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ON HOCHSCHILD AND CYCLIC HOMOLOGY OF CERTAIN HOMOGENEOUS SPACES

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1. INTRODUCTION AND FORMULATION OF RESULTS

Cyclic and Hochschild homologies $HC_*(X)$ and $HH_*(X)$ of a topological space X have appeared to be subject of wide interest since the papers of D. Burghelea [5], D. Burghelea and Z. Fiedorowicz [6], T. Goodwillie [12] were published. Since then many papers on this theme have been written, e.g. [11], [15], [22], [23]. Since $HH_*(X)$ can be identified to $H_*(X^{S^1})$ (the homology of a free loop space X^{S^1}) and $HC_*(X)$ to the homology of the associated bundle $ES^1 \times_{S^1} X^{S^1}$ [12], cyclic and Hochschild homologies provide a powerful technique for studying a free loop space (see e.g. [15]). It should be mentioned that the investigation of various topological invariants of X^{S^1} is very important in view of their role in mathematical physics [27]. Let us note also the papers on cyclic and Hochschild homologies of algebras, on k-formality and other topological applications [11], [7]–[9], [23]–[26].

In spite of the above mentioned facts there were few papers explicitly calculating $HH_*(X)$ and $HC_*(X)$ for a given topological space X. In [22] these calculations were done for any topological space X such that its cohomology algebra $H^*(X)$ is a truncated polynomial algebra of one variable.

Later R. Krasauskas [19] calculated $HC_*(X)$ for the complex quadric and complex Grassmannian $\mathrm{Gr}(2,m) = U(m)/U(2) \times U(m-2)$ provided $m \leqslant 6$.

The author's intention when writing this paper was to demonstrate a certain approach to obtain $HH_*(M)$ in a general case of homogeneous spaces $M = G/H$ of compact Lie groups. The results of the work permit us to calculate effectively $HH_*(M)$ when $M = G/H$ is a so called Cartan pair [21]. Note that Cartan pairs constitute a wide and important class of homogeneous spaces which covers e.g. all ho-

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mogeneous spaces of maximal rank ($\mathrm{rank}(G) = \mathrm{rank}(H)$), all symmetric Riemannian spaces [17], flag manifolds [18] and some other important classes.

The importance of calculating $HH_*(M)$ comes also from the fact that there exist "Connes long exact sequences"

$$(1)\quad \ldots \to HH_*(X,k) \to HC_*(X,k) \to HC_{*-2}(X,k) \to HH_{*-1}(X,k) \to \ldots,$$

$$(2)\quad \ldots \to HH^*(X,k) \to HC^*(X,k) \to HC^{*+2}(X,k) \to HH^{*+1}(X,k) \to \ldots$$

((1) for homology and (2) for cohomology). These sequences show that $HH_*(X)$ is a "step" in the calculation of $HC_*(X)$. The results obtained in this article about $HH_*(X)$ and (1), (2) are applied to obtain new results in calculating a cyclic homology of certain topological spaces, namely, we prove a theorem that is a direct generalization of theorems of M. Vigué-Poirrier and D. Burghelea [22] and calculates $HC_*(X)$ when X is any topological space with the cohomology algebra

$$H^*(X, \mathbf{Q}) \simeq \mathbf{Q}[X_1, X_2]/(f_1, f_2)$$

where $\mathbf{Q}[X_1, X_2]$ denotes the polynomial algebra of two variables and f_1, f_2 are polynomials forming a regular sequence. Here and everywhere below the symbol (f_1, f_2) denotes the ideal generated by f_1 and f_2. The paper presents also new examples of calculation of $HC_*(M)$ for certain compact Riemannian symmetric spaces.

Let us formulate the main results of the paper (the appropriate definitions and explanations are contained in the next section, but we try to use the traditional terminology and notation). As usual, for a graded vector space V its Poincaré polynomial will be denoted by $P_V(t)$.

Theorem 1. *Let $M = G/H$ be a simply connected compact homogeneous space generated by a Cartan pair (G, H).*

(i) *Then its Cartan algebra is quasiisomorphic to the graded commutative algebra (C, d) of the form*

$$(3)\quad \begin{aligned} &(C,d) = \big(\mathbf{Q}[X_1,\ldots,X_s] \otimes \wedge(y_1,\ldots,y_n), d\big),\\ &d(X_i) = 0, \quad i = 1,\ldots,s,\ \deg(X_i) = 2k_i,\ k_i \geqslant 1,\ s = \mathrm{rank}(H),\\ &d(y_j) = f_j(X_1,\ldots,X_s), \quad j = 1,\ldots,s,\ \deg(y_j) = 2l_j - 1,\ l_j \geqslant 1,\\ &d(y_k) = 0, \quad k = s+1,\ldots,n,\ \deg(y_k) = 2l_k - 1,\ l_k \geqslant 1,\ n = \mathrm{rank}(G). \end{aligned}$$

(ii) *Introduce the graded differential algebra of the form*

$$A = \mathbf{Q}[X_1,\ldots,X_s] \otimes \wedge(x_1,\ldots,x_s)$$

where $\wedge(x_1, \ldots, x_s)$ *denotes the exterior algebra over the vector space with the base* $x_1, \ldots, x_s$ *of the degrees* $\deg(x_p) = \deg(X_p) - 1$. *Denote*

$$a_j = \sum_{i=1}^{s} \partial f_j / \partial X_i \otimes x_i. \tag{4}$$

Then

$$\begin{aligned} P_{HH_*(M)}(t) = & \prod_{j=s+1}^{n} (1 + t^{2l_j - 1}) \prod_{j=1}^{s} (1 - t^{2l_j}) \prod_{j=s+1}^{n} (1 - t^{2l_j})^{-1} \\ & \times \left\{ \Big(\prod_{i=1}^{s} (1 + t^{2k_i - 1})(1 - t^{2k_i})^{-1} \prod_{j=1}^{n} (1 - t^{2l_j})^{-1} \Big) - (t-1) P_{B(L)}(t) \right\} \end{aligned} \tag{5}$$

where $B(L)$ *is a subspace of coboundaries of the graded commutative differential algebra* (L, d) *of the form* $(L, d) = (A \otimes \mathbf{Q}[Y_1, \ldots, Y_s, Y_{s+1}, \ldots, Y_n], d)$ *with* $d\big|_A = 0$ *and* $d(Y_i) = a_i$, $i = 1, \ldots, s$, $d(Y_j) = 0$, $j = s+1, \ldots, n$, $\deg(Y_p) = \deg(y_p) - 1$.

Remark. The numbers $2k_i$, $2l_j - 1$ can be effectively calculated [13]. Explicit calculations for various G and H can be found in [10], [2], [3], [13]. Formula (5) shows that the problem of calculating $P_{HH^*(M)}(t)$ is reduced to an algebraic one of calculating a subalgebra of coboundaries $B(L)$. Theorem 2 of the present paper shows that the explicit calculations in certain cases are possible, namely, we apply Theorem 1 to calculations of a cyclic homology. Observe also that the algebra L is a free algebra of simpler form then the initial algebra used for calculation of $P_{HH^*(M)}(t)$ in [22].

Theorem 2. *Let* M *be any simply connected topological space satisfying the condition:*

the cohomology algebra of M *is a commutative algebra of the form*

$$H^*(M, \mathbf{Q}) \simeq \mathbf{Q}[X_1, X_2]/(f_1, f_2), \tag{6}$$

where f_1 *and* f_2 *are polynomials forming a regular sequence.*

Then the Poincaré polynomial for the cyclic homology of M *is determined by the formulae*

$$P_{HC^*}(t) = (t+1) P_{HC^*_{(0)}}(t) + (t^{-1} + 1) P_{HC^*_{(2)}}(t) \tag{7}$$

$$P_{HC^*_{(0)}}(t) = 1/(1 - t^2), \quad P_{HC^*_{(2)}}(t) = t^{2k_1 + 2k_2 + 2} P_{H^*(M)}(t) \Big/ \prod_{j=1}^{s} (1 - t^{2l_j}) \tag{8}$$

where $2k_1$, $2k_2$, $2l_1 - 1$, $2l_2 - 1$ are the degrees of the even or odd generators of the minimal model of M, respectively.

Theorem 3. *Compact simply connected Riemannian symmetric spaces*

$$\begin{aligned} M_1 &= \mathrm{Sp}(3)/U(3), & M_2 &= AdE_6/T^1\cdot \mathrm{Spin}(10),\\ M_3 &= E_7/SU(2)\cdot \mathrm{Spin}(12), & M_4 &= F_4/SU(2)\times \mathrm{Sp}(3),\\ M_5 &= \mathrm{Sp}(m)/\mathrm{Sp}(2)\times \mathrm{Sp}(m-2) \end{aligned}$$

satisfy the conditions of Theorem 2.

Corollary (example of explicit calculation). *Poincaré polynomials for cyclic homologies of M_1 and M_2 from Theorem 3 are determined by the formula (8) where the exact expressions for $P_{HC^\bullet_{(2)}}(t)$ are of the form*

$$(9)\qquad (M_1)\qquad P_{HC^\bullet_{(2)}}(t) = t^{12}/((1-t^2)(1-t^6)),$$

$$(10)\qquad (M_2)\qquad P_{HC^\bullet_{(2)}}(t) = t^{14}(1+t^2+t^4+\ldots+t^{16})\,(1+t^8+t^{16})/((1-t^4)(1-t^6)).$$

R e m a r k . (1) When the present paper was under consideration in the journal, the paper [28] of mine appeared, where a weaker version of Theorem 2 was proved, namely formula (7) was obtained and the summand $P_{HC^\bullet_{(2)}}(t)$ was expressed by means of the Hochschild homology.

(2) Everywhere in the paper we consider Poincaré polynomials, but sometimes for brevity we use the words "homology" or "cohomology". Nevertheless we do not consider the algebra structure on $HH^*(M)$.

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2. Preliminaries

Everywhere in the paper we use the independent numbering of formulae. The cross-references are used in doubled form, the first number being the number of the section.

We consider the category $k\text{-}ADG_{(c)}$ of differential k-algebras, $\mathbf{N}$-graded, associative and commutative in the graded sense (see [20]) over the field k of zero characteristics (in most cases $k = \mathbf{Q}$) and of a finite type. If $(A, d) \in k\text{-}ADG_{(c)}$, then $H^*(A, d)$ denotes its cohomology algebra. If V is any graded vector space with the grading

$$V = \bigoplus_{i \in \mathbf{Z}} V_i$$

then the degree of a homogeneous element $v \in V$ is denoted by $\deg(v)$. We use the symbol $P_V(t)$ for the Poincaré polynomial of V. The definitions of the cyclic and Hochschild homologies of $(A, d) \in k\text{-}ADG_{(c)}$ ($HH_*(A, d)$, $HC_*(A, d)$) can be found in [5]. We reproduce here only the definition of the Hochschild and cyclic homologies of a topological space X. Denote by MX the Moore loop space on X [1], and by $C_*(MX)$ its algebra of singular k-chains.

Definition 1 ([5], [12]). Put by definition

$$HH_*(X) = HH_*(C_*(MX)), \quad HC_*(X) = HC_*(C_*(MX))$$

and call $HH_*(X)$ and $HC_*(X)$ respectively the Hochschild and the cyclic homology of the topological space X.

Now let us recall some facts connected with the cohomology of homogeneous spaces. Let G be a compact connected Lie group, H its closed subgroup. Everywhere below the Lie algebras of Lie groups G, H, ... are denoted by the corresponding Gothic letters $\mathfrak{G}$, $\mathfrak{H}$, ... Let $W \leqslant GL(V)$ be a discrete subgroup of $GL(V)$ generated by reflections. Let $k[V]$ denote the symmetric algebra over the vector space V. Consider the extension of W-action on $k[V]$ and denote the ring of W-invariants by $k[V]^W$. In particular, consider the maximal torus T of a Lie group G, its Weyl group $W(G, T)$ and the algebra

$$\mathbf{Q}[\mathfrak{T}_{\mathbf{Q}}]^{W(G,T)}$$

(here $\mathfrak{T}_{\mathbf{Q}}$ denotes the $\mathbf{Q}$-structure on $\mathfrak{T}$, that is $\mathfrak{T}_{\mathbf{Q}} = \{v \in \mathfrak{T}^{\mathbf{C}},\ \alpha(v) \in \mathbf{Q}$, for any root $\alpha \in$ root system $R(\mathfrak{G}^{\mathbf{C}}, \mathfrak{T}^{\mathbf{C}})\}$). The well-known Chevalley theorem implies

$$\mathbf{Q}[\mathfrak{T}_{\mathbf{Q}}]^{W(G,T)} \simeq \mathbf{Q}[f_1, \dots, f_n] \tag{1}$$

where f_i are the algebraically independent generators. Consider the homogeneous space G/H and choose the maximal tori T and T' in G and H in such a way that $T' \subset T$. Consider also the algebra of invariants

$$\mathbf{Q}[\mathfrak{T}'_{\mathbf{Q}}]^{W(H,T')} \simeq \mathbf{Q}[u_1, \ldots, u_s].$$

Denote by $\wedge(V)$ the exterior algebra over the vector space V. If the base x_1, ..., x_k is chosen, we use also the notation $\wedge(x_1, \ldots, x_k)$. If V is a graded vector space the vectors x_i have odd degrees $\deg(x_i) = 2l_i - 1$. As usual $\wedge_k(V)$ denotes the subspace of all the elements of degree k.

It is well-know that

$$H^*(G) \simeq \wedge(x_1, \ldots, x_n), \quad n = \mathrm{rank}(G),$$

where x_i are the primitive elements in $H^*(G)$.

Definition 2. The algebra $(C', d') \in \mathbf{Q}\text{-}ADG_{(c)}$ of the form

$$(C', d') = \left(\mathbf{Q}[\mathfrak{T}'_{\mathbf{Q}}]^{W(H,T')} \otimes \wedge(x_1, \ldots, x_n), d\right) \tag{2}$$

$$d(u) = 0 \text{ for any } u \in \mathbf{Q}[\mathfrak{T}'_{\mathbf{Q}}]^{W(H,T')}$$

$$d(x_i) = f_i\big|_{\mathfrak{T}'} = \tilde{f}_i(u_1, \ldots, u_s) \tag{3}$$

where f_i ($i = 1$, ..., $n = \mathrm{rank}(G)$) are defined by (1), is called a Cartan algebra of the homogeneous space G/H.

Remark 1. To obtain the above definition in the form (2)–(3) it is enough to combine isomorphisms in [13, p. 565] and the definition of Koszul's complex in [13, p. 420].

Remark 2. It was proved in [2, 13] that

$$H^*(M, \mathbf{Q}) \simeq H^*(C', d')$$

if $M = G/H$ with a reductive Lie group G.

The main object of study in the paper is the class of Cartan pairs. Let us introduce the appropriate definitions. Everywhere below for the ring of polynomials $\mathbf{Q}[X_1, \ldots, X_n]$ of variables X_1, ..., X_n we shall denote by the symbol $(f_1, \ldots, f_l)$ the ideal generated by polynomials $f_1, \ldots, f_l$.

Definition 3. A graded differential algebra $(C', d') \in \mathbf{Q}\text{-}ADG_{(c)}$ of the form (2)–(3) is called normal if $\tilde{f}_i \in \mathbf{Q}[u_1, \ldots, u_s]$ satisfy the conditions: (i) there exists a sequence $\tilde{f}_1, \ldots, \tilde{f}_k$ $(k \leqslant s)$ such that $(\tilde{f}_1, \ldots, \tilde{f}_s) = (\tilde{f}_1, \ldots, \tilde{f}_k)$ and for any $q \in \mathbf{Q}[u_1, \ldots, u_s]$ the implication

$$q \cdot \tilde{f}_i \in (\tilde{f}_1, \ldots, \tilde{f}_{i-1}) \Rightarrow q \in (\tilde{f}_1, \ldots, \tilde{f}_{i-1}), \qquad i \leqslant k \tag{4}$$

is valid (it means that the sequence $f_1, \ldots, f_s$ is regular), (ii) the elements $f_1, \ldots, f_s$ are decomposable, that is f_i are polynomials of only such variables u_j, for which $\deg(u_j) \leqslant \deg(x_i)$.

Definition 4. A homogeneous space G/H of a compact Lie group G is said to be generated by a Cartan pair (G, H) if its Cartan algebra (2)–(3) is normal in the sense of Definition 3.

Remark 3. We use the above definition in the form proposed by P. Rashevskii [21].

Proposition 1 [13]. (i) *Any compact homogeneous space G/H of maximal rank $((\mathrm{rank}(G) = \mathrm{rank}(H))$ is generated by a Cartan pair.*

(ii) *Any compact Riemannian symmetric space G/H is generated by a Cartan pair.*

Let (G, H) be a Cartan pair, $n = \mathrm{rank}(G)$, $s = \mathrm{rank}(H)$ and let $H^*(M, \mathbf{Q})$ be the cohomology algebra of $M = G/H$ with rational coefficients. According to Definition 2 the Cartan algebra of M is defined by (2). According to Definition 3 $\tilde{f}_i$ from (2)–(3) satisfy (4). It is well-known [2, 21] that any graded differential algebra satisfying (2)–(4) is quasiisomorphic to the algebra $(C, d) \in \mathbf{Q}\text{-}ADG_{(c)}$ of the form

$$\begin{aligned}
&(C, d) = (\mathbf{Q}[X_1, \ldots, X_s] \otimes \wedge(y_1, \ldots, y_n), d),\\
&d(X_i) = 0, \quad i = 1, \ldots, s, \ \deg(X_i) = 2k_i,\\
&d(y_j) = f_j(X_1, \ldots, X_s), \quad j = 1, \ldots, s, \ \deg(y_j) = 2l_j - 1,\\
&d(y_j) = 0, \quad j = s+1, \ldots, n, \ \deg(y_j) = 2l_j - 1
\end{aligned} \tag{5}$$

(the derivation d has the degree $(+1)$).

Remark 4. Recall that two graded differential algebras are quasiisomorphic if there exists a homomorphism between them inducing an isomorphism in cohomology.

Introduce the Poincaré polynomials

$$P_G(t) = (1 + t^{2l_1 - 1}) \ldots (1 + t^{2l_n - 1}),$$
$$P_H(t) = (1 + t^{2k_1 - 1}) \ldots (1 + t^{2k_s - 1})$$

and recall that [2], [13]

$$(6) \qquad P_{H^*(M)}(t) = \prod_{j=1}^{s} ((1 - t^{2l_j})/(1 - t^{2k_j})) \cdot \prod_{j=s+1}^{n} (1 + t^{2l_j - 1}).$$

Let us recall certain facts from rational homotopy theory, referring to [14], [20] for details. Recall that a graded differential algebra $(\mathfrak{M}, d)$ is called minimal if $\mathfrak{M}$ is a free graded commutative algebra satisfying the conditions

$$\mathfrak{M} = k[W^{\text{even}}] \otimes \wedge(W^{\text{odd}}),$$

(a) $W = \bigoplus_{\alpha \in I} W_\alpha$ (I is an ordered set);
(b) each of the spaces W_α consists of homogeneous elements;
(c) for any $\alpha \in I$, $d(W_\alpha) \subset S(\bigoplus_{\beta < \alpha} W_\beta)$
($S(K)$ denotes the subalgebra generated by K).

Definition 5 ([14], [20]). (i) Let $(A, d) \in k\text{-}ADG_{(c)}$. A minimal algebra $(\mathfrak{M}_A, D)$ is said to be the minimal model of (A, d) if there exists a homomorphism of graded differential algebras $\varrho \colon (\mathfrak{M}_A, D) \to (A, d)$ inducing isomorphism in the cohomology

$$\varrho^* \colon H^*(\mathfrak{M}_A, D) \to H^*(A, d).$$

(ii) Let X be a topological space, let $A_{\mathbf{Q}} \colon \mathcal{K} \to \mathbf{Q}\text{-}ADG_{(c)}$ be a functor from the category $\mathcal{K}$ of simplicial sets to the category $\mathbf{Q}\text{-}ADG_{(c)}$ constructed in [20] (that is, satisfying the simplicial de Rham theorem). The minimal model of the algebra $A_{\mathbf{Q}}(S_*(X)) \in \mathbf{Q}\text{-}ADG_{(c)}$ is called the minimal model of a topological space X and is denoted by

$$\mathfrak{M}_X = \mathfrak{M}_{A_{\mathbf{Q}}(S_*(X))}.$$

Here $S_*(X)$ is a simplicial set of all singular simplices of X.

3. Proof of Theorem 1

Lemma 1. *Let $M = G/H$ where (G, H) is a Cartan pair. Then there exists an isomorphism of graded differential algebras*

$$(1) \qquad (\mathfrak{M}_M, d) \simeq (C, d)$$

where (C, d) is defined by (2.5) and the polynomials $f_1, \ldots, f_s$ are defined by (2.3).

Proof. It is well-known (e.g. see [2], [21]) that

$$H^*(M,\mathbf{Q}) \simeq (\mathbf{Q}[X_1,\dots,X_s]/(f_1,\dots,f_s)) \otimes \wedge(y_{s+1},\dots,y_n)$$

where $f_1, \dots, f_s$ are the first s polynomials among $f_1, \dots, f_n$ in (2.3). As (G,H) is a Cartan pair, the ideal $(f_1,\dots,f_s)$ satisfies the conditions of Definition 3, which means that $(f_1,\dots,f_s)$ is the so called "Borel's ideal" (see [4]). Note that in fact our construction corresponds to replacing a Cartan algebra of G/H by the algebra $(C,d) \in \mathbf{Q}\text{-}ADG_{(c)}$ of the form (2.2)–(2.3) with the only change $d(y_{s+1}) = d(y_{s+2}) = \dots = d(y_n) = 0$. Here we have changed the notation for our convenience: x_i is denoted by y_i, u_i is denoted by X_i. Now it is enough to repeat the argument of [4] (§ 16) to obtain the desired isomorphism (1) ($y_{s+1}, \dots, y_n$ do not play a role). Lemma 1 is proved. □

Remark. In the case of homogeneous spaces it is possible to give an easier proof. Evidently, (C,d) is minimal and there exists a quasiisomorphism from (C,d) to the complex of de Rham forms on G/H (see [10], prop. V.4). Nevertheless, the author feels that the application of the Bousfield and Guggenheim result may be useful in other situations as well.

Now let us reproduce the formulation of a certain result of M. Vigué-Poirrier and D. Burghelea [22] used in our considerations. We introduce the following notation. For a given graded vector space $V = \bigoplus_{i\geqslant 2} V_i$ define $\bar{V}_i = V_{i+1}$ and $\bar{V} = \bigoplus_{i\geqslant 1} \bar{V}_i$. Let

$$\wedge(V) = \bigotimes_i \wedge(V_i)$$

where $\wedge(V_i)$ denotes either $k[V_i]$ when i is even or the exterior algebra when i is odd. Define the derivation

$$\beta\colon \wedge(V) \otimes \wedge(\bar{V}) \to \wedge(V) \otimes \wedge(\bar{V}) \tag{2}$$

by the equalities

$$\beta(v) = \bar{v},\ \beta(\bar{v}) = 0, \qquad v \in V,\ \bar{v} \in \bar{V}.$$

Introduce differential algebras

$$\begin{aligned} &(\mathcal{H},\delta) = \big(\wedge(V) \otimes \wedge(\bar{V}), \delta\big),\\ &\delta(v) = d(v),\ \delta(\bar{v}) = -\beta\big(d(v)\big), \qquad v \in V,\ \bar{v} \in \bar{V} \end{aligned} \tag{3}$$

and

$$\begin{aligned} &(\mathcal{B},\bar{\delta}) = \big(k[Z] \otimes \wedge(V) \otimes \wedge(\bar{V}), \bar{\delta}\big),\\ &\bar{\delta}(Z) = 0,\ \bar{\delta}(u) = \delta(u) + Z \cdot \beta(u), \qquad u \in \wedge(V) \otimes \wedge(\bar{V}) \end{aligned} \tag{4}$$

where Z is a new generator of degree 2.

The lemma below is a reformulation of the Vigué-Poirrier and Burghelea result [22].

Lemma 2. *Let $M = G/H$ where (G,H) is a Cartan pair. Then*

(i)

$$HH^*(M) \simeq H^*(\tilde{C}, \delta)$$

where

$$\begin{aligned}
&(\tilde{C}, \delta) = \big(\mathbf{Q}[X_1, \ldots, X_s] \otimes \wedge(x_1, \ldots, x_s) \otimes \wedge(y_1 \ldots, y_n) \otimes \mathbf{Q}[Y_1, \ldots, Y_n], \delta\big),\\
&\delta(X_i) = \delta(x_i) = 0, \quad i = 1, \ldots, s,\\
&\delta(y_j) = f_j(X_1, \ldots, X_s), \quad j = 1, \ldots, s,\\
&\delta(Y_j) = \sum_{i=1}^{s} (\partial f_j / \partial X_i) \otimes x_i, \quad j = 1, \ldots, s,\\
&\delta(Y_j) = 0, \quad j = s+1, \ldots, n.
\end{aligned}$$

(ii)

$$HC^*(M) = H^*(\tilde{R}, \eta)$$

where

$$\begin{aligned}
&(\tilde{R}, \eta) = \mathbf{Q}[Z] \otimes \tilde{C}, \eta),\\
&\eta(Z) = \eta(x_i) = 0, \quad i = 1, \ldots, s,\\
&\eta(X_i) = Z \otimes x_i, \quad i = 1, \ldots, s,\\
&\eta(y_j) = f_j, \quad j = 1, \ldots, s,\\
&\eta(Y_j) = \sum_{i=1}^{s} (\partial f_j / \partial X_i) \otimes x_i
\end{aligned}$$

and the degrees of the free generators X_i, x_i, Y_j, y_j are defined by the equalities

$$\deg(x_i) = \deg(X_i) - 1, \quad \deg(Y_j) = \deg(y_j) - 1.$$

P r o o f. The known result of [22] shows that if $(\mathfrak{M}_M, D) \simeq \big(\wedge(V), d\big)$, then

$$HH^*(M) \simeq H^*(\mathcal{H}, \delta),$$
$$HC^*(M) \simeq H^*(\mathcal{B}, \bar{\delta})$$

where $(\mathcal{H}, \delta)$ and $(\mathcal{B}, \bar{\delta})$ are defined by (3), (4). Using Lemma 1 one obtains $(\mathfrak{M}_M, D) \simeq (C, d)$ since (G,H) is a Cartan pair. Introducing the convenient for us notation x_i instead of $\overline{X}_i$ and Y_j instead of $\bar{y}_j$ one quickly obtains $\mathcal{H} = \overline{C}$, $\mathcal{B} = \tilde{R}$ and the assertions (i), (ii) of Lemma 2. Lemma 2 is proved. □

Consider subalgebras $C_1 \subset \tilde{C}$, $C_2 \subset \tilde{C}$ of the form

$$C_1 = \mathbf{Q}[X_1, \ldots, X_s] \otimes \wedge(y_1, \ldots, y_n),$$
$$C_2 = \mathbf{Q}[X_1, \ldots, X_s] \otimes \wedge(x_1, \ldots, x_s) \otimes \mathbf{Q}[Y_1, \ldots, Y_n].$$

Formulae (i) of Lemma 2 show that $\delta(C_1) \subset C_1$, $\delta(C_2) \subset C_2$ and therefore it is possible to construct the tensor product of graded differential algebras

$$(R, \eta) = (C_1, \delta|_{C_1}) \otimes (C_2, \delta|_{C_2}), \quad (S, \xi) = (\mathbf{Q}[X_1, \ldots, X_s], 0) \otimes (\tilde{C}, \delta).$$

Lemma 3. *The following equality is valid:*

$$P_{H^\bullet(R)}(t) = P_{H^\bullet(S)}(t). \tag{5}$$

P r o o f. Let c be a generator of R of the form

$$c = X_1^{k_1} \ldots X_s^{k_s} \otimes y_{j_1} \wedge \ldots \wedge y_{j_l} \otimes X_1^{k'_1} \ldots X_s^{k'_s} \otimes x_{i_1} \wedge \ldots \wedge x_{i_r} \otimes Y_1^{l_1} \ldots Y_n^{l_n}.$$

Then, evidently,

$$\begin{aligned} \eta(c) = X_1^{k_1} \ldots X_s^{k_s} \otimes \big(\pm \textstyle\sum f_{j_k} \otimes y_{j_1} \wedge \ldots \wedge \hat{y}_{j_k} \wedge \ldots y_{j_l} \big) \otimes X_1^{k'_1} \ldots \\ \ldots X_s^{k'_s} \otimes x_{i_1} \wedge \ldots \wedge x_{i_r} \otimes Y_1^{l_1} \ldots Y_n^{l_n} + \\ +(-1)^{a+b} X_1^{k_1} \ldots X_s^{k_s} \otimes y_{j_1} \wedge \ldots \wedge y_{j_l} \otimes X_1^{k'_1} \ldots X_s^{k'_s} \otimes x_{i_1} \wedge \ldots \\ \ldots x_{i_r} \otimes \Big(\sum_q l_q \cdot (\sum_p (\partial f_q / \partial X_p) \otimes x_p) Y_1^{l_1} \ldots Y_q^{l_q - 1} \ldots Y_n^{l_n} \Big) \end{aligned}$$

where $a = \deg(y_{j_1} \wedge \ldots \wedge y_{j_l})$, $b = \deg(x_{i_1} \wedge \ldots \wedge x_{i_r})$ and $\hat{}$ denotes the absence of the element. Considering the element $c' \in S$ of the form

$$c' = X_1^{k_1} \ldots X_s^{k_s} \otimes X_1^{k'_1} \ldots X_s^{k'_s} \otimes x_{i_1} \wedge \ldots \wedge x_{i_r} \otimes y_{j_1} \wedge \ldots \wedge y_{j_l} \wedge Y_1^{l_1} \ldots Y_n^{l_n}$$

one can easily verify, using the definition of ξ and anticommutativity of x_{i_p} and y_{j_q} that $\eta(c) = \xi(c')$ as polynomials (note that $R \simeq S$ as graded polynomial algebras). As usual, denote for any cochain complex K the subspace of q-dimensional cochains by K^q, the subspace of cocycles by $Z^q(K)$ and the subspace of coboundaries by $B^q(K)$. The equality $\eta(c') = \eta(c)$ shows that $\dim B^q(S) = \dim B^q(R)$. Using this equality and the evident formula

$$\dim H^q(K) = \dim K^q - \dim B^{q+1}(K) - \dim B^q(K)$$

one obtains (5). □

Lemma 4. *The following formula is valid:*

$$P_{HH^*(M)}(t) = \prod_{j=s+1}^{n} (1+t^{2l_j-1}) \cdot \prod_{j=1}^{s}(1-t^{2l_j}) \cdot P_{H^*(C_2)}(t). \tag{6}$$

Proof. The equality $(R,\eta) = (C_1, \delta|_{C_1}) \otimes (C_2, \delta|_{C_2})$ implies

$$P_{H^*(R)}(t) = P_{H^*(C_1)}(t) \cdot P_{H^*(C_2)}(t) = P_{H^*(M)}(t) \cdot P_{H^*(C_2)}(t) \tag{7}$$

since $(C_1, \delta|_{C_1})$ is quasiisomorphic to Cartan algebra of G/H. Analogously

$$P_{H^*(S)}(t) = P_{H^*(\tilde{C})}(t)/(1-t^{2k_1})\dots(1-t^{2k_s}). \tag{8}$$

Then Lemmas 2, 3, equalities (7), (8) and (2.6) imply (6). □

Proof of Theorem 1. (i) See formula (2.5) and the remark before. (ii) According to Lemma 4

$$P_{HH^*(M)}(t) = \prod_{j=s+1}^{n} (1+t^{2l_j-1}) \cdot \prod_{j=1}^{s}(1-t^{2l_j}) \cdot P_{H^*(C_2)}(t)$$

where

$$C_2 = \mathbf{Q}[X_1,\dots,X_s] \otimes \wedge(x_1,\dots,x_s) \otimes \mathbf{Q}[Y_1,\dots,Y_n],$$

$$\delta(X_i) = \delta(x_i) = 0, \quad i = 1,\dots,s,$$

$$\delta(Y_j) = -\sum_{j=1}^{s}(\partial f_i/\partial X_j) \otimes x_j,\ \delta(Y_{s+1}) = \dots = \delta(Y_n) = 0.$$

Thus C_2 can be represented in the form

$$(C_2,\delta) = (L,\delta) \otimes (\mathbf{Q}[Y_{s+1},\dots,Y_n],0)$$

where L is of the form satisfying the conditions of Theorem 1 for δ instead of d and $A = \mathbf{Q}[X_1,\dots,X_s] \otimes \wedge(x_1,\dots,x_s)$. Then evidently

$$\begin{aligned} P_{H^*(C_2)}(t) &= P_{H^*(L)}(t) \Big/ \prod_{j=s+1}^{n}(1-t^{2l_j}) \\ &= \left(P_L(t) - t\cdot P_{B(L)}(t) - P_{B(L)}(t)\right) \Big/ \prod_{j=s+1}^{n}(1-t^{2l_j}). \end{aligned}$$

Now, substituting the evident expression of $P_L(t)$ into the above formulae for $P_{HH^*(M)}(t)$ one obtains (1.5). □

4. Cyclic homology (proofs of Theorems 2, 3)

Proof of Theorem 2. As

$$H^*(M,\mathbb{Q}) = \mathbb{Q}[X_1,X_2]/(f_1,f_2)$$

with the regular sequence f_1, f_2 the well-known result of Bousfield and Guggenheim [4] implies

$$(\mathfrak{M}_M,D) = \big(\mathbb{Q}[X_1,X_2]\otimes\wedge(y_1,y_2)\big), \tag{1}$$

$$D(X_1)=D(X_2)=0,$$

$$D(y_j)=f_j(X_1,X_2),\quad j=1,2$$

where f_j are homogeneous polynomials. Note that (1) implies $\mathfrak{M}_M$ to be the same as in Theorem 1. This observation allows us to use all the previous calculations. Consider $(\tilde{C},\tilde{\delta})$ and $(\tilde{R},\tilde{\eta})$ as in Lemma 2. In our special case

$$\begin{aligned}
&(\tilde{C},\tilde{\delta}) = \big(\mathbb{Q}[X_1,X_2,Y_1,Y_2]\otimes\wedge(x_1,x_2,y_1,y_2),\tilde{\delta}\big),\\
&\tilde{\delta}(X_i)=\tilde{\delta}(x_i)=0,\quad i=1,2,\\
&\tilde{\delta}(Y_j) = -\big((\partial f_j/\partial X_1)\otimes x_1 + (\partial f_j/\partial X_2)\otimes x_2\big),\quad j=1,2,\\
&\tilde{\delta}(y_j)=f_j(X_1,X_2),\quad j=1,2.
\end{aligned}$$

Applying [16] it is easy to obtain

$$HH^*(M) = H^*(\overline{C},\bar{\delta}),$$

$$HC^*(M) = H^*(\overline{R},\bar{\delta})$$

where

$$\overline{C} = \mathbb{Q}[X_1,X_2,Y_1,Y_2]\otimes\wedge(x_1,x_2)/F,$$

$$F = \big(\tilde{\delta}(y_1),\tilde{\delta}(y_2)\big)\subset\tilde{C}$$

and a similar trick is applied to construct $\overline{R}$.

Use the following results from [22]:

1) there exists a short exact sequence

$$0\to\tilde{R}\xrightarrow{S}\tilde{R}\xrightarrow{\bar{\imath}}\tilde{C}\to 0 \tag{2}$$

where $S(r)=Z\cdot r$ for any $r\in R$, $\bar{\imath}\colon\tilde{R}\to\tilde{C}$ is a natural projection (recall that $\tilde{R}=\mathbb{Q}[Z]\otimes\tilde{C}$);

2) there exists an associated long exact sequence

$$\ldots \to HH^{*+1} \xrightarrow{B} HC^{*} \xrightarrow{J} HC^{*+2} \xrightarrow{I} HH^{*+2} \to \ldots \tag{3}$$

where $B([y]) = [1 \otimes \beta(y)]$ for cohomology classes $[y] \in H^*(\tilde{C}, \tilde{\delta})$, $[1 \otimes \beta(y)] \in H^*(\tilde{R}, \tilde{\eta})$. It is easy to verify that $B' = I \circ B$ has the degree (-1) and $(B')^2 = 0$. $\square$

Remark. The lemma below is due to R. Krasauskas. As the lemma was proved in his Ph.D. thesis [19] and may be not available for the reader, we briefly outline the proof.

Lemma 5 (R. Krasauskas [19]). *There exists an exact sequence*

$$0 \to \operatorname{im} J/\operatorname{im} J^2 \xrightarrow{a} H^*(HH^*, B') \xrightarrow{b} \ker J \cap \operatorname{im} J \to 0 \tag{4}$$

for the mappings a, b defined by the formulae

$$a([J(x)]) = [I(x)], \quad b([y]) = B(y). \tag{5}$$

Sketch of the proof. The correctness of (5) can be verified directly. The mapping a is a monomorphism, because if $I(x) = I \cdot B(y)$, then $I(x - B(y)) = 0$ and there exists z such that $x - B(y) = J(z)$. Thus $J(x) = J^2(z)$ and $[J(x)] = 0$ in $\operatorname{im} J/\operatorname{im} J^2$. The sequence (4) is exact, because $(b \cdot a)([J(x)]) = B \cdot I(x) = 0$ and those y for which $B(y) = 0$ can be represented in the form $y = I(x)$ and thus $y = a([J(x)])$. The mapping b is epimorphic, because the condition $z \in \ker J \cap \operatorname{im} J$ implies the existence of y such that $z = B(y)$ and $I \cdot B(y) = I(z) = 0$. Lemma 5 is proved. $\square$

Consider now the usual construction of reduced cohomologies $\widetilde{HH}^*$ and $\widetilde{HC}^*$. It is easy to verify that

$$HH^* \cong \mathbf{Q} \oplus \widetilde{HH}^*, \quad HC^* \cong \mathbf{Q}[Z] \oplus \widetilde{HC}^*. \tag{6}$$

Remark. In (3)–(6) and in several cases below we use the notation HH^* instead of $HH^*(M)$ to avoid clumsy formulae.

Lemma 6 ([23]). *For the reduced version of the exact sequence* (4) *the operator J is nilpotent.*

Consider the decomposition

$$\overline{C} = \overline{C}_0 \oplus \overline{C}_1 \oplus \overline{C}_2 \tag{7}$$

where $\overline{C}_2$ is generated by all the monomials containing the expression $x_1 \wedge x_2$, $\overline{C}_1$ is generated by the monomials of odd degrees and $\overline{C}_0$ is generated by the monomials of even degrees which do not contain $x_1 \wedge x_2$. The direct calculation shows that the following formulae hold:

$$\tag{8} \begin{aligned} \beta(X_1^{m+1}X_2^n \otimes Y_1^{s_1} \cdot Y_2^{s_2} \otimes x_2) &= (m+1)X_1^m X_2^n \otimes Y_1^{s_1} \cdot Y_2^{s_2} \otimes x_1 \wedge x_2, \\ \beta(X_1^m X_2^{n+1} \otimes Y_1^{s_1} \cdot Y_2^{s_2} \otimes x_1) &= (n+1)X_1^m X_2^n \otimes Y_1^{s_1} \cdot Y_2^{s_2} \otimes x_1 \wedge x_2. \end{aligned}$$

Put

$$z = X_1^{m+1}X_2^n \otimes Y_1^{s_1} \cdot Y_2^{s_2} \otimes x_2, \quad w = X_1^m X_2^{n+1} \otimes Y_1^{s_1} \cdot Y_2^{s_2} \otimes x_1.$$

Note that f_j from the definition of $(\overline{C}, \bar{\delta})$ are homogeneous polynomials and $f_j \in F$. Therefore

$$f_j = \sum_{lk_1+tk_2=\deg(f_j)} \alpha_1^j X_1^l X_2^t, \qquad \deg(X_1) = k_1, \ \deg(X_2) = k_2.$$

Calculating directly $\partial f_j/\partial X_1$, $\partial f_j/\partial X_2$ one can verify the formula

$$\begin{aligned} &\bar{\delta}(k_1 z + k_2 w) = \\ &= \sum_j s_j \cdot X_1^m X_2^n \big(\deg(f_j) \cdot \textstyle\sum \alpha_1^j X_1^l X_2^t\big) Y_1^{s_1} \dots Y_j^{s_j-1} \dots Y_k^{s_k} \otimes x_1 \wedge x_2 = 0 \\ &\qquad\qquad (k = 1,2, \ j = 1,2) \end{aligned}$$

in $\overline{C}$ since

$$\sum \alpha_1^j X_1^l X_2^t \in F.$$

On the other hand, comparing (8) and the above equality it is easy to notice that the mapping

$$\tag{9} B' \colon HH^*_{(1)} \to HH^*_{(2)}$$

is epimorphic since $B'([z]) = [\beta(z)]$ (here and everywhere below we denote by $HH^*_{(i)}$ and $HC^*_{(i)}$ the submodules in HH^* and HC^*, respectively, corresponding to the decomposition (7). Using Lemma 6 and (9) one obtains

$$\operatorname{im} J \cap HC^*_{(2)} = 0.$$

This equality shows that

$$\tag{10} I \colon HC^*_{(2)} \simeq HH^*_{(2)}$$

is an isomorphism. The exactness of (4) and Lemma 6 imply $\widetilde{HC}^*_{(0)} = 0$ and therefore

$$HC^*_{(0)} = \mathbf{Q}[Z]. \tag{11}$$

Consider the decomposition (7) for $\overline{R}$ and introduce the complex of graded modules

$$0 \to \overline{R}_0 \xrightarrow{\bar{\eta}} \overline{R}_1 \xrightarrow{\bar{\eta}} \overline{R}_2 \to 0.$$

It can be evidently derived from the definition of $\overline{R}_0$, $\overline{R}_1$, $\overline{R}_2$ that the Euler characteristics $\chi(R)$ of the above complex equals zero: $\chi(R) = 0$ since in any dimension p

$$\dim \overline{R}_0^p + \dim \overline{R}_2^p = \dim \overline{R}_1^p.$$

The equality $\chi(R) = 0$ and (10), (11) imply

$$\begin{aligned} P_{HC^*_{(1)}} &= t \cdot P_{HC^*_{(0)}}(t) + t^{-1} \cdot P_{HC^*_{(2)}}(t) \\ &= t/(1-t^2) + t^{-1} \cdot P_{HH^*_{(2)}}(t). \end{aligned} \tag{12}$$

Formula (12) shows that it remains to calculate $P_{HH^*_{(2)}}(t)$. Recall that $HH^*_{(2)}$ has been obtained from $\overline{C}_2$. The proofs of Lemmas 2–4 show that it is enough to apply the arguments of the above lemmas to subalgebras

$$(R_2, \eta) = (C_1, \delta|_{C_1}) \otimes (C_{2(2)}, \delta|_{C_{2(2)}})$$

and

$$(S_2, \xi) = \big(\mathbf{Q}[X_1, X_2], 0\big) \otimes (\tilde{C}_{2(2)}, \delta)$$

(here "(2)" denotes the component generated by the elements containing $x_1 \wedge x_2$). The appropriate subalgebra A from Theorem 1 (we apply it to $HH^*_{(2)}$, as in our case all the calculations are the same and depend only on the form of the minimal model) will be of the form

$$A = \mathbf{Q}[X_1, X_2] \otimes \wedge_2(x_1, x_2)$$

and therefore evidently

$$A = \operatorname{Ann}(a_1, a_2).$$

Then $P_{B(L)}$ is calculated in an evident way and using (3.7) in the proof of Lemma 4 one obtains

$$P_{HH^*_{(2)}}(t) = t^{2k_1+2k_2+2} P_{H^*(M)}(t) \Big/ \prod_{i=1}^{n} (1 - t^{2l_j})$$

and (1.7)–(1.8) are proved (taking into consideration (12)).

Proof of Theorem 3. Each of the cases is considered separately but the calculations do not differ essentially, therefore we reproduce them only in the cases corresponding to Corollary.

Consider the case $M = \mathrm{Sp}(3)/U(3)$. According to [3]

$$H^*(M,\mathbf{Q}) = S(x_1,x_2,x_3)/S^+(x_1^2,x_2^2,x_3^2)$$

where $S(x_1,x_2,x_3)$ denotes the algebra of all symmetric polynomials of the variables x_1, x_2, x_3, $S^+(x_1^2,x_2^2,x_3^2)$ denotes the ideal generated by the symmetric polynomials of positive degree, that is

$$H^*(M,\mathbf{Q}) = \mathbf{Q}[\sigma_1,\sigma_2,\sigma_3]/(\bar\sigma_1,\bar\sigma_2,\bar\sigma_2) \tag{13}$$

where σ_i are the elementary symmetric polynomials of the variables x_i, while $\bar\sigma_j$ are those of the variables x_j^2. By direct computation one obtains

$$\sigma_3^2 = \bar\sigma_3, \quad \sigma_2^2 = \bar\sigma_2 + 2\sigma_3\cdot\sigma_1, \quad \sigma_1^2 = \bar\sigma_1 + 2\sigma_2$$

and therefore it is easy to notice that the factor algebra (13) has two generators $u = \pi(\sigma_1)$, $v = \pi(\sigma_3)$ ($\pi\colon \mathbf{Q}[\sigma_1,\sigma_2,\sigma_3] \to H^*(M,\mathbf{Q})$ being a natural projection). Thus

$$H^*(M,\mathbf{Q}) \simeq \mathbf{Q}[X_1,X_2]/(f_1,f_2)$$

as M is a homogeneous space of maximal rank. Then Theorem 2 implies

$$P_{HC^\bullet_{(2)}}(t) = P_{H^\bullet(M)}(t)\cdot t^{2k_1+2k_2+2}\Big/\prod_j(1-t^{2l_j}).$$

According to (2.6)

$$P_{H^\bullet(M)}(t) = (1+t^2)(1+t^4)(1+t^6).$$

Isomorphism (13) shows that $(\mathfrak{M}_M, D)$ can be calculated by a theorem of Bousfield and Guggenheim [4], § 16 and $2l_1 = 8$, $2l_2 = 12$, $2k_1 = 4$, $2k_2 = 6$. Then

$$P_{HC^\bullet_{(2)}}(t) = P_{HH^\bullet_{(2)}}(t) = t^{12}/((1-t^2)(1-t^6))$$

and (1.9) is proved.

In the case $M = \mathrm{Ad}\,E_6/T^1\cdot\mathrm{Spin}(10)$ we restrict ourselves to the proof of the isomorphism $H^*(M,\mathbf{Q}) \simeq \mathbf{Q}[X_1,X_2]/(f_1,f_2)$. Note that M is generated by a Cartan pair according to Proposition 1. Thus its minimal model is defined by (2.2), (2.3). Using the calculation in [10]

$$\mathbf{Q}[\mathfrak{T}]^{W(G,T)} \simeq \mathbf{Q}[f_1,\ldots,f_6]$$

where

$$
\begin{aligned}
&f_1 = \lambda^2 + \lambda_1^2 + \ldots + \lambda_6^2, &&f_2 = \lambda^5 + \lambda_1^5 + \ldots + \lambda_6^5\\
&f_3 = \lambda^6 + \lambda_1^6 + \ldots + \lambda_6^6, &&f_4 = \lambda^8 + \lambda_1^8 + \ldots + \lambda_6^8\\
&f_5 = \lambda^9 + \lambda_1^9 + \ldots + \lambda_6^9, &&f_6 = \lambda^{12} + \lambda_1^{12} + \ldots + \lambda_6^{12}
\end{aligned}
\tag{14}
$$

and $\lambda_1, \ldots, \lambda_6$ are the weights of the representation $\varrho_0\colon A_5 \to GL(V)$ of minimal dimension, $(\pm)\lambda$ are the weights of the representation

$$A_1 \xrightarrow{i} A_5 + A_1 \xrightarrow{j} E_6 \to GL(V)$$

(i, j being the natural imbeddings). Analogously

$$\mathbf{Q}[\mathfrak{T}]^{W(H,T)} \simeq \mathbf{Q}[u_1, \ldots, u_6],$$

$$
\begin{aligned}
&u_1 = \lambda, &&u_2 = \lambda_1^2 + \ldots + \lambda_6^2, &&u_3 = \lambda_1^4 + \ldots + \lambda_6^4,\\
&u_4 = \lambda_1^6 + \ldots + \lambda_6^6, &&u_5 = \lambda_1^8 + \ldots + \lambda_6^8, &&u_6 = \lambda_1^5 + \ldots + \lambda_6^5.
\end{aligned}
\tag{15}
$$

Observe that (14) and (15) imply that $H^*(M)$ has not more than two generators while the expression for $P_{H^*(M)}(t)$

$$P_{H^*(M)}(t) = (1 + t^2 + t^4 + \ldots t^{16})(1 + t^8 + t^{16})$$

implies that there are exactly two generators. The proof is completed. □

Remark. The $W(G,T)$-invariant polynomials f_i for any compact simple Lie group G were calculated by many authors. We have used these calculations in the form of [10]. Namely, we consider the linear representation ϱ_0 of G of minimal dimension and denote by $\lambda_1, \ldots, \lambda_m$ the weights of ϱ_0. Then $\lambda_1, \ldots, \lambda_m$ can be regarded as "coordinates" in a Cartan subalgebra $\mathfrak{T}$ of $\mathfrak{G}$ (they define a vector of $\mathfrak{T}$ in spite of the fact that they may be linearly dependent). Thus f_i can be expressed as polynomials of variables $\lambda_1, \ldots, \lambda_m$.

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