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ON ONE PROBLEM IN THE THEORY OF PARTIAL MONOUNARY ALGEBRAS

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Let $\mathscr{K}$ be a weak variety (i.e. a class of all partial algebras of the same type which weakly satisfy a set E of equations). Further, let E' be the set of all equations satisfied by all total algebras belonging into the class $\mathscr{K}$. Define another class $\mathscr{K}^*$ of all partial algebras of the same type which weakly satisfy all equations of the set E'. It is easy to see that $\mathscr{K}^* \subseteq \mathscr{K}$. L. Rudak [1] proposed the following problem:

Problem. For which classes $\mathscr{K}$ of partial algebras the relation $\mathscr{K}^* = \mathscr{K}$ is valid?

In this paper the problem is investigated for partial monounary algebras. A necessary and sufficient condition (concerning E) is found under which $\mathscr{K}^* = \mathscr{K}$.

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1. Basic definitions and notation

A *type* (or *similarity type*) is a set F and a mapping ϱ of F into the set of nonnegative integers. The elements of F are called *operation symbols* of type ϱ. Further, $\mathbf{A} = (A, (f^{\mathbf{A}})_{f \in F})$ is a (*partial*) *algebra* of type ϱ if A is a nonempty set and $f^{\mathbf{A}}$ is a (partial) $\varrho(f)$-ary operation in A for every $f \in F$. Thus the word "algebra" will always be used in the sense "total algebra".

If p is a σ-term (for notions not defined here see [2]) and $\mathbf{A}$ is a (partial) algebra of type σ, $p^{\mathbf{A}}$ will denote the (partial) function induced in $\mathbf{A}$ by p and $\mathrm{dom}(p^{\mathbf{A}})$ will be its domain.

An *equation* of type σ is a word of the form $p \approx q$ where p and q are σ-terms.

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Let $\mathbf{A}$ be an algebra and $p \approx q$ an equation (both of type σ), and suppose that p and q are n-ary. If for any n-tuple $\overline{a} \in A^n$ we have $p^{\mathbf{A}}(\overline{a}) = q^{\mathbf{A}}(\overline{a})$ then we say that $p \approx q$ is *satisfied* in $\mathbf{A}$ and we write $\mathbf{A} \models p \approx q$.

Let $\mathbf{A}$ be a partial algebra and $p \approx q$ an equation (both of type σ), and suppose that p and q are n-ary. We say that the equation $p \approx q$ is *weakly satisfied* in $\mathbf{A}$ (and we write $\mathbf{A} \models_w p \approx q$) if for any n-tuple $\overline{a} \in A^n$ we have: if $\overline{a} \in \operatorname{dom}(p^{\mathbf{A}}) \cap \operatorname{dom}(q^{\mathbf{A}})$, then $p^{\mathbf{A}}(\overline{a}) = q^{\mathbf{A}}(\overline{a})$. (For this definition cf. [5].) In other words, one can say that $p \approx q$ is weakly satisfied in a partial algebra $\mathbf{A}$ if the following holds: if both $p^{\mathbf{A}}$ and $q^{\mathbf{A}}$ are defined on $\overline{a} \in A^n$, then they are equal.

Let E be a set of equations of type σ and $\mathscr{K}$ a class of algebras of type σ. Denote by $\mathscr{T}_\sigma$ the class of all algebras of type σ. We define

$$\operatorname{Eq}(\mathscr{K}) = \{p \approx q \colon \mathbf{A} \models p \approx q \quad \text{for all} \quad \mathbf{A} \in \mathscr{K}\},$$
$$\operatorname{Md}(E) = \{\mathbf{A} \in \mathscr{T}_\sigma \colon \mathbf{A} \models p \approx q \quad \text{for all} \quad p \approx q \in E\}.$$

Now let $\mathscr{K}$ be a class of partial algebras of type σ and let E be as above. Denote by $\mathscr{P}_\sigma$ the class of all partial algebras of type σ. We define

$$\mathscr{K}^T = \{\mathbf{A} \in \mathscr{K} \colon \mathbf{A} \text{ is an algebra}\},$$
$$\operatorname{Md}_w(E) = \{\mathbf{A} \in \mathscr{P}_\sigma \colon \mathbf{A} \models_w p \approx q \quad \text{for all} \quad p \approx q \in E\}.$$

Thus $\operatorname{Md}_w(E)$ is a class of all partial algebras of the same type which weakly satisfy a set E of equations.

Let E be a set of equations of type σ. We denote by $\operatorname{Cl}(E)$ the smallest set of equations of type σ containing E and closed under trivial equations, symmetry, transitivity, substitutions and congruences (i.e. $\operatorname{Cl}(E)$ is the set of all equations which are provable from E using Birkhoff's rules). We write $\operatorname{Cl}(e_1, \dots, e_n)$ instead of $\operatorname{Cl}(\{e_1, \dots, e_n\})$; analogously we write $\operatorname{Md}(e_1, \dots, e_n)$, $\operatorname{Md}_w(e_1, \dots, e_n)$.

Denote $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Some auxiliary results

2.1. Lemma. *Let E be a set of equations of the same type, $\mathscr{K} = \operatorname{Md}_w(E)$ and $E' = \operatorname{Eq}(\mathscr{K}^T)$. Then $E' = \operatorname{Cl}(E)$.*

Proof. It is easy to see that $\mathscr{K}^T = \operatorname{Md}(E)$. Thus $E' = \operatorname{Eq}(\mathscr{K}^T) = \operatorname{Eq}(\operatorname{Md}(E))$. According to the well known Birkhoff's theorem we have $\operatorname{Eq}(\operatorname{Md}(E)) = \operatorname{Cl}(E)$ and hence $E' = \operatorname{Cl}(E)$. □

Now—using the above lemma—we can reformulate our problem as follows:

Let σ be a fixed type. For which sets E of equations of type σ the following equality holds:

$$\mathrm{Md}_w(E) = \mathrm{Md}_w(\mathrm{Cl}(E))\,?$$

Note that this equality does not hold in general, as the following example shows.

2.2. Example. Consider partial algebras with one unary operation f (i.e. partial monounary algebras) and let $E = \{f^2(x) \approx f(x), f^3(x) \approx x\}$ be a set of equations. It is easy to see that in total algebras one can deduce an equation $f(x) \approx x$ from the set E. Indeed, the equations

$$f(x) \approx f^2(x),\;\; f^2(x) \approx f^3(x),\;\; f^3(x) \approx x$$

follow from E by symmetry and substitution. Using transitivity we get the desired equation. Thus we have $f(x) \approx x \in \mathrm{Cl}(E)$.

On the other hand, a partial algebra $\mathbf{A}$ with a two-element carrier set $\{a, b\}$ and a partial operation $f^{\mathbf{A}}$ defined only on a with $f^{\mathbf{A}}(a) = b$ is in the class $\mathrm{Md}_w(E)$, but is not in $\mathrm{Md}_w(\mathrm{Cl}(E))$ (because $\mathbf{A}$ does not weakly satisfy $f(x) \approx x$).

2.3. Lemma. *Let e be an equation and E a set of equations of the same type as e. Then the following conditions are equivalent:*

(i) $e \in \mathrm{Cl}(E)$;

(ii) $\mathrm{Cl}(e) \subseteq \mathrm{Cl}(E)$;

(iii) $\mathrm{Eq}(\mathrm{Md}(e)) \subseteq \mathrm{Eq}(\mathrm{Md}(E))$;

(iv) $\mathrm{Md}(E) \subseteq \mathrm{Md}(e)$.

Proof. Easy. We recall that by Birkhoff's theorem $\mathrm{Cl}(e) = \mathrm{Eq}(\mathrm{Md}(e))$ and $\mathrm{Cl}(E) = \mathrm{Eq}(\mathrm{Md}(E))$. □

From now on we will consider only a monounary type. We suppose throughout that f is a unary operation symbol and x, y are different variables. There are two types of equations:

(1) $f^i(x) \approx f^j(x)$,

(2) $f^i(x) \approx f^j(y)$,

where $i, j \in \mathbb{N}_0$. (For a positive integer m and any variable z the symbol $f^m(z)$ has a natural meaning; $f^0(z)$ means z). The equations of type (1) are called regular equations, those of type (2) are nonregular.

The following lemmas 2.4–2.6 can be deduced from [3] and [4].

2.4. Lemma. *Let $i, j \in \mathbb{N}_0$, $i \leqslant j$. Then* $\mathrm{Md}(f^i(x) \approx f^j(y)) = \mathrm{Md}(f^i(x) \approx f^i(y))$.

2.5. Lemma. *Let $r, s, i, j \in \mathbb{N}_0$, $l, m \in \mathbb{N}$.*
(i) *If* $\mathrm{Md}(f^r(x) \approx f^r(y)) = \mathrm{Md}(f^s(x) \approx f^s(y))$, *then* $r = s$.
(ii) *If* $\mathrm{Md}(f^i(x) \approx f^{i+l}(x)) = \mathrm{Md}(f^j(x) \approx f^{j+m}(x))$, *then* $i = j$ *and* $l = m$.

2.6. Lemma. *Let $r, s, i, j \in \mathbb{N}_0$, $l, m \in \mathbb{N}$. Then*
(i) $\mathrm{Md}(f^r(x) \approx f^r(y)) \cap \mathrm{Md}(f^s(x) \approx f^s(y)) = \mathrm{Md}(f^{\min(r,s)}(x) \approx f^{\min(r,s)}(y))$;
(ii) $\mathrm{Md}(f^r(x) \approx f^r(y)) \cap \mathrm{Md}(f^i(x) \approx f^{i+l}(x)) = \mathrm{Md}(f^{\min(r,i)}(x) \approx f^{\min(r,i)}(y))$;
(iii) $\mathrm{Md}(f^i(x) \approx f^{i+l}(x)) \cap \mathrm{Md}(f^j(x) \approx f^{j+m}(x)) =$
$\mathrm{Md}(f^{\min(i,j)}(x) \approx f^{\min(i,j)+(l,m)}(x))$, *where (l, m) is the greatest common divisor of l and m.*

2.7. Corollary. *Let $r, s, i, j \in \mathbb{N}_0$, $l, m \in \mathbb{N}$. Then*
(i) $\mathrm{Md}(f^r(x) \approx f^r(y)) \subseteq \mathrm{Md}(f^s(x) \approx f^s(y))$ *if and only if* $r \leqslant s$;
(ii) $\mathrm{Md}(f^r(x) \approx f^r(y)) \subseteq \mathrm{Md}(f^i(x) \approx f^{i+l}(x))$ *if and only if* $r \leqslant i$;
(iii) $\mathrm{Md}(f^i(x) \approx f^{i+l}(x)) \subseteq \mathrm{Md}(f^j(x) \approx f^{j+m}(x))$ *if and only if* $i \leqslant j$ *and* l/m.

Proof. The assertion follows from 2.5 and 2.6. □

2.8. Proposition. *Let $r, i, j \in \mathbb{N}_0$, $i < j$, $s \in \mathbb{N}$. Then* $f^i(x) \approx f^j(x) \in \mathrm{Cl}(f^r(x) \approx f^{r+s}(x))$ *if and only if* $i \geqslant r$ *and* $s/j - i$.

Proof. According to 2.3, $f^i(x) \approx f^j(x) \in \mathrm{Cl}(f^r(x) \approx f^{r+s}(x))$ if and only if $\mathrm{Md}(f^r(x) \approx f^{r+s}(x)) \subseteq \mathrm{Md}(f^i(x) \approx f^j(x))$. Since $i < j$, we have $j - i \in \mathbb{N}$ and $\mathrm{Md}(f^i(x) \approx f^j(x)) = \mathrm{Md}(f^i(x) \approx f^{i+(j-i)}(x))$. We can use 2.7(iii). □

2.9. Proposition. *Let $r, i, j \in \mathbb{N}_0$, $i \leqslant j$. Then*
(i) $f^i(x) \approx f^j(x) \in \mathrm{Cl}(f^r(x) \approx f^r(y))$ *if and only if* $i \geqslant r$ *or* $i = j$;
(ii) $f^i(x) \approx f^j(y) \in \mathrm{Cl}(f^r(x) \approx f^r(y))$ *if and only if* $i \geqslant r$.

Proof. (i) If $i = j$, then $f^i(x) \approx f^j(x)$ is a trivial equation and hence $f^i(x) \approx f^j(x) \in \mathrm{Cl}(f^r(x) \approx f^r(y))$. Now let $i < j$. By 2.3, $f^i(x) \approx f^j(x) \in \mathrm{Cl}(f^r(x) \approx f^r(y))$ if and only if $\mathrm{Md}(f^r(x) \approx f^r(y)) \subseteq \mathrm{Md}(f^i(x) \approx f^j(x))$. But $\mathrm{Md}(f^i(x) \approx f^j(x)) = \mathrm{Md}(f^i(x) \approx f^{i+(j-i)}(x))$, where $j - i \in \mathbb{N}$, and using 2.7(ii) we get the desired assertion.

(ii) Again, by applying 2.3 we have $f^i(x) \approx f^j(y) \in \mathrm{Cl}(f^r(x) \approx f^r(y))$ if and only if $\mathrm{Md}(f^r(x) \approx f^r(y)) \subseteq \mathrm{Md}(f^i(x) \approx f^j(y))$. From 2.4 it follows that the last inclusion is true if and only if $\mathrm{Md}(f^r(x) \approx f^r(y)) \subseteq \mathrm{Md}(f^i(x) \approx f^i(y))$. Then 2.7(i) completes the proof. □

3.1. Lemma. *If E is empty or consists of trivial equations only, then* $\mathrm{Md}_w(E) = \mathrm{Md}_w(\mathrm{Cl}(E))$.

Proof. Every partial monounary algebra weakly satisfies any trivial equation, so $\mathrm{Md}_w(E)$ is the class of all partial monounary algebras, whenever the assumptions of the lemma are fulfilled. Then $\mathrm{Cl}(E)$ is the set of all trivial equations and hence $\mathrm{Md}_w(\mathrm{Cl}(E))$ is the class of all partial monounary algebras, too. □

From now on let E be an arbitrary fixed set of equations.

3.2. Assumption. Suppose (from now up to 3.10) that E satisfies the following three conditions:

(i) E is nonempty;

(ii) E does not contain any trivial equation;

(iii) if $f^i(x) \approx f^j(z) \in E$, where $i, j \in \mathbb{N}_0$, $z \in \{x, y\}$, then $i \leqslant j$.

Denote $\mathscr{K} = \mathrm{Md}_w(E)$ and $\mathscr{K}^* = \mathrm{Md}_w(\mathrm{Cl}(E))$. It is easy to see that $\mathscr{K}^* \subseteq \mathscr{K}$. The question is: under which conditions the relation $\mathscr{K}^* = \mathscr{K}$ is valid?

3.3. Definition. Put

$$k = \min\{i \in \mathbb{N}_0\colon \text{ there are } j \in \mathbb{N}_0, z \in \{x, y\} \text{ such that } f^i(x) \approx f^j(z) \in E\}.$$

The set E is nonempty, therefore such a k ($\in \mathbb{N}_0$) exists.

We distinguish two cases:

(1) E contains only regular equations.

We put

$$n = \text{g.c.d.}\{j - i\colon i, j \in \mathbb{N}_0 \text{ are such that } f^i(x) \approx f^j(x) \in E\}.$$

Such an n ($\in \mathbb{N}$) exists because in this case all equations in E are nontrivial and regular. We define $e(E)$ as the equation $f^k(x) \approx f^{k+n}(x)$.

(2) E contains a nonregular equation.

In this case we define $e(E)$ as the equation $f^k(x) \approx f^k(y)$.

The equation $e(E)$ will be called the *basic equation* to the set E.

Notice that the basic equation to the set E need not belong to E. Let $E = \{x \approx f^3(x), f(x) \approx f^2(x)\}$. Then $k = 0$, $n = 1$ and so $e(E)$ is the equation of the form $x \approx f(x)$. We see that $e(E) \notin E$.

3.4. Proposition. $\mathrm{Cl}(e(E)) = \mathrm{Cl}(E)$.

Proof. We distinguish two cases:

(1) E is the set of regular equations.

Then $e(E)$ is the equation $f^k(x) \approx f^{k+n}(x)$, where $k \in \mathbb{N}_0$, $n \in \mathbb{N}$. Let $f^i(x) \approx f^j(x)$ ($i \in \mathbb{N}_0$, $j \in \mathbb{N}$) be any equation of E. By the definition of $e(E)$, $k \leqslant i$ and $n/j - i$. Then 2.8 implies $f^i(x) \approx f^j(x) \in \mathrm{Cl}(f^k(x) \approx f^{k+n}(x)) = \mathrm{Cl}(e(E))$. We have proved $E \subseteq \mathrm{Cl}(e(E))$ and thus $\mathrm{Cl}(E) \subseteq \mathrm{Cl}(e(E))$.

Conversely, it suffices to show that $\mathrm{Md}(E) \subseteq \mathrm{Md}(e(E))$ (see 2.3). According to 3.3 there exist $i_1, j_1 \in \mathbb{N}_0$ such that $f^{i_1}(x) \approx f^{j_1}(x) \in E$ and $i_1 = k$. Further, there exist $m \in \mathbb{N}$, $i_2, j_2, i_3, j_3, \ldots, i_m, j_m \in \mathbb{N}_0$ such that $f^{i_2}(x) \approx f^{j_2}(x)$, $f^{i_3}(x) \approx f^{j_3}(x)$, ..., $f^{i_m}(x) \approx f^{j_m}(x) \in E$ and $n = \text{g.c.d.}\{j_2 - i_2, j_3 - i_3, \ldots, j_m - i_m\}$ (it is true even in the case when E is infinite).

Let $\mathbf{A} \in \mathrm{Md}(E)$. Then $\mathbf{A} \in \mathrm{Md}(f^{i_l}(x) \approx f^{j_l}(x))$ for $l = 1, \ldots, m$. So we have

$$\mathbf{A} \in \bigcap_{l=1}^{m} \mathrm{Md}(f^{i_l}(x) \approx f^{j_l}(x)) = \bigcap_{l=1}^{m} \mathrm{Md}(f^{i_l}(x) \approx f^{i_l+(j_l-i_l)}(x)),$$

where $i_l \in \mathbb{N}_0$ and $j_l - i_l \in \mathbb{N}$ for all $l \in \{1, \ldots, m\}$. Using 2.6(iii) (repeatedly) we get

$$\mathbf{A} \in \mathrm{Md}(f^{\min\{i_1,\ldots,i_m\}}(x) \approx f^{\min\{i_1,\ldots,i_m\}+\text{g.c.d.}\{j_1-i_1,j_2-i_2,\ldots,j_m-i_m\}}(x)).$$

Obviously $\min\{i_1, \ldots, i_m\} = k$ and $\text{g.c.d.}\{j_1 - i_1, j_2 - i_2, \ldots, j_m - i_m\} = n$ (see the definition of k and n). Hence $\mathbf{A} \in \mathrm{Md}(f^k(x) \approx f^{k+n}(x)) = \mathrm{Md}(e(E))$ and therefore $\mathrm{Md}(E) \subseteq \mathrm{Md}(e(E))$.

(2) E contains a nonregular equation.

In this case $e(E)$ is the equation $f^k(x) \approx f^k(y)$. Let $f^i(x) \approx f^j(x) \in E$, where $i \in \mathbb{N}_0$, $j \in \mathbb{N}$. According to 3.3 we have $k \leqslant i$. Then 2.9(i) implies that $f^i(x) \approx f^j(x) \in \mathrm{Cl}(f^k(x) \approx f^k(y)) = \mathrm{Cl}(e(E))$. Similarly, if $f^r(x) \approx f^s(y) \in E$ ($r, s \in \mathbb{N}_0$) then by 3.3 we get $k \leqslant r$ and using 2.9(ii) we obtain $f^r(x) \approx f^s(y) \in \mathrm{Cl}(f^k(x) \approx f^k(y)) = \mathrm{Cl}(e(E))$. Thus $E \subseteq \mathrm{Cl}(e(E))$ and this yields $\mathrm{Cl}(E) \subseteq \mathrm{Cl}(e(E))$.

It remains to prove the opposite inclusion. By the definition of k there exist $i, j \in \mathbb{N}_0$, $z \in \{x, y\}$ such that $f^i(x) \approx f^j(z) \in E$ and $i = k$. If $z = y$, then we have $f^k(x) \approx f^j(y) \in E$ and thus $\mathrm{Cl}(f^k(x) \approx f^j(y)) \subseteq \mathrm{Cl}(E)$. Then by 2.4 $\mathrm{Md}(f^k(x) \approx f^j(y)) = \mathrm{Md}(f^k(x) \approx f^k(y))$ and hence $\mathrm{Eq}(\mathrm{Md}(f^k(x) \approx f^j(y))) = \mathrm{Eq}(\mathrm{Md}(f^k(x) \approx f^k(y)))$, which means $\mathrm{Cl}(f^k(x) \approx f^j(y)) = \mathrm{Cl}(f^k(x) \approx f^k(y))$. We obtain $\mathrm{Cl}(f^k(x) \approx f^k(y)) \subseteq \mathrm{Cl}(E)$ and thus $\mathrm{Cl}(e(E)) \subseteq \mathrm{Cl}(E)$. If $z = x$, then we have $f^k(x) \approx f^j(x) \in E$. Note that $j > k$. Since E contains a nonregular equation, there exist $r, s \in \mathbb{N}_0$ such that $f^r(x) \approx f^s(y) \in E$. Clearly $k \leqslant r$. Let $\mathbf{A} \in \mathrm{Md}(E)$. Then

$\mathbf{A} \in \mathrm{Md}(f^k(x) \approx f^j(x))$ and $\mathbf{A} \in \mathrm{Md}(f^r(x) \approx f^s(y))$. Therefore $\mathbf{A} \in \mathrm{Md}(f^k(x) \approx f^j(x)) \cap \mathrm{Md}(f^r(x) \approx f^s(y))$. But $\mathrm{Md}(f^k(x) \approx f^j(x)) \cap \mathrm{Md}(f^r(x) \approx f^s(y)) = \mathrm{Md}(f^(x) \approx f^{k+(j-k)}(x)) \cap \mathrm{Md}(f^r(x) \approx f^r(y)) = \mathrm{Md}(f^{\min(k,r)}(x) \approx f^{\min(k,r)}(y)) = \mathrm{Md}(f^k(x) \approx f^k(y)) = \mathrm{Md}(e(E))$ by virtue of 2.4 and 2.6(ii). We have proved that $\mathrm{Md}(E) \subseteq \mathrm{Md}(e(E))$, and 2.3 yields $\mathrm{Cl}(e(E)) \subseteq \mathrm{Cl}(E)$. □

3.5. Corollary. *Let* **A** *be a partial monounary algebra. If* **A** *does not weakly satisfy* $e(E)$, *then* $\mathbf{A} \notin \mathscr{K}^*$.

Proof. If **A** does not weakly satisfy $e(E)$, then $\mathbf{A} \notin \mathrm{Md}_w(e(E))$. Since obviously $\mathrm{Md}_w(\mathrm{Cl}(e(E))) \subseteq \mathrm{Md}_w(e(E))$, we have $\mathbf{A} \notin \mathrm{Md}_w(\mathrm{Cl}(e(E)))$ as well. By 3.4, $\mathrm{Cl}(e(E)) = \mathrm{Cl}(E)$, thus we get $\mathbf{A} \notin \mathrm{Md}_w(\mathrm{Cl}(E)) = \mathscr{K}^*$. □

For $i, j \in \mathbb{N}_0$ we denote $[i, j] = \{l \in \mathbb{N}_0 : i \leqslant l \leqslant j\}$.

3.6. Lemma. *If* E *is a set of regular equations and* $e(E) \notin E$, *then* $\mathscr{K}^* \neq \mathscr{K}$.

Proof. Suppose that E is a set of regular equations. Then $e(E)$ is the equation $f^k(x) \approx f^{k+n}(x)$, where $k \in \mathbb{N}_0$, $n \in \mathbb{N}$. Consider a partial monounary algebra $\mathbf{A} = (A, f)$ (if no misunderstanding can occur, we write f instead of $f^{\mathbf{A}}$) such that

$A = [0, k+n]$,

$f(i) = i + 1$ for $i \in [0, k+n-1]$, $f(k+n)$ is not defined.

The equation $f^k(x) \approx f^{k+n}(x)$ is not weakly satisfied in **A**, because $f^k(0) = k \neq k + n = f^{k+n}(0)$. Thus **A** does not weakly satisfy $e(E)$, and 3.5 implies that $\mathbf{A} \notin \mathscr{K}^*$. We will show that $\mathbf{A} \in \mathscr{K}$.

Let $f^i(x) \approx f^j(x) \in E$, where $i, j \in N_0$. Then $i < j$ and according to the definition of k we have $k \leqslant i$. Similarly $n \leqslant j - i$. Thus $k + n \leqslant i + (j - i) = j$ and the equality $k + n = j$ holds if and only if $i = k$, $j - i = n$. The assumption $f^k(x) \approx f^{k+n}(x)$ $(= e(E)) \notin E$ implies that $k + n < j$. This yields that f^j is not defined on any element of **A**. Then obviously $f^i(x) \approx f^j(x)$ is weakly satisfied in **A**. So **A** weakly satisfies each equation of E and hence $\mathbf{A} \in \mathrm{Md}_w(E) = \mathscr{K}$. □

3.7. Lemma. *If* $e(E) \notin E$, *then* $\mathscr{K}^* \neq \mathscr{K}$.

Proof. According to the previous lemma it suffices to consider the case when E contains a nonregular equation. In such a case $e(E)$ is the equation $f^k(x) \approx f^k(y)$. Let $\mathbf{A} = (A, f)$ be a partial monounary algebra such that

$$A = [0,1] \times [0,k],$$
$$f((i,j)) = (i, j+1) \quad \text{for } i \in [0,1], j \in [0, k-1],$$
$$f((0,k)), f((1,k)) \quad \text{are not defined.}$$

(Notice that if $k = 0$, then f is not defined anywhere in $\mathbf{A}$.) $\mathbf{A}$ does not weakly satisfy the equation $f^k(x) \approx f^k(y)$, because $f^k((0,0)) = (0,k) \neq (1,k) = f^k((1,0))$. Thus $\mathbf{A} \notin \mathscr{K}^*$ in view of 3.5. We will show that $\mathbf{A} \in \mathscr{K}$.

Let $f^i(x) \approx f^j(y) \in E$. Then $k \leqslant i \leqslant j$, and $k = j$ only in the case when $k = i = j$. But then we have $f^k(x) \approx f^k(y) \in E$, i.e. $e(E) \in E$, which is a contradiction with the assumption. Therefore $k < j$ and we can see that f^j is not defined in $\mathbf{A}$ and hence $f^i(x) \approx f^j(y)$ is clearly weakly satisfied in $\mathbf{A}$.

Let $f^r(x) \approx f^s(x) \in E$. Then $r < s$ and $k \leqslant r$. Thus $k < s$, which means that f^s is not defined in $\mathbf{A}$. Then $f^r(x) \approx f^s(x)$ is weakly satisfied in $\mathbf{A}$.

We have shown that each equation of E is weakly satisfied in $\mathbf{A}$, hence $\mathbf{A} \in \mathscr{K}$.

□

3.8. Lemma. *If E is a set of regular equations and $e(E) \in E$, then $\mathscr{K}^* = \mathscr{K}$.*

Proof. Let $\mathbf{A} = (A, f) \in \mathscr{K}$. We will show that $\mathbf{A} \in \mathscr{K}^*$ (the relation $\mathscr{K}^* \subseteq \mathscr{K}$ is always true). We need to prove that $\mathbf{A}$ weakly satisfies all equations of $\mathrm{Cl}(E)$.

Let $i, j \in \mathbb{N}_0$ be such that $f^i(x) \approx f^j(x) \in \mathrm{Cl}(E)$. Without loss of generality we may suppose that $i < j$ because $\mathbf{A}$ weakly satisfies the equation $f^i(x) \approx f^j(x)$ if and only if it weakly satisfies the equation $f^j(x) \approx f^i(x)$. Since E is a set of regular equations, $e(E)$ is the equation $f^k(x) \approx f^{k+n}(x)$. By 3.4 $\mathrm{Cl}(E) = \mathrm{Cl}(f^k(x) \approx f^{k+n}(x))$, thus $f^i(x) \approx f^j(x) \in \mathrm{Cl}(f^k(x) \approx f^{k+n}(x))$. From 2.8 it follows that $k \leqslant i$ and $n/j - i$. Then there exist $d \in \mathbb{N}$, $l \in \mathbb{N}_0$ with $j - i = dn$ and $i = k + l$.

Let $a \in A$ be such that $f^i(a)$ and $f^j(a)$ are defined. It suffices to show that $f^i(a) = f^j(a)$. We have

$$f^i(a) = f^{k+l}(a), f^j(a) = f^{i+(j-i)}(a) = f^{k+l+dn}(a). \tag{1}$$

Since $f^j(a) = f^{k+l+dn}(a)$ is defined, we conclude that $f^{k+l+(d-1)n}(a)$ is defined. By virtue of the relation $k + l + dn = k + n + l + (d-1)n$ we get

$$f^{k+l+dn}(a) = f^{k+n+l+(d-1)n}(a) = f^{k+n}(f^{l+(d-1)n}(a)), \tag{2}$$

$$f^{k+l+(d-1)n}(a) = f^k(f^{l+(d-1)n}(a)). \tag{3}$$

Thus we have $f^{l+(d-1)n}(a) \in A$ and $f^k(f^{l+(d-1)n}(a))$, $f^{k+n}(f^{l+(d-1)n}(a))$ are defined. By the assumption of the lemma $e(E) \in E$, so $\mathbf{A}$ ($\in \mathscr{K}$) weakly satisfies $f^k(x) \approx f^{k+n}(x)$. Then $f^k(f^{l+(d-1)n}(a)) = f^{k+n}(f^{l+(d-1)n}(a))$. According to (2) and (3) we have proved that $f^{k+l+(d-1)n}(a) = f^{k+l+dn}(a)$. Repeating this process we get $f^{k+l}(a) = f^{k+l+dn}(a)$ and hence $f^i(a) = f^j(a)$, using (1). □

3.9. Lemma. *If* $e(E) \in E$, *then* $\mathscr{K}^* = \mathscr{K}$.

Proof. It suffices to consider the case when E contains a nonregular equation (see the previous lemma). In this case $e(E)$ is the equation $f^k(x) \approx f^k(y)$. Let $\mathbf{A} = (A, f) \in \mathscr{K}$. We will show that $\mathbf{A} \in \mathscr{K}^*$.

Let $f^i(x) \approx f^j(y)$, where $i, j \in \mathbb{N}_0$, be an arbitrary but fixed nonregular equation of $\mathrm{Cl}(E)$. We may suppose $i \leqslant j$. By 3.4 we have $\mathrm{Cl}(E) = \mathrm{Cl}(f^k(x) \approx f^k(y))$ and hence $f^i(x) \approx f^j(y) \in \mathrm{Cl}(f^k(x) \approx f^k(y))$. From 2.9(ii) it follows that $k \leqslant i$.

Let $a, b \in A$ be such that $f^i(a)$, $f^j(b)$ are defined. We will prove that $f^i(a) = f^j(b)$. Since $i \geqslant k$ and $j \geqslant i$, there exist $l, m \in \mathbb{N}_0$ such that $i = k + l$, $j = k + m$. Then $f^i(a) = f^{k+l}(a) = f^k(f^l(a))$, $f^j(b) = f^{k+m}(b) = f^k(f^m(b))$, where $f^k(f^l(a))$, $f^k(f^m(b))$ are defined and thus $f^l(a)$, $f^m(b)$ are defined. Partial algebra $\mathbf{A}$ belongs to $\mathscr{K}$, so $\mathbf{A}$ weakly satisfies each equation of E, especially $e(E) \in E$, and hence $\mathbf{A}$ weakly satisfies $f^k(x) \approx f^k(y)$. Since $f^l(a)$, $f^m(b) \in A$ and $f^k(f^l(a))$, $f^k(f^m(b))$ are defined, we obtain $f^k(f^l(a)) = f^k(f^m(b))$. Therefore $f^i(a) = f^j(b)$. We have proved that $\mathbf{A}$ weakly satisfies each nonregular equation of $\mathrm{Cl}(E)$.

Now consider a regular equation $f^r(x) \approx f^s(x) \in \mathrm{Cl}(E)$ $(r, s \in \mathbb{N}_0)$. We may suppose $r < s$. Since $\mathrm{Cl}(E) = \mathrm{Cl}(f^k(x) \approx f^k(y))$, we have $f^r(x) \approx f^s(x) \in \mathrm{Cl}(f^k(x) \approx f^k(y))$. By 2.9(i) $r \geqslant k$ and then it follows from 2.9(ii) that $f^r(x) \approx f^s(y) \in \mathrm{Cl}(f^k(x) \approx f^k(y))$. According to the first part of the proof $f^r(x) \approx f^s(y)$ is weakly satisfied in $\mathbf{A}$. Clearly also $f^r(x) \approx f^s(x)$ is weakly satisfied in $\mathbf{A}$. □

3.10. Lemma. *Let E contain a nontrivial equation. Then there exists a set of equations $\hat{E}$ such that* $\mathrm{Md}_w(E) = \mathrm{Md}_w(\mathrm{Cl}(E))$ *if and only if* $\mathrm{Md}_w(\hat{E}) = \mathrm{Md}_w(\mathrm{Cl}(\hat{E}))$ *and $\hat{E}$ satisfies* 3.2.

Proof. Obviously $\mathrm{Md}_w(E) = \mathrm{Md}_w(E_0)$ and $\mathrm{Cl}(E) = \mathrm{Cl}(E_0)$, where E_0 is the set of all nontrivial equations of E; thus $\mathrm{Md}_w(E) = \mathrm{Md}_w(\mathrm{Cl}(E))$ if and only if $\mathrm{Md}_w(E_0) = \mathrm{Md}_w(\mathrm{Cl}(E_0))$ and E_0 satisfies 3.2(i) and 3.2(ii). We put

$$\begin{aligned}\hat{E} = \{&f^i(x) \approx f^j(z)\colon f^i(x) \approx f^j(z) \in E_0, i, j \in \mathbb{N}_0, i \leqslant j, z \in \{x, y\}\}\\ \cup\{&f^j(x) \approx f^i(z)\colon f^i(x) \approx f^j(z) \in E_0, i, j \in \mathbb{N}_0, i > j, z \in \{x, y\}\}.\end{aligned}$$

Then $\mathrm{Md}_w(E_0) = \mathrm{Md}_w(\hat{E})$ and $\mathrm{Cl}(E_0) = \mathrm{Cl}(\hat{E})$, and therefore $\mathrm{Md}_w(E_0) = \mathrm{Md}_w(\mathrm{Cl}(E_0))$ if and only if $\mathrm{Md}_w(\hat{E}) = \mathrm{Md}_w(\mathrm{Cl}(\hat{E}))$. It is not difficult to see that $\hat{E}$ satisfies 3.2. □

3.11. Theorem. *Let E be a set of equations of monounary type,* $\mathscr{K} = \mathrm{Md}_w(E)$, $\mathscr{K}^* = \mathrm{Md}_w(\mathrm{Cl}(E))$.

(i) *If E is empty or consists of trivial equations only, then* $\mathscr{K}^* = \mathscr{K}$.

(ii) *If E contains a nontrivial equation and satisfies* 3.2 *(according to* 3.10 *we may assume this without loss of generality), then $\mathscr{K}^* = \mathscr{K}$ if and only if the basic equation to the set E belongs to E.*

Proof. The assertion (i) follows immediately from 3.1 and the assertion (ii) from 3.7 and 3.9. □

3.12. Example. Let $E = \{f^3(x) \approx f^5(x), f^2(x) \approx f^2(y), f^3(x) \approx f^6(x)\}$. By Definition 3.3, $e(E)$ is the equation $f^2(x) \approx f^2(y)$ and thus $e(E) \in E$. Then $\mathscr{K}^* = \mathscr{K}$ by 3.10.

Now let $E = \{x \approx f^2(x), f(x) \approx f^3(x)\}$. In this case $e(E)$ is the equation $x \approx f(x)$, $e(E) \notin E$, and thus $\mathscr{K}^* \neq \mathscr{K}$.

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