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A REMARK ON CONFLUENT CAUCHY AND CONFLUENT LOEWNER MATRICES

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1. Introduction

Cauchy matrices are matrices (rectangular in general) with elements $1/(y_i - z_j)$, corresponding to two sequences of interpolation nodes

$$y^S = (y_0, \dots, y_{n_1-1}), \quad z^S = (z_0, \dots, z_{n_2-1}) \tag{1}$$

(y_i, z_j are $n_1 + n_2$ mutually distinct nodes, complex in general). For a given function φ defined at least at the points y_i, z_j we denote by $L_\varphi \in \mathcal{L}(y^S, z^S)$ the n_1-by-n_2 Loewner matrix with elements $\big(\varphi(y_i) - \varphi(z_j)\big)/(y_i - z_j)$ (where $\mathcal{L}(y^S, z^S)$ is the class of all such matrices for fixed sequences y^S and z^S). Then the Cauchy matrix is a special case of Loewner matrices, obtained for $\varphi(y_i) = 1 \forall i$ and $\varphi(z_j) = 0 \forall j$. Evidently the rational function

$$\varphi(x) = \frac{b^S(x)}{a^S(x) + b^S(x)} \tag{2}$$

where

$$b^S(x) = \prod (x - z_j), a^S(x) = \prod (x - y_i) \tag{3}$$

serves as an example. Denoting the Cauchy matrix by C_{y^S, z^S} we obtain

$$C_{y^S, z^S} = L_{\frac{b^S}{a^S + b^S}} \in \mathcal{L}(y^S, z^S).$$

(Let us remark that the existence of a rational function φ of Mac-Millan degree[1] n for the conditions $\varphi(y_i) = 1$, $\varphi(z_j) = 0$ follows from the fact that C_{y^S, z^S} is nonsingular and by Loewner's theory connecting Loewner matrices with interpolation.)

[1] The Mac-Millan degree of a rational function is the maximum of the degrees of its numerator and denominator.

The present note solves the problem whether the confluent Cauchy matrix, introduced in [5] (see Definition 1 below) has the analogous property of being a special case of confluent Loewner matrices. Theorem 5 in Section 3 proves the validity of the formally identical equality

$$C_{y,z} = L_{\frac{b}{a+b}} \in \mathcal{L}(y,z).$$

2. Notation and preliminaries

Besides the sequences of simple interpolation nodes 1, we introduce the multiple-nodes sequences

$$(4)\qquad \begin{gathered} y = ([y_0, \varrho_0], \ldots, [y_{r-1}, \varrho_{r-1}]), \\ y_i \neq y_{i'} \quad \text{if } i \neq i', \quad \sum_{i=0}^{r-1} \varrho_i = n_1, \end{gathered}$$

$$(5)\qquad \begin{gathered} z = ([z_0, \sigma_0], \ldots, [z_{s-1}, \sigma_{s-1}]), \\ z_j \neq z_{j'} \quad \text{if } j \neg j', \quad \sum_{j=0}^{s-1} \sigma_j = n_2 \end{gathered}$$

(ϱ_i, σ_j are positive integers—multiplicities). We introduce also the corresponding polynomials

$$(6)\qquad a(x) = \prod_{i=0}^{r-1} (x - y_i)^{\varrho_i}, \quad b(x) = \prod_{j=0}^{s-1} (x - z_j)^{\sigma_j}.$$

Definition 1. If $y_i \neq z_j \,\forall i = 0, \ldots, r-1$ and $\forall j = 0, \ldots, s-1$ then we introduce the confluent Cauchy matrix $C_{y,z}$ (of dimension n_1-by-n_2) [5]:

$$(7)\qquad C_{y,z} = \left(C_{i,j}\right)_{\substack{i=0,\ldots,r-1 \\ j=0,\ldots,s-1}}$$

$$(8)\qquad \begin{aligned} C_{ij} &= \left(\binom{k+l}{k} \frac{(-1)^k}{(y_i - z_j)^{k+l+1}}\right)_{\substack{k=0,\ldots,\varrho_i-1 \\ l=0,\ldots,\sigma_j-1}} \\ &= \left(\frac{\partial^{k+l}}{k!l!\partial\eta^k\zeta^l}\left[\frac{1}{\eta - \zeta}\right]_{\substack{\eta=y_i \\ \zeta=z_j}}\right), \end{aligned}$$

Remark 2. G. Heinig in his paper [3] introduced another generalization of Cauchy matrices, of the form

$$\left(\frac{c_i^T d_j}{y_i - z_j}\right)$$

where c_i, d_j are k-term vectors ($k \ll n$). Such matrices have connections with vector interpolation.

Definition 3. If $\varphi(x)$ is a function such that the values $\varphi^{(k)}(y_i)$, $i = 0, \ldots, r-1$, $k = 0, \ldots, \varrho_i - 1$ and $\varphi^{(l)}(z_j)$, $j = 0, \ldots, s-1$, $l = 0, \ldots, \sigma_j - 1$, exist then we introduce the confluent Loewner matrix $L_\varphi \in \mathcal{L}(y,z)$ (of dimension n_1-by-n_2) (see the "generalized Loewner matrix" in [4]) by

$$L_\varphi = (L_{ij})_{\substack{i=0,\ldots,r-1\\ j=0,\ldots,s-1}}, \tag{9}$$

$$L_{ij} = \Big([\underbrace{y_i, \ldots, y_i}_{(k+1)\text{times}}, \underbrace{z_j, \ldots, z_j}_{(l+1)\text{times}}]_\varphi\Big)_{\substack{k=0,\ldots,\varrho_i-1\\ l=0,\ldots,\sigma_i-1}}. \tag{10}$$

Here $[\ldots]_\varphi$ denotes the divided difference—see e.g. [2]. We admit $y_i = z_j$ for some i and j. If, however, $y_i \neq z_j$ then

$$[y_i, \ldots, y_i, z_j, \ldots, z_j]_\varphi = \frac{\partial^{k+l}}{k!l!\partial\eta^k\partial\zeta^l}\left[\frac{\varphi(\eta)-\varphi(\zeta)}{\eta-\zeta}\right]_{\substack{\eta=y_i\\ \zeta=z_j}} \tag{11}$$

Remark 4. 1. The same definition was introduced one year before [4] in [1], up to the constants $1/k!l!$.

2. The author decided here to change the name from "generalized" to "confluent" Loewner matrices since this corresponds better to the interpolation connections.

3. The result

Theorem 5. *If the sequences of interpolation nodes fulfil the condition*

$$y_i \neq z_j, \quad i = 0, \ldots, r-1, j = 0, \ldots, s-1 \tag{12}$$

then the confluent Loewner matrix

$$L_{\frac{b}{a+b}} \in \mathcal{L}(y,z)$$

is defined and equals the confluent Cauchy matrix $C_{y,z}$.

The proof will be very easy if we use the following lemma:

Lemma 6. *Let* k *and* l *be positive integers and let the function* φ *have derivatives up to orders* k, l *at the points* η_0, ζ_0 *respectively* ($\eta_0 \neq \zeta_0$). *Then the partial derivative*

$$\frac{\partial^{k+l}}{\partial\eta^k \partial\zeta^l}\left[\frac{\varphi(\eta)-\varphi(\zeta)}{\eta-\zeta}\right]_{\substack{\eta=\eta_0\\ \zeta=\zeta_0}}$$

exists and can be expressed in the form

$$\frac{1}{(\eta_0-\zeta_0)^{k+l+1}}\Bigg[\sum_{\kappa=1}^{k}\varphi^{(\kappa)}(\eta_0)p_{k,l,\kappa}(\eta_0,\zeta_0)$$
$$+\sum_{\lambda=1}^{l}\varphi^{(\lambda)}(\zeta_0)q_{k,l,\lambda}(\eta_0,\zeta_0)+(-1)^k(k+l)!\big(\varphi(\eta_0)-\varphi(\zeta_0)\big)\Bigg]$$

where $p_{k,l,\kappa}$, $q_{k,l,\lambda}$ *are polynomials in two variables.*

The proof is easy by induction.

Now we shall return to the proof of Theorem :

Proof. Let us denote

$$\varphi(x)=\frac{b(x)}{a(x)+b(x)}.$$

Then

$$\varphi'(x)=\frac{a(x)b'(x)-a'(x)b(x)}{\big(a(x)+b(x)\big)^2}.$$

This shows that $\varphi'(x)$ is divisible by $(x-y_i)^{\varrho_i-1}$ and by $(x-z_j)^{\sigma_j-1}$. As an easy consequence we get that

$$\varphi(y_i)=1,\quad \varphi^{(\kappa)}(y_i)=0,\quad \kappa=1,\dots,\varrho_i-1,$$
$$\varphi^{(\lambda)}(z_j)=0,\quad \lambda=0,\dots,\sigma_j-1.$$

This together with Lemma 6 proves Theorem 5. □

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