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OSCILLATORY PROPERTIES OF A DIFFERENTIAL INCLUSION OF ORDER $n > 1$ AND THE ASYMPTOTIC EQUIVALENCE

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We will consider two differential inclusions

$$y^{(n)}(t) \in F\big(t, y(t)\big), \tag{1}$$

$$x^{(n)}(t) \in G\big(t, x(t)\big), \tag{2}$$

where $n > 1$, $t \in J = [t_0, \infty)$, $t_0 \geqslant 0$ and F and G are multifunctions which fulfil the assumptions

(H$_1$) $F(G)\colon J \times \Re \to \{\text{nonempty convex compact subsets of } \Re\}$, $F(G)$ is upper semicontinuous on $J \times \Re$;

(H$_2$) $F(t,x)x < 0$ $\big(G(t,x)x < 0\big)$ for each $(t,x) \in J \times \Re$, $x \neq 0$;

(H$_3$) $F(t,x)x > 0$ $\big(G(t,x)x > 0\big)$ for each $(t,x) \in J \times \Re$, $x \neq 0$;

(H$_4$) $F(t,0) = \{0\}$ $\big(G(t,0) = \{0\}\big)$ for each $t \in J$.

We note that $F(t,x)x > 0$ (< 0) means that for each $z \in F(t,x)$ we have $zx > 0$ (< 0). The same is meant by $G(t,x)x > 0$ (< 0).

Under a solution $y(t) \in (1)$ ($x(t) \in (2)$) we will understand a solution which exists on J and is such that

$$\sup\big\{|y(t)|\colon t \geqslant t_1\big\} > 0 \quad \text{for all } t_1 \geqslant t_0$$
$$\big(\sup\big\{|x(t)|\colon t \geqslant t_1\big\} > 0 \quad \text{for all } t_1 \geqslant t_0\big).$$

The notion of an oscillatory and nonoscillatory solution will be used in the usual sense. It is easy to see that nonoscillatory solutions have the following properties. Let $z(t)$ be a nonoscillatory solution of (1) (or of (2)). It means that $z(t) \neq 0$ on some interval $[t_1, \infty)$, $t_1 \geqslant t_0$. Taking into consideration the assumptions (H_2), (H_4) we get $z(t)z^{(n)}(t) < 0$ on $[t_1, \infty)$ and $z(t)z^{(n)}(t) > 0$ on $[t_1, \infty)$ if (H_3) and (H_4) are satisfied. This implies the existence of such $t_2 \geqslant t_1$ that $z^{(i)}(t)$, $i = 0, 1, \ldots, n$ has a constant sign on $[t_2, \infty)$. Therefore each $z^{(i)}(t)$, $i = 0, 1, \ldots, n-1$, is monotone on $[t_2, \infty)$ and $\lim_{t\to\infty} z^{(i)}(t)$, $i = 0, 1, \ldots, n-1$ exists in the extended sense, i.e. $\lim_{t\to\infty} |z^{(i)}(t)|$ is finite or $+\infty$.

Thus, for nonoscillatory solutions the following two cases are possible:

(a) $\lim_{t\to\infty} |z^{(i)}(t)| = \infty$, $i = 0, 1, 2, \ldots, n-1$;

(b) there exists $k \in \{0, 1, \ldots, n-1\}$ such that

$$\begin{aligned}
&\lim_{t\to\infty} z^{(k)}(t) \text{ is finite,}\\
&\lim_{t\to\infty} z^{(i)}(t) = \infty \operatorname{sgn} z(t) \quad \text{for } i = 0, 1, \ldots, k-1,\\
&\lim_{t\to\infty} z^{(i)}(t) = 0 \quad \text{for } i = k+1, \ldots, n-1.
\end{aligned}$$

We note that the case (a) can occur only if the assumptions (H_3) and (H_4) are satisfied (see [1], Remark 1).

Definition 1. We will say that the inclusion (1) ((2)) has the property A if for n even each solution of (1) (of (2)) is oscillatory and for n odd each solution of (1) (of (2)) is either oscillatory or monotonically tends to zero for $t \to \infty$ with all its derivatives of orders less than n.

Definition 2. We will say that the inclusion (1) ((2)) has the property B if for n even each solution of (1) (of (2)) is either oscillatory or monotonically tends to zero for $t \to \infty$ with all its derivatives of orders less than n or monotonically tends to $+\infty$ or $-\infty$ for $t \to \infty$ with all its derivatives of orders less than n, and for n odd each solution of (1) (of (2)) is either oscillatory or monotonically tends to $+\infty$ or $-\infty$ with all its derivatives of orders less than n.

For conditions with guarantee the validity of the property A (or B) of a differential inclusion see e.g. [1]. We note that the inclusions (1) ((2)) can have the property B only if (H_3) is satisfied.

The aim of this paper is to discuss how the property A (or property B) of a differential inclusion of order $n > 1$ depends on the perturbation of the right hand side of this inclusion. More precisely, we will prove that the perturbations for which the differential inclusion and its perturbed inclusion are asymptotically equivalent maintain the property A (property B). A similar problem for differential equations was discussed in [4].

Definition 3. We will say that the differential inclusions (1) and (2) are asymptotically equivalent if for each solution $y(t) \in (1)$ there exists a solution $x(t) \in (2)$ such that

$$\lim_{t\to\infty} \big(y^{(i)}(t) - x^{(i)}(t)\big) = 0, \qquad i = 0, 1, \dots, n-1$$

and conversely.

In what follows we assume that F and G fulfil the assumptions (H_1), (H_2), (H_4) or (H_1), (H_3), (H_4).

Theorem 1. *Let* (1) *and* (2) *be asymptotically equivalent. Then either both* (1) *and* (2) *possess or both do not possess the property* A.

Proof. Assume that the inclusion (1) has the property A and that (2) has not the property A. It means that (2) has a nonoscillatory solution, denote it $x_1(t)$, such that for some $k \in \{0, 1, \dots, n-1\}$ we have $\lim x_1^{(k)}(t) = c_k$ as $t \to \infty$ and $0 < |c_k| \leqslant \infty$. Let $y_1(t)$ be the solution of (1) which corresponds to $x_1(t)$ in the asymptotic equivalence of (1) and (2). Then $\lim \big(x_1^{(k)}(t) - y_1^{(k)}(t)\big) = 0$ as $t \to \infty$. However,

$$\begin{aligned}\lim_{t\to\infty} y_1^{(k)}(t) &= \lim_{t\to\infty} \big[\big(y_1^{(k)}(t) - x_1^{(k)}(t)\big) + x_1^{(k)}(t)\big] \\ &= \lim_{t\to\infty} \big[y_1^{(k)}(t) - x_1^{(k)}(t)\big] + \lim_{t\to\infty} x_1^{(k)}(t) = c_k \neq 0,\end{aligned}$$

which contradicts the property A of (1).

If we assume that (2) has the property A and (1) has not the property A we get a similar contradiction. □

Theorem 2. *Let* (1) *and* (2) *be asymptotically equivalent. Then either both* (1) *and* (2) *possess or both do not possess the property* B.

Proof. Assume that (1) has the property B and that (2) has not the property B. It means that (2) has a nonoscillatory solution, say $x_2(t)$, such that for some $k \in \{0, 1, \dots, n-1\}$ we have $\lim\limits_{t\to\infty} x_2^{(k)}(t) = c_2$, $0 < |c_2| < \infty$. Let y_2 be the solution of (1) which corresponds to $x_2(t)$ in the asymptotic equivalence between (1) and (2). Then

$$\begin{aligned}c_2 = \lim_{t\to\infty} x_2^{(k)}(t) &= \lim_{t\to\infty} \big[\big(x_2^{(k)}(t) - y_2^{(k)}(t)\big) + y_2^{(k)}(t)\big] \\ &= \lim_{t\to\infty} \big[x_2^{(k)}(t) - y_2^{(k)}(t)\big] + \lim_{t\to\infty} y_2^{(k)}(t) = \lim_{t\to\infty} y_2^{(k)}(t),\end{aligned}$$

which proves that (1) has not the property B. However, this contradicts the assumption. Similarly, the assumptions that (2) has the property B and (1) has not the property B lead to a contradiction. □

Now we will focus our attention on conditions which will guarantee the asymptotic equivalence between the inclusions (1) and (2).

Definition 4. Let $D \subset \Re$. Then $|D| = \sup\{|d| : d \in D\}$.

Lemma 1. *Let $F(t,x)$ satisfy the condition* (H$_1$). *Let $u(t)$ be a continuous function on J. Denote $z(t) = \max\{F(t,u(t)\}$, $t \in J$. Then $z(t) \in F(t,u(t))$, $z(t)$ is upper semicontinuous on J and Lebesgue measurable.*

Proof. Let $t_0 < t_1$. Then the continuity of $u(t)$ and (H$_1$) imply that for each $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|u(t) - u(t_1)| < \delta_2$ for each $t \in (t_1 - \delta_1, t_1 + \delta_1)$, and for each $(t,u(t)) \in (t_1 - \delta_1, t_1 + \delta_1) \times (u(t_1) - \delta_2, u(t_1) + \delta_2)$ we obtain $F(t,u(t)) \subset F(t_1,u(t_1)) + \varepsilon$. Consequently, $z(t) = \max\{F(t,u(t))\} \leqslant \max\{F(t_1,u(t_1))\} + \varepsilon = z(t_1) + \varepsilon$ for each $t \in (t_1 - \delta_1, t_1 + \delta_1)$. It means that $z(t)$ is upper semicontinuous at t_1. A similar argument gives the upper semicontinuity from the right of $z(t)$ at t_0.

Now, $z(t)$ being upper semicontinuous on J, it is Borel measurable on J and consequently, also Lebesgue measurable on J. □

Theorem 3. *Let $F(t,x)$, $G(t,x)$ satisfy* (H$_1$). *Moreover, let the following conditions be satisfied:*

(H$_5$) *There exists a continuous function $V(t,z) \colon J \times [0,\infty) \to [0,\infty)$ nondecreasing in z for each fixed $t \in J$ such that* $|F(t,u_1) - G(t,u_2)| \leqslant V(t,|u_1 - u_2|)$ *for each $u_1, u_2 \in \Re$.*

(H$_6$) $$\int_{t_0}^{\infty} t^{n-1} V(t,c)\,\mathrm{d}t < \infty \quad \text{for each } c \geqslant 0.$$

(H$_7$) $$\liminf_{c\to\infty} c^{-1} \int_{t_0}^{\infty} t^{n-1} V(t,c)\,\mathrm{d}t = 0.$$

Then the inclusions (1) *and* (2) *are asymptotically equivalent.*

Proof. Let $x(t)$ be a solution of (2). Our aim is to prove the existence of such a solution $y(t)$ of (1) that

$$\lim_{t\to\infty} u^{(i)}(t) = \lim_{t\to\infty} \left[y^{(i)}(t) - x^{(i)}(t)\right] = 0, \qquad i = 0,1,\ldots,n-1. \tag{3}$$

Put $y(t) = x(t) + u(t)$. Then $u(t)$ is a solution of the inclusion

$$u^{(n)}(t) \in F(t, x(t) + u(t)) - G(t, x(t)) \tag{4}$$

satisfying (3). Denote $M(x(t)+u(t)) = \{$the set of all measurable selectors of the function $H_x(u(t)) = F(t, x(t)+u(t)) - G(t, x(t))\}$. Following Lemma 1 we know that $M(x(t)+u(t))$ is not empty. From (4) we get the existence of such a measurable selector $v(t) \in M(x(t)+u(t))$ that

$$u^{(n)}(t) = v(t), \qquad t \in J, \tag{5}$$

and by (3), (H_5) and (H_6) we obtain

$$\int_{t_0}^{\infty} t^{n-1} v(t)\,\mathrm{d}t < \infty, \tag{6}$$

$$u(t) = -\int_t^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} v(s)\,\mathrm{d}s\,. \tag{7}$$

Thus,

$$u(t) \in \Big\{ -\int_t^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} z(s)\,\mathrm{d}s : z(t) \in M\big(x(t)+u(t)\big)\Big\}. \tag{8}$$

Let $C_0(J)$ be the Banach space of all functions $u(t)$ defined and continuous on J such that $\lim\limits_{t\to\infty} u(t) = 0$ with the norm $\|u(t)\| = \sup\limits_J |u(t)|$. We see from (8) that $u(t)$ is a fixed point of the multivalued operator T defined on $C_0(J)$ by

$$Tu(t) = \Big\{ -\int_t^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} z(s)\,\mathrm{d}s : z(t) \in M\big(x(t)+u(t)\big)\Big\}. \tag{9}$$

If $\|u\| = \beta$, then from (H_5) and (H_6) we get $|z(t)| \leqslant V(t,\beta)$ and

$$\begin{aligned}\Big|\int_t^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} z(s)\,\mathrm{d}s\Big| &\leqslant \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} |z(s)|\,\mathrm{d}s\\ &\leqslant \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} V(s,\beta)\,\mathrm{d}s < \infty.\end{aligned}$$

Thus T is well defined.

Let $S_K = \{u(t) \in C_0(J) \mid \|u(t) \leqslant K\}$. We will prove that for the given solution $x(t)$ of (2) there exists $K_0 > 0$ such that $TS_{K_0} \subset S_{K_0}$, T is upper semicontinuous on S_{K_0}, TS_{K_0} is relatively compact and $Tu(x)$ is nonempty, closed and convex for all $u(x) \in S_{K_0}$. It means that the fixed point theorem of Bohnenblust and Karlin will be applicable. The existence of such $K_0 > 0$ that $TS_{K_0} \subset S_{K_0}$ follows from (H_7).

Let $u(t) \in S_K$, $K > 0$. Then $\|u(t)\| = \beta \leqslant K$. Let $h(t) \in Tu(t)$. Then there exists $v(t) \in M(x(t)+u(t))$ such that

$$h(t) = -\int_t^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} v(s)\,\mathrm{d}s\,, \qquad t \in J$$

and

$$|h(t)| \leqslant \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}|v(s)|\,\mathrm{d}s \leqslant \int_{t_0}^\infty \frac{s^{n-1}}{(n-1)!}V(s,K)\,\mathrm{d}s = L < \infty.$$

Thus, TS_K is a set of continuous functions uniformly bounded by the constant L. For $|h'(t)|$ we get

$$|h'(t)| \leqslant \int_t^\infty \frac{(s-t)^{n-2}}{(n-2)!}|v(s)|\,\mathrm{d}s \leqslant \int_{t_0}^\infty \frac{s^{n-2}}{(n-2)!}V(s,k)\,\mathrm{d}s = L_1 < \infty$$

which means that the functions of TS_K are equicontinuous. Moreover, for each $\varepsilon > 0$ there exists $t_0(\varepsilon) \geqslant t_0$ such that for $t_0(\varepsilon) \leqslant t_1 \leqslant t_2$ we have

$$\begin{aligned}|h(t_2) - h(t_1)| &\leqslant |h(t_2)| + |h(t_1)| \\ &\leqslant \int_{t_2}^\infty \frac{s^{n-1}}{(n-1)!}V(s,K)\,\mathrm{d}s + \int_{t_1}^\infty \frac{s^{n-1}}{(n-1)!}V(s,K)\,\mathrm{d}s \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for $t_0(\varepsilon)$ such that $\int_{t_0(\varepsilon)}^\infty \frac{s^{n-1}}{(n-1)!}V(s,K)\,\mathrm{d}s < \frac{\varepsilon}{2}$.

From this fact, from the uniform boundedness and from the equicontinuity of all functions of TS_K we conclude that TS_K as well as $Tu(t)$ are relatively compact in the topology of $C_0(J)$.

Let $u_n(t)$, $n = 1,2,\ldots,u(t)$ be from $C_0(J)$ and let $\{u_n(t)\}$ converge to $u(t)$ in $C_0(J)$, i.e. uniformly on J. Then the set $\{u_n(t),\ n = 1,2,\ldots,u(t)\}$ is bounded in $C_0(J)$. Therefore, there exists $K > 0$ such that $u_n(t) \in S_K$, $n = 1,2,\ldots,u(t) \in S_K$ and TS_K is relatively compact. Let $h_n(t) \in Tu_n(t)$, $n = 1,2,\ldots$. Evidently, $h_n(t) \in TS_K$, $n = 1,2,\ldots$. For each $h_n(t)$ there exists $v_n(t) \in M\big(x(t) + u_n(t)\big)$ such that

$$h_n(t) = -\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}v_n(s)\,\mathrm{d}s\,, \qquad t \geqslant t_0.$$

Let $\hat{L}_1(J)$ be the space of all functions $f(t)$ defined on J such that

$$\int_{t_0}^\infty \frac{(t-t_0)^{n-1}}{(n-1)!}|f(t)|\,\mathrm{d}t < \infty.$$

We see that $v_n(t) \in \hat{L}_1(J)$ and

$$\int_{t_0}^\infty \frac{(t-t_0)^{n-1}}{(n-1)!}|v_n(s)|\,\mathrm{d}s \leqslant \int_{t_0}^\infty \frac{(t-t_0)^{n-1}}{(n-1)!}V(s,k)\,\mathrm{d}s < \infty.$$

Thus, $\{v_n(t)\}$ is bounded in $\hat{L}_1(J)$. Moreover, if $\{E_m\}$, $E_m \subset J$, is a nonincreasing sequence of measurable sets such that $\bigcap_{m=1}^{\infty} E_m = \emptyset$ (empty set) then

$$\lim_{m\to\infty} \left| \int_{E_m} \frac{(t-t_0)^{n-1}}{(n-1)!} v_n(t)\,\mathrm{d}t \right| \leqslant \lim_{m\to\infty} \int_{E_m} \frac{(t-t_0)^{n-1}}{(n-1)!} V(t,K)\,\mathrm{d}t = 0.$$

Then (see [2], Th. IV.8.9) it is possible to choose from $\{v_n(t)\}$ a subsequence $\{v_{n_k}(t)\}$ which weakly converges to some $v(t) \in L_1(J)$. It follows from (H_1) that $H(t, x+u) = F(t, x+u) - G(t, x)$ is upper semicontinuous in u for each fixed t and x. Furthermore, because $\{u_{n_k}\}$ converges uniformly to $u(t)$ and $v_{n_k}(t) \in H\big(x(t) + u_{n_k}(t)\big)$, for given $\varepsilon > 0$, $t \in J$ and $x(t)$ there exists $0 < N = N\big(T, \varepsilon, x(t)\big)$ such that for any $n_k \geqslant N$ we have

$$H\big(t, x(t) + u_{n_k}(t)\big) \subset O_\varepsilon\Big(H\big(t, x(t) + u(t)\big)\Big)$$

where $O_\varepsilon\big(H\big(t, x(t) + u(t)\big)\big)$ is an ε-neighbourhood of the set $H\big(t, x(t) + u(t)\big)$. It means that for all $n_k \geqslant N$ we have $v_{n_k}(t) \in O_\varepsilon\big(H\big(t, x(t) + u(t)\big)\big)$. Then (see [2], Corollary V.3.14) it is possible to construct such convex combinations from v_{n_k}, $n_k \geqslant N$, denote them $g_m(t)$, $m = 1, 2, \ldots$, that the sequence $\{g_m(t)\}$ converges to $v(t)$ in $\hat{L}_1(J)$. Furthermore, by the Riesz theorem there exists a subsequence $\{g_{m_i}\}$ of $\{g_m(t)\}$ which converges to $v(t)$ a.e. on J. From the convexity of $O_\varepsilon\big(H\big(t, x(t) + u(t)\big)\big)$ and from the fact that $v_{n_k} \in O_\varepsilon\big(H\big(t, x(t) + u(t)\big)\big)$ we conclude that $g_{m_i} \in O_\varepsilon\big(H\big(t, x(t) + u(t)\big)\big)$, $i = 1, 2, \ldots$ and therefore, $v(t) \in \overline{O}_\varepsilon\big(H\big(t, x(t) + u(t)\big)\big)$. For $\varepsilon \to 0$ we get $v(t) \in H\big(t, x(t) + u(t)\big)$.

Recall that $t \in J$ was a fixed point and that $H\big(t, x(t) + u(t)\big)$ is a compact convex subset of $\Re$. Furthermore, from Lebesgue's dominated convergence theorem we get that

$$h(t) = -\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} v(s)\,\mathrm{d}s\,, \qquad t \geqslant t_0$$

is well defined and $h(t) \in Tu(t)$, $t \in J$.

It follows from the weak convergence of $\{v_{n_k}(t)\}$ to $v(t)$ in $\hat{L}_1(J)$ that the subsequence $\{h_{n_k}(t)\} \subset \{h_n(t)\}$, i.e. for $t \in J$

$$h_{n_k}(t) = \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} v_{n_k}(s)\,\mathrm{d}s\,, \qquad t \geqslant t_0$$

converges to $h(t)$ a.e. on J. However, the functions $h_{n_k}(t)$ belong to the relatively compact set TS_K. Therefore, there exists a subsequence of the sequence $\{h_{n_k}(t)\}$ which converges to a function $\bar{h}(t)$ uniformly on J. It means that $\bar{h}(t) = h(t) \in Tu(t)$ a.e. on J. This completes the proof of the upper semicontinuity of the operator T.

It follows from upper semicontinuity of T that $Tu(t)$, $u(t) \in C(J)$, is closed. Furthermore, from (H_1) and Lemma 1 we get that $M(x(t)+u(t))$ is nonempty and convex and consequently $Tu(t)$ is also nonempty and convex. Thus, T maps S_K into $cf(S_K)$. All conditions for the application of the Bohnenblust and Karlin theorem are satisfied. We have proved that for each solution $x(t)$ of (2) there exists $u(t)$ such that $y(t) = x(t) + u(t)$ is a solution of (1) such that $\lim\limits_{t\to\infty} u^{(i)}(t) = \left[y^{(i)}(t) - x^{(i)}(t)\right] = 0$, $i = 0, 1, \ldots, n-1$.

To complete the proof of the theorem we only need to change the role of F and G and $x(t)$ and $y(t)$. □

Example. Consider the equations

$$y^{(n)} + f(t)|y|^{\alpha} \operatorname{sgn} y = 0 \tag{10}$$

where $f(t)$ is continuous and positive on $J = [t_0, \infty)$ and

$$x^{(n)} \in G(t, x) \tag{2}$$

where $G(t,x)$ satisfies (H_1), (H_2), (H_4). Assume that $F(t,y) = -f(t)|y|^{\alpha} \operatorname{sgn} y$ and $F(t,y)$ and $G(t,x)$ satisfy (H_5)–(H_7). Then, following Theorem 3, (2) and (10) are asymptotically equivalent.

The condition

$$\int_{t_0}^{\infty} t^{\gamma} f(t)\,\mathrm{d}t = \infty, \qquad \gamma = \begin{cases} (n-1)\alpha, & \text{for } 0 < \alpha < 1 \\ (n-1), & \text{for } \alpha > 1 \end{cases} \tag{11}$$

is necessary and sufficient for the equation (10) to have the property A. Then from Theorem 1 we obtain that the inclusion (2) has the property A, too.

Now, condition (H_5) yields

$$\left|G(t,y) + f(t)|y|^{\alpha} \operatorname{sgn} y\right| \leqslant V(t,0).$$

Let $z \in G(t,y)$. Then

$$\left|z + f(t)|y|^{\alpha} \operatorname{sgn} y\right| \leqslant V(t,0)$$

or

$$-f(t)|y|^{\alpha} \operatorname{sgn} y - V(t,0) \leqslant z \leqslant -f(t)|y|^{\alpha} \operatorname{sgn} y + V(t,0).$$

We have the following result:

Let $0 < \alpha$, $\alpha \neq 1$.

$$F(t,y) = -f(t)|y|^\alpha \operatorname{sgn} y,$$
$$G(t,x) \subset \big[-f(t)|x|^\alpha \operatorname{sgn} x - V(t,0), -f(t)|x|^\alpha \operatorname{sgn} x + V(t,0)\big],$$

$G(t,x)$ satisfies (H_1), (H_2), (H_4); *$V(t,c)$ is continuous on $J \times [0,\infty)$ and nondecreasing in c for each fixed $t \in J$ and satisfies* (H_6) *and* (H_7). *Then the condition* (11) *is necessary and sufficient for the inclusion* (2) *to have the property* A.

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