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# FILIPPOV'S OPERATION AND SOME ATTRIBUTES

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Let $f\colon \mathbb{R} \to \mathbb{R}$ be given. The *Filippov* of $f$ is defined as follows:

$$\mathscr{F}[f](x) = \bigcap_{\varepsilon>0} \bigcap_{Z:\mu(Z)=0} \overline{\operatorname{conv}} f(B_\varepsilon(x) \setminus Z),$$

where $\mu$ denotes Lebesgue measure, $\overline{\operatorname{conv}} A$ represents the closure of the convex hull of the set $A$ and $B_\varepsilon(x)$ represents the open ball of radius $\varepsilon$ about the point $x$. The Filippov is used in defining a generalized solution of the ordinary differential equation $x' = f(x)$, particularly in the case in which $f$ is discontinuous. Information concerning the Filippov can be found in many references, including [1] through [10]. (Actually, Filippov's operation and the notion of a Filippov solution were defined for nonautonomous differential equations. However, for the present paper, consideration of the nonautonomous case essentially has only the effect of introducing an unnecessary parameter $t$ into our results.) In this paper, we treat $\mathscr{F}$ as a function, mapping real-valued functions into set-valued functions, and investigate the properties of $\mathscr{F}$. Such results add to our understanding of this operation. We note that there is an alternate definition of the Filippov (for $f \in L^\infty$), equivalent [5] to the previous one, that we will frequently use:

$$\mathscr{F}[f](x) = \{y\colon \lim_{\varepsilon\to 0} \operatorname*{ess\,inf}_{B_\varepsilon(x)} f \leqslant y \leqslant \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f\}.$$

We first consider choosing an appropriate domain for $\mathscr{F}$. Certainly, there are a number of possibilities, but we require that the domain be restricted to $f$'s which are useful for differential equations in the following sense. It can be shown that the functions in $L^\infty$ are precisely the ones which satisfy the classical local existence theorem for Filippov solutions in the case of $x' = f(x)$, namely Theorem 4 in [5]. Hence, we choose $L^\infty$ as the domain for $\mathscr{F}$, using $\|\cdot\|$ to denote the usual norm on $L^\infty$.

We now discuss the selection of a codomain for the function $\mathscr{F}$. We recall two standard definitions (see [1]). We shall say $F\colon \mathbb{R} \to \mathscr{P}(\mathbb{R})$ (= power set of $\mathbb{R}$) is *bounded* if and only if $\sup_{x\in\mathbb{R}}\{\sup\{|y|\colon y \in F(x)\}\} < \infty$. Also $F$ is said to be *upper semicontinuous* if and only if for each $x \in \mathbb{R}$ and for each open set $N$ containing $F(x)$ there exists an open set $M$ containing $x$ such that $F(M) \subseteq N$. We choose for the codomain the set $\mathscr{B} \equiv \{F\colon \mathbb{R} \to \mathscr{P}(\mathbb{R}) \mid F$ is upper semicontinuous, $F$ is closed-interval valued and $F$ is bounded$\}$. $\mathscr{B}$ can be made into a metric space by defining

$$D(F,G) = \sup_{x\in\mathbb{R}}\{h(F(x),G(x))\},$$

where $F,G \in \mathscr{B}$ and $h$ represents the Hausdorff distance between the two sets $F(x)$ and $G(x)$. It follows easily that $\mathscr{F}[L^\infty] \subseteq \mathscr{B}$ using well-known facts such as for $f \in L^\infty$, $\mathscr{F}[f]\colon \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is upper semicontinuous. We note that $D(F,G)$ is alternatively given by

$$D(F,G) = \sup_{x\in\mathbb{R}}\{\max\{|\min F(x) - \min G(x)|, |\max F(x) - \max G(x)|\}\},$$

where, for example, $\max F(x) \equiv \max\{y\colon y \in F(x)\}$.

We now consider questions involving the range of $\mathscr{F}$. In the results which follow, we shall make use of the following definition. Let $F\colon \mathbb{R} \to \mathscr{P}(\mathbb{R})$. Then the *Filippov* of $F$ is defined by

$$\mathscr{F}[F](x) = \bigcap_{\varepsilon>0}\ \bigcap_{Z:\mu(Z)=0} \overline{\text{conv}} \bigcup_{y\in B_\varepsilon(x)\setminus Z} F(y).$$

(Note that the purpose of this is to extend the Filippov so that it can be applied to set-valued functions. We emphasize that in Corollary 1 and Theorems 2, 6, 7 and 9 that the domain of $\mathscr{F}$, as mentioned earlier, is $L^\infty$.) Results from Jarnik's paper [8] allow us to completely characterize the range of $\mathscr{F}$.

**Theorem 1.** *Let $F\colon \mathbb{R} \to \mathscr{P}(\mathbb{R})$. Then, there exists $f \in L^\infty$ such that $\mathscr{F}[f] = F$ if and only if $F$ satisfies the following conditions:*

1) *$F$ is upper semicontinuous,*
2) *There exists $M > 0$ such that $F(x) \subseteq [-M, M]$ for all $x \in \mathbb{R}$, and*
3) *$\mathscr{F}[F] = F$.*

P r o o f. Suppose there exists $f \in L^\infty$ such that $\mathscr{F}[f] = F$. As noted above, $\mathscr{F}[L^\infty] \subseteq \mathscr{B}$, thus $F$ satisfies 1) and 2), while $F$ satisfies property 3) by (7) in [8]. Now assuming that $F$ satisfies 1), 2) and 3), the existence of $f \in L^\infty$ such that $\mathscr{F}[f] = F$ follows from the main result in [8] using a simple, but tedious, change of scale, which we omit for brevity. $\square$

**Corollary 1.** *$\mathscr{F}$ is not onto $\mathscr{B}$.*

Proof. Define $F \in \mathscr{B}$ by

$$F(x) = \begin{cases} \{0\} & \text{for } x \neq 0, \\ [0,1] & \text{for } x = 0. \end{cases}$$

Clearly, $\mathscr{F}[F] \equiv \{0\}$, so $\mathscr{F}[F] \neq F$. Thus, by property 3) of Theorem 1 $F$ is not the Filippov of an $L^\infty$ function. □

For our next result concerning the range of $\mathscr{F}$, we shall need the following.

**Lemma 1.** *For all $F, G \in \mathscr{B}$, we have $D(\mathscr{F}[F], \mathscr{F}[G]) \leqslant D(F,G)$ and hence $\mathscr{F}$, thought of as mapping $\mathscr{B} \to \mathscr{B}$, is continuous.*

Proof. It can easily be shown that $\mathscr{F}[\mathscr{B}] \subseteq \mathscr{B}$. Define $a(x) = \lim\limits_{\varepsilon\to 0} \operatorname*{ess\,inf}\limits_{B_\varepsilon(x)} F$, $b(x) = \lim\limits_{\varepsilon\to 0} \operatorname*{ess\,sup}\limits_{B_\varepsilon(x)} F$, $c(x) = \lim\limits_{\varepsilon\to 0} \operatorname{ess\,inf}_{B_\varepsilon(x)} G$ and $d(x) = \lim\limits_{\varepsilon\to 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} G$. Then, $D(\mathscr{F}[F], \mathscr{F}[G]) = \sup\limits_{\mathbb{R}}\{\max\{|a-c|, |b-d|\}\}$. Similarly, define $e(x) = \min F(x)$, $f(x) = \max F(x)$, $g(x) = \min G(x)$ and $h(x) = \max G(x)$. Then, $D(F,G) = \sup\limits_{\mathbb{R}}\{\max\{|e-g|, |f-h|\}\}$. In the proof of the following claim, we shall use the fact that for $A \subseteq \mathbb{R}$,

$$|\operatorname{ess\,sup}_A f - \operatorname*{ess\,sup}_A h| \leqslant \operatorname*{ess\,sup}_A |f-h|.$$

This is easily verified, so we omit the proof.

*Claim 1*: For all $x \in \mathbb{R}$, $|b(x) - d(x)| \leqslant \sup\limits_{\mathbb{R}} |f-h|$.

Proof of Claim 1

$$\begin{aligned} |b(x) - d(x)| &= |\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} F - \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} G| \\ &= |\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f - \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} h| \\ &\leqslant \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} |f-h| \\ &\leqslant \operatorname*{ess\,sup}_{\mathbb{R}} |f-h| \leqslant \sup_{\mathbb{R}} |f-h|. \end{aligned}$$

*Claim 2*: For all $x \in \mathbb{R}$, $|a(x) - c(x)| \leqslant \sup\limits_{\mathbb{R}} |e-g|$.

We omit the proof of Claim 2 since it is similar to that of Claim 1. We now have, applying Claims 1 and 2, for all $x \in \mathbb{R}$,

$$\begin{aligned}\max\{|a(x)-c(x)|,|b(x)-d(x)|\} &\leqslant \max\{\sup_{\mathbb{R}}|e-g|,\sup_{\mathbb{R}}|f-h|\}\\ &= \sup_{\mathbb{R}}\max\{|e-g|,|f-h|\}\end{aligned}$$

and thus

$$\sup_{\mathbb{R}}\max\{|a-c|,|b-d|\} \leqslant \sup_{\mathbb{R}}\max\{|e-g|,|f-h|\},$$

i.e., $D(\mathscr{F}[F],\mathscr{F}[G]) \leqslant D(F,G)$. □

**Theorem 2.** *The range of $\mathscr{F}\colon L^\infty \to \mathscr{B}$ is closed and unbounded in $(\mathscr{B}, D)$.*

Proof. Let $\{F_n\}_{n=1}^\infty \subseteq \mathscr{F}[L^\infty]$ and $F_n \to F$ in $(\mathscr{B}, D)$. Lemma 1 implies that $\mathscr{F}[F_n] \to \mathscr{F}[F]$ in $(\mathscr{B}, D)$. By Theorem 1, for each $n \in \mathbb{N}$, $\mathscr{F}[F_n] = F_n$, hence $F_n \to \mathscr{F}[F]$. Since limits are unique in $(\mathscr{B}, D)$, we have $F = \mathscr{F}[F]$. Also, since $F \in \mathscr{B}$, we have that i) $F$ is upper semicontinuous and ii) there exists $M > 0$ such that $F(x) \subseteq [-M, M]$ for all $x \in \mathbb{R}$. Thus, applying Theorem 1 to $F$, we have $F \in \mathscr{F}[L^\infty]$ and hence $\mathscr{F}$ has closed range. Now, for each $n \in \mathbb{N}$, define $f_n \in L^\infty$ by $f_n(x) \equiv n$, and also define $f_\infty \in L^\infty$ by $f_\infty(x) \equiv 0$. It follows that $\sup\limits_{n\in\mathbb{N}} D(\mathscr{F}[f_n], \mathscr{F}[f_\infty]) = \infty$, and hence the range is unbounded. □

We now consider the question of whether or not $\mathscr{F}$ is one-to-one. In [2], the following appeared.

**Theorem 3.** *Let $f, g \in \mathscr{C} \equiv \{h \in L^\infty$: there exists a set $A_h$ of full measure such that $h|_{A_h}$ is continuous$\}$. If $\mathscr{F}[f] = \mathscr{F}[g]$, then $f = g$ (in $L^\infty$).*

To complete the one-to-one question, we shall need the following lemmas, the first of which is proven, for example, in [8].

**Lemma 2.** *Let $A \subseteq \mathbb{R}$ be Lebesgue measurable with $\mu A > 0$. Then, there exist Lebesgue measurable sets $D$ and $E$ such that $D \cap E = \emptyset$, $D \cup E = A$ and for all $\varepsilon > 0$, for all $x \in A$ with $\mu(B_\varepsilon(x) \cap A) > 0$, we have both $\mu(D \cap B_\varepsilon(x)) > 0$ and $\mu(E \cap B_\varepsilon(x)) > 0$. ($D$ and $E$ are known as "metrically dense" subsets of $A$.)*

**Lemma 3.** *$f \in \mathscr{C}$ if and only if $\mathscr{F}[f]$ is a singleton a.e.*

The proof is given in [2]. We are now able to prove the following, which, in a sense, tells us that the set $\mathscr{C}$ in Theorem 3 is the largest subset of $L^\infty$ on which $\mathscr{F}$ is one-to-one.

**Theorem 4.** *Let $f \in L^\infty \setminus \mathscr{C}$. Then, there exists some $g \in L^\infty$ such that $\mathscr{F}[f] = \mathscr{F}[g]$ but $f \neq g$ (in $L^\infty$).*

Proof. Define $\overline{f}, \underline{f}\colon \mathbb{R} \to \mathbb{R}$ by

$$\overline{f}(x) = \lim_{\varepsilon \to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f \text{ and } \underline{f}(x) = \lim_{\varepsilon \to 0} \operatorname*{ess\,inf}_{B_\varepsilon(x)} f.$$

Since $f \notin \mathscr{C}$, by Lemma 3 there exists a set $D \subseteq \mathbb{R}$ of positive measure such that for all $x \in D$, $\underline{f}(x) < \overline{f}(x)$. It follows from Lusin's Theorem that there exists a set $F$ with the following properties:

1) $F \subseteq D$,
2) $F$ is closed,
3) $\mu(F) > 0$, and
4) $\overline{f}$ and $\underline{f}$ are both continuous relative to $F$.

Let $A$ and $B$ be disjoint metrically dense subsets of $F$ such that $A \cup B = F$. (Such sets exist by Lemma 2.) Define $k_1, k_2\colon \mathbb{R} \to \mathbf{R}$ by

$$k_1(x) = \begin{cases} f(x) & \text{for } x \notin F, \\ \overline{f}(x) & \text{for } x \in A, \\ \underline{f}(x) & \text{for } x \in B, \end{cases} \qquad k_2(x) = \begin{cases} f(x) & \text{for } x \notin F, \\ \underline{f}(x) & \text{for } x \in A, \\ \overline{f}(x) & \text{for } x \in B. \end{cases}$$

Clearly, $k_1$ and $k_2$ disagree on $F$, a set of positive measure. Without loss of generality, assume $f$ and $k_1$ disagree on a set of positive measure, and let $g(x) = k_1(x)$ for all $x \in \mathbb{R}$.

*Claim*: $\mathscr{F}[f] = \mathscr{F}[g]$.

*Case 1*: $x \notin F$.

Since $F$ is closed, there exists an open interval $I \subseteq \mathbb{R} \setminus F$ containing $x$. We note that $f(y) = g(y)$ for all $y \in I$, thus $\mathscr{F}[f](x) = \mathscr{F}[g](x)$.

*Case 2*: $x \in F$ and $x$ is a point of density of $F$.

We want to show

$$\lim_{\varepsilon \to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} g = \lim_{\varepsilon \to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f.$$

Note that $\lim_{\varepsilon \to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x) \cap F} \overline{f} = \overline{f}(x)$ since $\overline{f}$ is continuous on $F$, and

$$\lim_{\varepsilon \to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x) \cap F^c} f \leqslant \lim_{\varepsilon \to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f = \overline{f}(x).$$

We then have

$$\begin{aligned}\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} g &= \lim_{\varepsilon\to 0} \max\{\operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F} g,\ \operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F^c} g\} \\ &\leqslant \lim_{\varepsilon\to 0} \max\{\operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F} \overline{f},\ \operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F^c} f\} \\ &= \max\{\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F} \overline{f},\ \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)\cap F^c} f\} \\ &= \overline{f}(x) = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f.\end{aligned}$$

For the opposite inequality, we note that for all $\varepsilon > 0$ and for all $Z \subseteq \mathbb{R}$ with $\mu(Z) = 0$, we have

$$\sup_{B_\varepsilon(x)\setminus Z} g \geqslant \sup_{(B_\varepsilon(x)\setminus Z)\cap A} g = \sup_{(B_\varepsilon(x)\setminus Z)\cap A} \overline{f} \geqslant \overline{f}(x),$$

since $\overline{f}$ is continuous on $A$ and $x$ is a point of density of $F$. Thus, for all $\varepsilon > 0$,

$$\operatorname*{ess\,sup}_{B_\varepsilon(x)} g \geqslant \overline{f}(x)$$

and so

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} g \geqslant \overline{f}(x) = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x)} f.$$

The fact that $\lim_{\varepsilon\to 0} \operatorname*{ess\,inf}_{B_\varepsilon(x)} g = \lim_{\varepsilon\to 0} \operatorname*{ess\,inf}_{B_\varepsilon(x)} f$ is handled analogously.

We note that we need not handle Case 3, in which $x \in F$ but $x$ is not a point of density of $F$ since these form a set of measure zero. The result follows since it can be shown that if $\mathscr{F}[f]$ and $\mathscr{F}[g]$ agree on a set of full measure, then $\mathscr{F}[f](x) = \mathscr{F}[g](x)$ for all $x \in \mathbb{R}$. □

We now investigate whether or not $\mathscr{F}$ is continuous. It not only turns out to be continuous, but is "Lipschitz," in the following sense.

**Theorem 5.** *For all $f, g \in L^\infty$, we have $D(\mathscr{F}[f], \mathscr{F}[g]) \leqslant \|f - g\|$.*

The proof is similar to that of Lemma 1, so for brevity, we omit it. Examples can be given to show that the Lipschitz constant of 1 is sharp, and also that, in general, the inequality cannot be replaced with equality.

We now investigate other topological properties of $\mathscr{F}$.

**Theorem 6.** *$\mathscr{F}$ is not an open map.*

P r o o f. Let $U \subseteq L^\infty$ be an open ball containing the zero function in $L^\infty$ (call it $f$). Define $F \in \mathscr{B}$ by $F(x) = \{0\}$ for all $x \in \mathbb{R}$. Note that $\mathscr{F}[f] = F$ so $F \in \mathscr{F}[U]$. Let $\varepsilon > 0$. Define $G \in \mathscr{B}$ by

$$G(x) = \begin{cases} \{0\} & \text{for } x \neq 0 \\ [0, \frac{\varepsilon}{2}] & \text{for } x = 0. \end{cases}$$

Clearly, $G$ lies in the $\varepsilon$-ball centered at $F$. $G$ cannot be in $\mathscr{F}[U]$ since Filippovs which agree almost everywhere agree everywhere. Thus, $F$ is not an interior point of $\mathscr{F}[U]$, so $\mathscr{F}[U]$ is not open. □

**Theorem 7.** *$\mathscr{F}$ is not a closed map. (Here, closed map is intended to mean images of closed sets are closed.)*

P r o o f. Let $A$ and $\mathbb{R}\backslash A$ be metrically dense in $\mathbb{R}$ and let $A_n$ and $B_n$, $n = 1, 2, \ldots$, be metrically dense in $[n, n+1] \setminus A$ with $A_n \cup B_n = [n, n+1] \setminus A$ and $A_n \cap B_n = \emptyset$. For each $n \in \mathbb{N}$, define $f_n \in L^\infty$ by

$$f_n(x) = \begin{cases} (1 + \frac{1}{n})\chi_A(x) & \text{for } x \notin [n, n+1] \\ 1 + \frac{1}{n} & \text{for } x \in [n, n+1] \cap A \\ \frac{1}{2} & \text{for } x \in A_n \\ 0 & \text{for } x \in B_n \end{cases}$$

Then, $\mathscr{F}[f_n](x) \equiv [0, 1 + \frac{1}{n}]$. Define $F \in \mathscr{B}$ by $F(x) = [0, 1]$ for all $x \in \mathbb{R}$. Clearly, $\mathscr{F}[f_n] \to F$ in $(\mathscr{B}, D)$, but $\mathscr{F}[f_n] \neq F$ for each $n \in \mathbb{N}$. Let $K \equiv \{f_n\}_{n=1}^\infty \subseteq L^\infty$. If $n \neq m$, $\|f_n - f_m\| \geqslant \frac{1}{2}$. Thus, there can be no Cauchy sequences and hence no convergent sequences in $K$ (except, of course, those which are eventually constant), so $K$ is closed. However, $F \in \overline{\mathscr{F}[K]} \setminus \mathscr{F}[K]$. Therefore, $\mathscr{F}$ is not a closed map. □

Another property often considered concerning functions is monotonicity. The notion of order depends on the particular application. To study monotonicity in this more abstract context, we can define a partial order $\preceq$ on the domain $L^\infty$ by:

$$f \preceq g \text{ iff } f(x) \leqslant g(x) \text{ for almost all } x \in \mathbb{R}$$

where $f, g \in L^\infty$. We next define a partial order on the codomain $\mathscr{B}$, which we also denote by $\preceq$:

$$F \preceq G \text{ iff for all } x \in \mathbb{R}, \text{ we have both}$$
$$\min F(x) \leqslant \min G(x) \text{ and } \max F(x) \leqslant \max G(x)$$

where $F, G \in \mathscr{B}$.

**Theorem 8.** *Let $f, g \in L^\infty$. If $f \preceq g$, then $\mathscr{F}[f] \preceq \mathscr{F}[g]$.*

The proof is trivial so we omit it. Although the condition $f \preceq g$ is sufficient for $\mathscr{F}[f] \preceq \mathscr{F}[g]$, it is not necessary. Such an example is provided by $k_1$ and $k_2$ in the proof of Theorem 4.

**Theorem 9.** *$\mathscr{F}$ is not strictly monotone.*

Proof. Let $A$ and $\mathbb{R} \setminus A$ be metrically dense in $\mathbb{R}$ and let $E$ and $A \setminus E$ be metrically dense in $A$. Define $f \in L^\infty$ by

$$f(x) = \begin{cases} 0 & \text{for } x \notin A, \\ \frac{1}{2} & \text{for } x \in E, \\ 1 & \text{for } x \in A \setminus E. \end{cases}$$

Clearly, $f$ is strictly less than $g \equiv \chi_A$. We note that $\mathscr{F}[g] \equiv [0,1]$ by the metric density of $A$ and $A^c$. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Metric density implies that the sets $(\mathbb{R} \setminus A) \cap B_\varepsilon(x)$ and $(A \setminus E) \cap B_\varepsilon(x)$ each have positive measure. Thus, $\mathscr{F}[f] \equiv [0,1]$. Therefore, $\mathscr{F}[f] = \mathscr{F}[g]$. □

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