteq H$.

Proof. Assume that $H \cap D \neq \{0\}$. Hence there is $d_1 \in D \cap H$ with $d_1 \neq 0$. Let $d_2 \in D, d_2 \neq d_1, d_2 \neq 0$. Then either $(0, d_1, d_2) \in C$ or $(0, d_2, d_1) \in C$. Therefore in view of $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$ we obtain that $d_2 \in \mathbf{H}$ and thus $D \subseteq H$. □

4.3. Lemma. *Let* $\mathbf{G}, \mathbf{H}, \mathbf{H}'$ *and* $\mathbf{D}$ *be as in 4.2. Then either* $D \subseteq H$ *or* $D \subseteq H'$.

Proof. By way of contradiction, assume that neither $D \subseteq H$ nor $D \subseteq H'$ is valid. Then in view of 4.2

(1) $$D \cap H = \{0\}$$

and

(2) $$D \cap H' = \{0\}.$$

There exist d_1 and d_2 in D such that

(3) $$(0, d_1, d_2) \in C.$$

Next, there are uniquely determined elements $a_1, a_2 \in H$ and $a'_1, a'_2 \in H'$ with

$$d_i = a_i + a'_i \quad (i = 1, 2).$$

In view of (3) we have $d_1 \neq d_2$ and hence by applying (1) we get $a_1 \neq a_2$. Also, $d_i \neq 0$ for $i = 1, 2$ and hence $a_1 \neq 0 \neq a_2$. This yields that

(4a) $$(0, a_1, a_2) \in C.$$

Analogously we obtain that $a'_1 \neq a'_2, a'_1 \neq 0 \neq a'_2$ and

(4b) $$(0, a'_1, a'_2) \in C.$$

There is a subgroup $\mathbf{D}'$ of G such that $\mathbf{G} = \mathbf{D} \times \mathbf{D}'$. Hence for each $t \in G$ there are uniquely determined elements $t(D) \in D$ and $t(D') \in D'$ such that $t = t(D) + t(D')$.

In particular, we have $a_1 = a_1(D) + a_1(D')$, whence $d_1 - a_1' = a_1(D) + a_1(D')$ and thus

$$-a_1(D) + d_1 = a_1(D') + a_1'. \tag{5}$$

From $-a_1(D) + d_1 \in D$ we obtain that $(-a_1(D) + d_1)(D') = 0$.
Thus (5) yields that

$$a_1'(D') = -a_1(D'). \tag{6}$$

Analogously we obtain

$$a_2'(D') = -a_2(D'). \tag{7}$$

If $a_1'(D') = 0$, then according to (6) we have $a_1(D') = 0$, thus $a_1 = a_1(D) \in D$, which is a contradiction (cf. (1)). Therefore

$$a_1'(D') \neq 0. \tag{8}$$

From (4b) and from $\mathbf{G} = \mathbf{D} \times \mathbf{D}'$ we infer that

$$(0, a_1'(D'), a_2'(D')) \in C_0 \tag{9}$$

is valid. Now, (8) and (9) yield that

$$(0, a_1'(D'), a_2'(D')) \in C. \tag{10}$$

In view of (6), (7) and (10) we have

$$(0, -a_1(D'), -a_2(D')) \in C. \tag{11}$$

In particular, the elements $0, a_1(D')$ and $a_2(D')$ are distinct. Thus according to (4a)

$$(0, a_1(D'), a_2(D')) \in C, \tag{12}$$

which contradicts (11). □

4.4. Lemma. *Let $\mathbf{G} = \prod_{i\in I} \mathbf{A}_i$ and assume that all A_i are ℓc-groups. Suppose that $\mathbf{D}$ is a direct factor of $\mathbf{G}$ and that $\mathbf{D}$ is an ℓc-group. Then there is $i(0) \in I$ such that $\mathbf{D} = \mathbf{A}_{i(0)}$.*

Proof. By way of contradiction, assume that $\mathbf{D} \neq \mathbf{A}_{i(0)}$ for each $i(0) \in I$. Thus $D \neq A_{i(0)}$ for each $i(0) \in I$. Let $i \in I$. If $D \cap A_i \neq \{0\}$, then from 4.3 we infer that $D \subseteq A_i$ and, at the same time, $A_i \subseteq D$. Therefore $D = A_i$, which is a contradiction. Hence $D \cap \mathbf{A}_i = \{0\}$ for each $i \in I$. Put $\mathbf{G}_i' = \prod_{j\in I\setminus\{i\}} \mathbf{A}_j$ for each $i \in I$. According to 4.3 the relation $D \subseteq G_i'$ is valid for each $i \in I$. But $\bigcap_{i\in I} G_i' = \{0\}$ and thus $D = \{0\}$, which is a contradiction. □

4.5. Theorem. *Let* $\mathbf{G}$ *be an ec-group. If* $\mathbf{G}$ *can be represented as a direct product of* ℓc*-groups, then this representation is unique.*

Proof. This is a corollary of 4.4. □

4.6. Lemma. *Let* $\mathbf{G} = \mathbf{A} \times \mathbf{B}$ *and let* $\mathbf{D}$ *be a direct factor of* $\mathbf{G}$ *such that* $\mathbf{D}$ *is an* ℓc*-group,* $\mathbf{G} = \mathbf{D} \times \mathbf{D}'$*. Assume that there is* $0 \neq a \in A$ *with* $a = d + d', d \in D, d' \in D', d \neq 0$*. Then* $D \subseteq A$*.*

Proof. By way of contradiction, suppose that D fails to be a subset of A. Then in view of 4.3 we have $D \subseteq B$. Next, according to 3.3 there exists a direct decomposition $\mathbf{B} = \mathbf{B}_1 \times \mathbf{D}$. Hence $\mathbf{G} = \mathbf{A} \times \mathbf{B}_1 \times \mathbf{D}$. Now, 3.9 yields that $\mathbf{A} \times \mathbf{B}_1 = \mathbf{D}'$. Thus the component of the element $a \in G$ in $\mathbf{D}$ (with respect to the direct decomposition $\mathbf{G} = \mathbf{D} \times \mathbf{D}'$) is the same as the component of a in $\mathbf{D}$ with respect to the direct decomposition $\mathbf{G} = \mathbf{A} \times \mathbf{B}_1 \times \mathbf{D}$, whence $d = 0$, which is a contradiction. □

4.7. Lemma. *Let* $\mathbf{G} = \prod_{i \in I} \mathbf{G}_i$*, where all* $\mathbf{G}_i$ *are* ℓc*-groups. Let* $\mathbf{A}$ *be a nonzero direct factor of* $\mathbf{G}$*. Then there is a nonzero subset* $I(1)$ *of* I *such that* $\mathbf{A} = \prod_{i \in I(1)} \mathbf{G}_i$*.*

Proof. There exists a direct decomposition $\mathbf{G} = \mathbf{A} \times \mathbf{B}$. Put $I(1) = \{i \in I : G_i \subseteq A\}$ and $I(2) = \{i \in I : G_i \subseteq B\}$. Then $I(1) \cap I(2) = \emptyset$; next, according to 4.3 we have $I(1) \cup I(2) = I$. If $I(1) = \emptyset$, then according to 4.2 the relation $A \cap G_i = \{0\}$ is valid for each $i \in I(1)$. Thus by applying the same notation as in the proof of 4.4 we infer that $A \subseteq G_i'$ for each $i \in I$ and hence $A = \{0\}$, which is a contradiction. Therefore $I(1) \neq \emptyset$. Put

$$\mathbf{P} = \prod_{i \in I(1)} \mathbf{G}_i, \quad \mathbf{Q} = \prod_{i \in I(2)} \mathbf{G}_i.$$

Hence $\mathbf{G} = \mathbf{P} \times \mathbf{Q}$.

Let $a \in A$, $a \neq 0$. Thus there exists $i \in I$ with $a(i) \neq 0$. In view of 4.6, each such i belongs to $I(1)$ and hence $a \in P$. Therefore $A \subseteq P$. Analogously we can verify that $B \subseteq Q$.

Let $p \in P$. There are uniquely determined elements $a_1 \in A$ and $b_1 \in B$ such that $p = a_1 + b_1$. Because of $A \subseteq P$ and $B \subseteq Q$ we infer that $b_1 = 0$ and $a_1 = p$. Hence $P \subseteq A$. Summarizing, we conclude that $\mathbf{A} = \mathbf{P}$. □

4.8. Theorem. *Let* $\mathbf{G}$ *be an ec-group possessing a direct product decomposition* $\mathbf{G} = \prod_{i \in I} \mathbf{G}_i$*, where all* $\mathbf{G}_i$ *are* ℓc*-groups. Then* $F(\mathbf{G})$ *is an atomic Boolean algebra.*

Proof. In view of 4.7, $F(\mathbf{G})$ is a Boolean algebra. Then according to 4.1, $F(\mathbf{G})$ is atomic. □

5.1. Let $\mathbb{R}$ be the additive group of all reals with the natural linear order $\leqslant$. Next, let $C_{\leqslant}$ be the cyclic order on $\mathbb{R}$ defined by means of the linear order $\leqslant$. We denote by G the set of all triples (x, y, z) with $x, y, z \in \mathbb{R}$. The operation $+$ on G is defined componentwise. Let us define a ternary relation C on G as follows. Let $a_i = (x_i, y_i, z_i) \in G$, $i = 1, 2, 3$. We put $(a_1, a_2, a_3) \in C$ if (i) $(x_1, x_2, x_3) \in C_{\leqslant}$, and (ii) $y_1 = y_2 = y_3$, $z_1 = z_2 = z_3$. Then $(G, +, C)$ is a cyclically ordered group, whence $(G, +, C_0)$ is a ec-group. $(G, +, C)$ fails to be a dc-group (e.g., if $a_1 = (0,0,0)$, $a_2 = (0,1,0)$, $a_3 \in G$, then neither $(a_1, a_2, a_3) \in C$ nor $(a_1, a_3, a_2) \in C$ is valid).

5.2. If $(G, \leqslant, +)$ is a linearly ordered group, then $(G, C_{\leqslant}, +)$ is an ℓc-group. It is well-known that there exist ℓc-groups which cannot be constructed in this way (cf. e.g. [2]). Each ℓc-group is a dc-group.

5.3. Let G be the set of all pairs (x, y) with $x, y \in \mathbb{R}$. The operation $+$ on G is defined componentwise. The ternary relation C on G is defined as follows. Let $a_i = (x_i, y_i)$, $i = 1, 2, 3$. We put $(a_1, a_2, a_3) \in C$ if the following conditions are satisfied (we can consider a_i to be points in a plane; the relation $C_{\leqslant}$ has the same meaning as in 5.1):

(i) a_1, a_2 and a_3 are distinct and situated on a line;
(ii) either $(x_1, x_2, x_3) \in C_{\leqslant}$, or $x_1 = x_2 = x_3$ and $(y_1, y_2, y_3) \in C_{\leqslant}$.

Then $(G, +, C_0)$ is a dc-group which fails to be an ℓc-group.

5.4. Let G_0 be the set of all real functions defined on $\mathbb{R}$. Next, let G be the set of all $f \in G_0$ having the property that the set of all points in which f fails to be continuous is finite. The operation $+$ on G is defined componentwise. Let $C_{\leqslant}$ be as in 5.1. Next, let C be the set of all triples $(f_1, f_2, f_3) \in G^3$ such that (i) f_1, f_2 and f_3 are distinct, and (ii) for each $i \in \mathbb{R}$ the relation $(f_1(i), f_2(i), f_3(i)) \in (C_{\leqslant})_0$ is valid. Then $\mathbf{G} = (G, C_0, +)$ is an ec-group. The system $F(\mathbf{G})$ of all direct factors of $\mathbf{G}$ is infinite and has no atom.

5.5. Let $(G, +, C_0) = \mathbf{G}$ be as in 5.1. Put

$$A = \{(x, y, 0) \colon x, y \in \mathbb{R}\}, \quad B = \{(0, 0, z) \colon z \in \mathbb{R}\},$$

$$D = \{(0, y, z) \colon y, z \in \mathbb{R} \quad \text{and} \quad y = z\}.$$

Next, let $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ be the corresponding ec-groups (with the extended cyclic order inherited from $\mathbf{G}$).

Then we have

$$\mathbf{G} = \mathbf{A} \times \mathbf{B} \tag{1}$$

and

$$\mathbf{G} = \mathbf{A} \times \mathbf{D}, \tag{2}$$

but $\mathbf{B} \neq \mathbf{D}$. Hence the cancellation law for direct products does not hold in general. Next, if $g \in G$, then the component of g in A with respect to the direct decomposition (1) need not be equal to the component of g in A with respect to the direct decomposition (2).

Let us consider the partially ordered set $F(\mathbf{G})$. For $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ the notation $\mathbf{X} \wedge \mathbf{Y} = \mathbf{Z}$ will mean that $\mathbf{Z}$ is the greatest lower bound of the system $\{X, Y\}$ (and dually for $\vee$). We have

$$\mathbf{A} \wedge \mathbf{B} = \mathbf{A} \wedge \mathbf{D} = \mathbf{O},$$

$$\mathbf{A} \vee \mathbf{B} = \mathbf{A} \vee \mathbf{B} = \mathbf{G}.$$

Hence $F(\mathbf{G})$ fails to be a distributive lattice.

5.6. Let (G, C_0) be an ec-set and suppose that $(G, +)$ is a group such that

(i) whenever $a, b, c, x, y \in G$ and $(a, b, c) \in C$, then

$$(x + a + y, x + b + y, x + c + y) \in C.$$

If, moreover, (G, C) is a cycle, then from the well-known representation theorem (cf. [7]) we easily obtain that also the following condition is valid:

(ii) whenever $(a, b, c) \in C_0$ then $(-c, -b, -a) \in C_0$.

If we do not assume that (G, C) is a cycle, then the condition (ii) need not hold. Indeed, let $(G, +)$ be as in 5.1. Let us now define a ternary relation C on G as follows. Put $a_1 = (1, 0, 0)$, $a_2 = (0, 1, 0)$, $a_3 = (0, 0, 1)$. For $b_1, b_2, b_3 \in G$ we put $(b_1, b_2, b_3) \in C$ if there exist $z \in G$ and a cyclic permutation $(j(1), j(2), j(3))$ of $(1, 2, 3)$ such that $b_i = a_{j(i)} + z$ is valid for $i = 1, 2, 3$.

Then (G, C) is a cyclically ordered set and for the ec-set (G, C_0) the condition (i) is satisfied. We have $(b_1, b_2, b_3) \in C_0$, but $(-b_3, -b_2, -b_1)$ does not belong to C_0.

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