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Aleš Nekvinda

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CHARACTERIZATION OF TRACES OF THE WEIGHTED SOBOLEV SPACE $W^{1,p}(\Omega, d_M^\varepsilon)$ ON M

ALEŠ NEKVINDA, Praha

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1. INTRODUCTION

In this paper we shall use the following notation. Let $N > 0$, $k \geqslant 0$ be integers. Let ε, p be real numbers, $1 < p < \infty$. Denote by p' the conjugate exponent, i.e. $p' = \frac{p}{p-1}$. Let Ω be a non-empty, open, bounded subset of $\mathbf{R}^N$. Let M be a closed subset of $\partial\Omega$ and let $d_M(x)$ be the distance function, $d_M(x) = \operatorname{dist}(x, M)$. For simplicity we shall write $d(x)$ instead of $d_M(x)$. For an integer m, $1 \leqslant m \leqslant N$, we set $Q_m = (0,1)^m$.

Definition 1.1. We shall write $(\Omega, M) \in B(k, N)$ for $1 \leqslant k \leqslant N-1$, $N \geqslant 2$ if and only if there exists a bilipschitz mapping

$$B\colon Q_N \to \Omega$$

such that $B(\overline{Q}_k) = M$.

By $C^\infty(\overline{\Omega})$ we denote the set of real functions u defined on $\overline{\Omega}$ such that the derivatives $D^\alpha u$ can be continuously extended to $\overline{\Omega}$ for all multiindices α.

Define the weighted Sobolev space $W^{1,p}(\Omega, d^\varepsilon)$ as the closure of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|u|W^{1,p}(\Omega, d^\varepsilon)\| = \left(\int_\Omega |u(x)|^p d^\varepsilon(x)\,\mathrm{d}x + \int_\Omega \sum_{i=1}^N |D_i u(x)|^p d^\varepsilon(x)\,\mathrm{d}x\right)^{1/p},$$

where $D_i u = \frac{\partial u}{\partial x_i}$ stands for the generalized derivative of the function u. The space $H^{1,p}(\Omega, d^\varepsilon)$ is the class of all functions locally integrable on Ω, with a finite norm

$$\|u|H^{1,p}(\Omega, d^\varepsilon)\| = \left(\int_\Omega |u(x)|^p d^{\varepsilon-p}(x)\,\mathrm{d}x + \int_\Omega \sum_{i=1}^N |D_i u(x)|^p d^\varepsilon(x)\,\mathrm{d}x\right)^{1/p}.$$

Now, let $(\Omega, M) \in B(k, N)$. Let $0 < s < 1$. Let us recall the definition of the Slobodeckij space $W^{s,p}(M)$ as the set of all functions u defined on M with a finite norm

$$\|u|W^{s,p}(M)\| = \Big(\int\limits_M |u(x)|^p \, \mathrm{d}x + \int\limits_M\int\limits_M \frac{|u(x)-u(y)|^p}{|x-y|^{k+sp}} \, \mathrm{d}x \, \mathrm{d}y\Big)^{1/p}.$$

It is well known (see [1] and [4]) that for $\varepsilon \leqslant k - N$ or $\varepsilon > p + k - N$ the space $W^{1,p}(\Omega, d^\varepsilon)$ is isomorphically and topologically equivalent to the space $H^{1,p}(\Omega, d^\varepsilon)$. The space $H^{1,p}(\Omega, d^\varepsilon)$ has zero traces on M in the case $\varepsilon \leqslant k - N$ while for $\varepsilon > p + k - N$ the traces on M have no sense in general.

In this paper we show that in the case $k - N < \varepsilon < p + k - N$ the class of traces on M of the space $W^{1,p}(\Omega, d^\varepsilon)$ for $(\Omega, M) \in B(k, N)$ is equal to the space $W^{1-\frac{N-k+\varepsilon}{p},p}(M)$. Section 2 contains the direct trace theorems, in Section 3 we find a corresponding extension operator.

2. Direct theorems

Lemma 2.1. *Let $M = \{0\}$, $-N < \varepsilon < p - N$. Then there exists a constant $c > 0$ dependent only on ε, p, N such that for all functions $u \in C^\infty(\overline{Q}_N)$ the inequality*

$$|u(0)| \leqslant c\|u|W^{1,p}(Q_N, d^\varepsilon)\| \tag{2.1}$$

holds.

Proof. Let $u \in C^\infty(\overline{Q}_N)$. For $x \in Q_N$ we have

$$u(x) = u(0) + \int\limits_0^1 \sum_{i=1}^N x_i D_i u(tx) \, \mathrm{d}t,$$

and so

$$|u(0)| \leqslant |u(x)| + \int\limits_0^1 \sum_{i=1}^N x_i |D_i u(tx)| \, \mathrm{d}t.$$

Multiplying the last inequality by $d^{\varepsilon/p}(x)$ and integrating over Q_N we get

$$\begin{aligned} |u(0)| \int\limits_{Q_N} d^{\varepsilon/p}(x) \, \mathrm{d}x &\leqslant \int\limits_{Q_N} |u(x)| d^{\varepsilon/p}(x) \, \mathrm{d}x \\ &+ \int\limits_0^1 \int\limits_{Q_N} \sum_{i=1}^N x_i |D_i u(tx)| d^{\varepsilon/p}(x) \, \mathrm{d}x \, \mathrm{d}t = I_1 + I_2. \end{aligned} \tag{2.2}$$

Since $\varepsilon/p > -N$, we have $c_1 = \int_{Q_N} d^{\varepsilon/p}(x)\,\mathrm{d}x < \infty$.

The Hölder inequality yields

$$I_1 \leqslant \Big(\int\limits_{Q_N} \mathrm{d}x\Big)^{1/p'} \Big(\int\limits_{Q_N} |u(x)|^p d^{\varepsilon}(x)\,\mathrm{d}x\Big)^{1/p} \leqslant \|u|W^{1,p}(Q_N, d^{\varepsilon})\|. \tag{2.3}$$

Let us estimate the integral I_2. By the assumptions there exists α, such that $\varepsilon + N - 1 < \alpha p < p - 1$. We use the substitution $tx = y$ and the Hölder inequality and obtain

$$\begin{aligned} I_2 &\leqslant \int\limits_0^1 \int\limits_{(0,t)^N} \sum_{i=1}^N |D_i u(y)| d^{\varepsilon/p}(y) t^{-N-\varepsilon/p}\,\mathrm{d}y\,\mathrm{d}t \\ &\leqslant \Big(\int\limits_0^1 t^{-\alpha p'}\mathrm{d}t\Big)^{1/p'} \Big(\int\limits_0^1 \Big[\int\limits_{(0,t)^N} \sum_{i=1}^N |D_i u(y)| d^{\varepsilon/p}(y) t^{\alpha-N-\varepsilon/p} dy\Big]^p dt\Big)^{1/p}. \end{aligned}$$

Since $-\alpha p' > -1$, we have $\big(\int_0^1 t^{-\alpha p'}\,\mathrm{d}t\big)^{1/p'} = c_2 < \infty$.

Using again the Hölder inequality we obtain

$$\begin{aligned} I_2 &\leqslant \Big(\int\limits_0^1 t^{p\alpha-N-\varepsilon}\mathrm{d}t\Big)^{1/p} \Big(\int\limits_{(0,1)^N} \sum_{i=1}^N |D_i u(y)|^p d^{\varepsilon}(y)\,\mathrm{d}y\Big)^{1/p} \\ &\leqslant c_3\|u|W^{1,p}(Q_N, d^{\varepsilon})\|, \end{aligned} \tag{2.4}$$

where $c_3 = c_2\big(\int_0^1 t^{p\alpha-N-\varepsilon}\mathrm{d}t\big)^{1/p} < \infty$, because $p\alpha - N - \varepsilon > -1$. Now (2.1) follows from (2.2), (2.3) and (2.4). □

Lemma 2.2. *Let $1 \leqslant k \leqslant N-1$, $k - N < \varepsilon < p + k - N$ and let $M = [0,1]^k$. Then there exists a unique bounded linear operator*

$$T\colon W^{1,p}(Q_N, d^{\varepsilon}) \to L^p(M)$$

satisfying

$$Tu(x) = u(x_1, x_2, \ldots, x_k, 0, \ldots, 0),\ x \in Q_N,$$

for every function $u \in C^{\infty}(\overline{Q}_N)$.

Proof. Let $u \in C^{\infty}(\overline{Q}_N)$. Fix $x_1, \ldots, x_k \in (0,1)$ and define a function v by

$$v(x_{k+1}, \ldots, x_N) = u(x),\ (x_{k+1}, \ldots, x_N) \in (0,1)^{N-k}.$$

Obviously, $v(x_{k+1},\dots,x_N)\in W^{1,p}(Q_{N-k},d^\varepsilon)$.

By Lemma 2.1 there exists a constant c dependent only on ε, p, N, k such that

$$|u(x_1,x_2,\dots,x_k,0,\dots,0)|^p\leqslant c\Bigg(\int\limits_{Q_{N-k}}|u(x)|^p d^\varepsilon(x)\,\mathrm{d}x_{k+1}\dots\mathrm{d}x_N + \int\limits_{Q_{N-k}}\sum_{i=k+1}^{N}|D_i u(x)|^p d^\varepsilon(x)\,dx_{k+1}\dots\mathrm{d}x_N\Bigg).$$

Integrating over $x_1,\dots,x_k$ we obtain

$$\|u(x_1,\dots,x_k,0,\dots,0)|L^p(M)\|^p\leqslant c\|u|W^{1,p}(Q_N,d^\varepsilon)\|^p.$$

The operator T is now the unique bounded linear extension of the mapping $u\mapsto u(x_1,x_2,\dots,x_k,0,\dots,0)$. □

Using a similar argument we can prove a more general assertion:

Theorem 2.3. *Let $1\leqslant k\leqslant N-1$, $k-N<\varepsilon<p+k-N$ and $(\Omega,M)\in B(k,N)$. Then there exists a unique bounded linear operator*

$$T\colon W^{1,p}(\Omega,d^\varepsilon)\to L^p(M)$$

such that

$$Tu=u\big|_M$$

for all $u\in C^\infty(\overline{\Omega})$.

Now we shall prove that the trace operator is a bounded mapping from $W^{1,p}(\Omega,d^\varepsilon)$ into $W^{1-\frac{N-k+\varepsilon}{p},p}(M)$ if $(\Omega,M)\in B(k,N)$ and $k-N<\varepsilon<p+k-N$.

The case $k=N-1$.

Lemma 2.4 (see [2]). *Let $-1<\varepsilon<p-1$ and let a be a real number such that $0<a<1$. Then there exists a constant $c>0$ independent of a, such that for all functions $u\in C^\infty([0,a])$ and $v\in C^\infty([a,1])$ the inequalities*

$$\int_0^a\Big|\frac{1}{a-x}\int_x^a u(t)\,\mathrm{d}t\Big|^p(a-x)^\varepsilon\,\mathrm{d}x\leqslant c\int_0^a|u(x)|^p(a-x)^\varepsilon\,\mathrm{d}x,$$

$$\int_a^1\Big|\frac{1}{x-a}\int_a^x v(t)\,\mathrm{d}t\Big|^p(x-a)^\varepsilon\,\mathrm{d}x\leqslant c\int_a^1|v(x)|^p(x-a)^\varepsilon\,\mathrm{d}x$$

hold.

Lemma 2.5. *Let $\Omega = \{(x,y) \in \mathbf{R}^2 \colon x \in (0,1), y \in (0,x)\}$, $M = \{(x,x)\colon 0 \leqslant x \leqslant 1\}$ and let $-1 < \varepsilon < p-1$. Then there exists a constant $c > 0$ such that*

$$\tag{2.5} \begin{aligned} &\int_0^1\int_0^t \frac{|u(t,t)-u(\tau,\tau)|^p}{|t-\tau|^{p-\varepsilon}}\,\mathrm{d}\tau\,\mathrm{d}t \\ &\leqslant c\int_0^1\int_0^t \big(|D_1u(t,\tau)|^p + |D_2u(t,\tau)|^p\big)(t-\tau)^\varepsilon\,\mathrm{d}\tau\mathrm{d}t \end{aligned}$$

for all $u \in C^\infty(\overline{\Omega})$.

Proof. Using the Fubini theorem we obtain

$$\begin{aligned} &\int_0^1\int_0^t \frac{|u(t,t)-u(\tau,\tau)|^p}{|t-\tau|^{p-\varepsilon}}\,\mathrm{d}\tau\,\mathrm{d}t \\ &\leqslant 2^{p-1}\Bigg[\int_0^1\Bigg(\int_0^t\Bigg|\frac{1}{t-\tau}\int_\tau^t D_2u(t,\xi)\,\mathrm{d}\xi\Bigg|^p(t-\tau)^\varepsilon\,\mathrm{d}\tau\Bigg)\,\mathrm{d}t \\ &+\int_0^1\Bigg(\int_\tau^1\Bigg|\frac{1}{t-\tau}\int_\tau^t D_1u(\xi,\tau)\,\mathrm{d}\xi\Bigg|^p(t-\tau)^\varepsilon\,\mathrm{d}t\Bigg)\,\mathrm{d}\tau\Bigg] \\ &= 2^{p-1}[I_1+I_2]. \end{aligned}$$

According to Lemma 2.4 we have

$$\begin{aligned} I_1 &\leqslant c\int_0^1\int_0^t |D_2u(t,\tau)|^p(t-\tau)^\varepsilon\,\mathrm{d}\tau\,\mathrm{d}t, \\ I_2 &\leqslant c\int_0^1\int_\tau^1 |D_1u(t,\tau)|^p(t-\tau)^\varepsilon\,\mathrm{d}t\,\mathrm{d}\tau. \end{aligned}$$

The inequality (2.5) follows. □

Lemma 2.6. *Let $N \geqslant 2$, $-1 < \varepsilon < p-1$. Define $A_i(u)$ by*

$$A_i(u) = \underbrace{\int_0^1\int_0^1 \dots \int_0^1}_{(N-2)\text{-fold}} \Big(\int_0^1\int_0^1 |u(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_{N-1}) - u(x_1,\dots,x_{i-1},\tau,x_{i+1},\dots,x_{N-1})|^p \frac{\mathrm{d}\tau\,\mathrm{d}t}{|t-\tau|^{p-\varepsilon}}\Big) dx_1\,\mathrm{d}x_2\dots\mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1}\dots\mathrm{d}x_{N-1}.$$

Then there exists a positive constant c such that

$$\|u|W^{1-\frac{1+\varepsilon}{p},p}(Q_{N-1})\|^p \leqslant c\Big(\|u|L^p(Q_{N-1})\|^p + \sum_{i=1}^{N-1} A_i(u)\Big)$$

for all functions $u \in L^p(Q_{N-1})$ such that $A_i(u) < \infty$ for $i = 1,2,\dots,N-1$.

Proof. The proof can proceed in a way similar to that of the proof of Lemma 6.8.10 in [3]. □

Lemma 2.7. *Let $N \geqslant 2$, $-1 < \varepsilon < p-1$ and $M = [0,1]^{N-1}$. Then there exists a constant $c > 0$ such that*

$$\|u(x_1,\dots,x_{N-1},0)|W^{1-\frac{1+\varepsilon}{p},p}(Q_{N-1})\| \leqslant c\,\|u|W^{1,p}(Q_N,d^\varepsilon)\|$$

for all $u \in C^\infty(\overline{Q}_N)$.

Proof. Let $u \in C^\infty(\overline{Q}_N)$. Set $v(x_1,x_2,\dots,x_{N-1}) = u(x_1,x_2,\dots,x_{N-1},0)$. According to Lemma 2.6 and Lemma 2.2 we have

$$\big\|u(x_1,\dots,x_{N-1},0)|W^{1-\frac{1+\varepsilon}{p},p}(Q_{N-1})\big\|^p = \big\|v|W^{1-\frac{1+\varepsilon}{p},p}(Q_{N-1})\big\|^p$$
$$\leqslant c_1\Big(\big\|v|L^p(Q_{N-1})\big\| + \sum_{i=1}^{N-1} A_i(v)\Big) \leqslant c_2\Big(\big\|u|W^{1,p}(Q_N,d^\varepsilon)\big\|^p + \sum_{i=1}^{N-1} A_i(v)\Big).$$

Obviously, it remains to estimate $A_i(v)$, $i = 1, 2, \dots, N-1$. Fix $i = 1, \dots, N-1$. Set $v_i(x_1,\dots,x_N) = u(x_1,\dots,x_{N-1},x_i - x_N)$ in the domain $0 \leqslant x_j \leqslant 1$ for

$j = 1, \ldots, N-1$ and $0 \leqslant x_N \leqslant x_i$. Lemma 2.5 and the direct calculation yield

$$A_i(v) = 2 \int_0^1 \ldots \int_0^1 \Big(\int_0^1 \int_0^t |u(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, 0)$$
$$- u(x_1, \ldots, x_{i-1}, \tau, x_{i+1}, \ldots, x_{N-1}, 0)|^p \frac{\mathrm{d}\tau\, \mathrm{d}t}{|t-\tau|^{p-\varepsilon}} \Big)$$
$$\mathrm{d}x_1 \ldots \mathrm{d}x_{i-1}\, \mathrm{d}x_{i+1} \ldots \mathrm{d}x_{N-1}$$
$$= 2 \int_0^1 \ldots \int_0^1 \Big(\int_0^1 \int_0^t |v_i(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, t)$$
$$- v_i(x_1, \ldots, x_{i-1}, \tau, x_{i+1}, \ldots, x_{N-1}, \tau)|^p \frac{\mathrm{d}\tau\, \mathrm{d}t}{|t-\tau|^{p-\varepsilon}} \Big)$$
$$\mathrm{d}x_1 \ldots \mathrm{d}x_{i-1}\, \mathrm{d}x_{i+1} \ldots \mathrm{d}x_{N-1}$$
$$\leqslant 2c \int_0^1 \ldots \int_0^1 \Big(\int_0^1 \int_0^t (|D_i v_i(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, \tau)|^p$$
$$+ |D_N v_i(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, \tau)|^p)(t-\tau)^\varepsilon \, \mathrm{d}\tau\, \mathrm{d}t \Big)$$
$$\mathrm{d}x_1 \ldots \mathrm{d}x_{i-1}\, \mathrm{d}x_{i+1} \ldots \mathrm{d}x_{N-1}$$
$$\leqslant 2^{p+1} c \int_0^1 \ldots \int_0^1 \Big(\int_0^1 \int_0^t (|D_i u(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, t-\tau)|^p$$
$$+ |D_N u(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{N-1}, t-\tau)|^p)(t-\tau)^\varepsilon \, \mathrm{d}\tau\, \mathrm{d}t \Big)$$
$$\mathrm{d}x_1 \ldots \mathrm{d}x_{i-1}\, \mathrm{d}x_{i+1} \ldots \mathrm{d}x_{N-1}.$$

The substitution $s = t - \tau$ gives

$$A_i(v) \leqslant 2^{p+1} c \| u | W^{1,p}(Q_N, d^\varepsilon) \|^p, \quad i = 1, 2, \ldots, N-1,$$

which completes the proof. □

As an immediate consequence we have

Theorem 2.8. *Let $N \geqslant 2$, $-1 < \varepsilon < p-1$ and $(\Omega, M) \in B(N-1, N)$. Then there exists a unique bounded linear operator*

$$T \colon W^{1,p}(\Omega, d^\varepsilon) \to W^{1-\frac{1+\varepsilon}{p}, p}(M)$$

such that

$$Tu = u\big|_M$$

for all $u \in C^\infty(\overline{Q}_N)$.

The case $1 \leqslant k \leqslant N-2$.

Lemma 2.9. *Let* $K, L, s > 0$. *Then there exists* $c > 0$ *such that*

$$\frac{c}{r^{1+s}} \leqslant \int_0^L \int_0^L \frac{1}{(r + |\varrho_1 - \varrho_2|)^{2+s}} \, d\varrho_1 \, d\varrho_2$$

for every $r \in (0, K]$.

Proof. Obviously,

$$\int_0^L \int_0^L \frac{d\varrho_1 \, d\varrho_2}{(r + |\varrho_1 - \varrho_2|)^{2+s}} = 2 \int_0^L \int_{\varrho_2}^L \frac{d\varrho_1 \, d\varrho_2}{(r + \varrho_1 - \varrho_2)^{2+s}} = 2I. \tag{2.6}$$

Direct calculation yields

$$I = \frac{1}{1+s} \frac{L}{r^{1+s}} + \frac{1}{s(s+1)} \Big(\frac{1}{(r+L)^s} - \frac{1}{r^s} \Big).$$

By the Lagrange Mean Value Theorem there exists $\xi \in (r, r+L)$ such that

$$I = \frac{L}{1+s} \Big(\frac{1}{r^{1+s}} - \frac{\xi^{s-1}}{r^s (r+L)^s} \Big).$$

If $s \leqslant 1$ we have

$$\begin{aligned} I &\geqslant \frac{L}{1+s} \Big(\frac{1}{r^{1+s}} - \frac{r^{s-1}}{r^s (r+L)^s} \Big) = \frac{L}{1+s} \frac{1}{r^{1+s}} \Big(1 - \big(\frac{r}{r+L} \big)^s \Big) \\ &\geqslant \frac{L}{1+s} \Big(1 - \Big(\frac{K}{K+L} \Big)^s \Big) \frac{1}{r^{1+s}}. \end{aligned}$$

For $s \geqslant 1$ we get

$$\begin{aligned} I &\geqslant \frac{L}{1+s} \Big(\frac{1}{r^{1+s}} - \frac{(r+L)^{s-1}}{r^s (r+L)^s} \Big) = \frac{L}{1+s} \frac{1}{r^{1+s}} \Big(1 - \frac{r}{r+L} \Big) \\ &\geqslant \frac{L}{1+s} \Big(1 - \frac{K}{K+L} \Big) \frac{1}{r^{1+s}}. \end{aligned}$$

Thus, (2.6) holds with $c = 2 \min\Big(\frac{L}{1+s} \Big(1 - \big(\frac{K}{K+L} \big)^s \Big), \frac{L}{1+s} \big(1 - \frac{K}{K+L} \big) \Big)$. □

Lemma 2.10. *Let $N \geqslant 2$, $k-N<\varepsilon<p+k-N$ and $M=[0,1]^k$. Then there exists a unique bounded linear operator*

$$T\colon W^{1,p}(Q_N,d^\varepsilon)\to W^{1-\frac{N-k+\varepsilon}{p},p}(M)$$

such that

$$Tu=u\big|_M$$

for all functions $u\in C^\infty(\overline{Q}_N)$.

Proof. Let $u\in C^\infty(\overline{Q}_N)$. We introduce general cylindrical coordinates (x',r,ϱ), $x'=(x_1,\dots,x_k)$, $\varrho=(\varrho_1,\dots,\varrho_{N-k-1})$, by

$$\begin{aligned}
x_1&=x_1,\\
&\dots\\
x_k&=x_k,\\
x_{k+1}&=r\cos\varrho_1,\\
x_{k+2}&=r\sin\varrho_1\cos\varrho_2,\\
&\dots\\
x_N&=r\sin\varrho_1\dots\sin\varrho_{N-k-2}\cos\varrho_{N-k-1}.
\end{aligned}$$

Let $\Omega'=\{(x',r,\varrho)\colon x'\in(0,1)^k, r\in(0,1),\varrho\in(\frac{\pi}{6},\frac{\pi}{3})^{N-k-1}\}$. Obviously $M=\{(x',r,\varrho')\in\overline{\Omega'}\colon r=0\}$. Let Ω be the set of points on $\mathbf{R}^N$ whose cylindrical coordinates (x',r,ϱ) belong to Ω'. Then $\Omega\subset Q_N$, $M\subset\overline{\Omega}$, $d(x)=r$ and there exist two constants $c_1,c_2>0$ such that

$$c_1r^{N-k-1}\leqslant J(x',r,\varrho)\leqslant c_2r^{N-k-1},\ (x',r,\varrho)\in\Omega', \tag{2.7}$$

where $J(x',r,\varrho)$ is the Jacobian of the mapping $(x',r,\varrho)\mapsto x$. Define a function v by

$$v(x',r,\varrho)=u(x).$$

It is easy to see that $v\in C^\infty(\overline{\Omega'})$ and

$$\begin{aligned}
\Big|\frac{\partial v}{\partial x_i}(x',r,\varrho)\Big|&=\Big|\frac{\partial u}{\partial x_i}(x)\Big| \quad\text{for}\quad i=1,2,\dots,k,\\
\Big|\frac{\partial v}{\partial r}(x',r,\varrho)\Big|&\leqslant\sum_{j=k+1}^{N}\Big|\frac{\partial u}{\partial x_j}(x)\Big|,\\
\Big|\frac{\partial v}{\partial \varrho_j}(x',r,\varrho)\Big|&\leqslant r\sum_{i=k+1}^{N}\Big|\frac{\partial u}{\partial x_i}(x)\Big| \quad\text{for}\quad j=1,\dots,N-k-1.
\end{aligned}$$

These inequalities and (2.7) yield

$$(2.8)\quad I(v) := \int\limits_{\Omega'} |v(x',r,\varrho)|^p r^{N-k-1} r^{\varepsilon}\,\mathrm{d}x'\,\mathrm{d}r\,\mathrm{d}\varrho$$

$$+ \int\limits_{\Omega'} \Big(\sum_{i=1}^{k} \Big|\frac{\partial v}{\partial x_i}(x',r,\varrho)\Big|^p + \Big|\frac{\partial v}{\partial r}(x',r,\varrho)\Big|^p\Big) r^{N-k-1} r^{\varepsilon}\,\mathrm{d}x'\,\mathrm{d}r\,\mathrm{d}\varrho$$

$$+ \int\limits_{\Omega'} \sum_{i=1}^{N-k-1} \Big|\frac{\partial v}{\partial \varrho_i}(x',r,\varrho)\Big|^p r^{N-k-1} r^{\varepsilon-p}\,\mathrm{d}x'\,\mathrm{d}r\,\mathrm{d}\varrho$$

$$\leqslant c_3 \|u|W^{1,p}(Q_N,d^{\varepsilon})\|^p.$$

Set $\varepsilon_1 = \varepsilon - N - k - 1$ and define the anisotropic weighted space $V^{1,p}(\Omega', r^{\varepsilon_1})$ as the closure of $C^{\infty}(\overline{\Omega'})$ with respect to the norm $\|.\,|V\| = \big(I(.)\big)^{1/p}$. Obviously

$$(2.9)\qquad V^{1,p}(\Omega', r^{\varepsilon_1}) \hookrightarrow W^{1,p}(\Omega', r^{\varepsilon_1}).$$

Putting $\varepsilon = \varepsilon_1$ in Theorem 2.8 we obtain the existence of a constant $c_4 > 0$ such that

$$(2.10)\qquad c_4 \|v|W^{1,p}(\Omega', r^{\varepsilon_1})\|^p$$

$$\geqslant \int\limits_{(0,1)^k} \int\limits_{(0,1)^k} \int\limits_{(\frac{\pi}{6},\frac{\pi}{3})^{N-k-1}} \int\limits_{(\frac{\pi}{6},\frac{\pi}{3})^{N-k-1}} \frac{|v(x',0,\varrho) - v(y',0,\psi)|^p}{(|x'-y'| + |\varrho - \psi|)^{N+p-2-\varepsilon_1}}\,\mathrm{d}\varrho\,\mathrm{d}\psi\,\mathrm{d}x'\,\mathrm{d}y'.$$

Denote the right-hand side of (2.10) by I_1. Since $v(x',0,\varrho) = u(x_1,\ldots,x_k,0,\ldots,0) = u(x',0)$, we can write

$$(2.11)\qquad I_1 = \int\limits_{(0,1)^k} \int\limits_{(0,1)^k} |u(x',0) - u(y',0)|^p J(|x'-y'|)\,\mathrm{d}x'\,\mathrm{d}y',$$

where

$$J(|x'-y'|)$$

$$\geqslant c_5 \int\limits_{(\frac{\pi}{6},\frac{\pi}{3})^{N-k-1}} \int\limits_{(\frac{\pi}{6},\frac{\pi}{3})^{N-k-1}} \frac{\mathrm{d}\varrho_1 \ldots \mathrm{d}\varrho_{N-k-1}\,\mathrm{d}\psi_1 \ldots \mathrm{d}\psi_{N-k-1}}{(|x'-y'| + |\varrho_1 - \psi_2| + \cdots + |\varrho_{N-k-1} - \psi_{N-k-1}|)^{N+p-2-\varepsilon_1}}.$$

Using $(N-k-1)$-times Lemma 2.9 we arrive at the estimate

$$(2.12)\qquad J(|x'-y'|) \geqslant \frac{c_6}{|x'-y'|^{p+k-1-\varepsilon_1}} = \frac{c_6}{|x'-y'|^{p+2k-N-\varepsilon}}.$$

From (2.11) and (2.12) we obtain

$$I_1 \geqslant c_6 \int\limits_{(0,1)^k} \int\limits_{(0,1)^k} \frac{|u(x',0) - u(y',0)|^p}{|x' - y'|^{p+2k-N-\varepsilon}} \, dx' \, dy'.$$

The estimates (2.8), (2.9), (2.10) and (2.12) imply the assertion of the lemma. □

Again, it is not difficult to prove the following more general statement:

Theorem 2.11. *Let $N \geqslant 2$, $k - N < \varepsilon < p + k - N$ and let $(\Omega, M) \in B(k, N)$. Then there exists a unique bounded linear operator*

$$T\colon W^{1,p}(\Omega, d^\varepsilon) \to W^{1-\frac{N-k+\varepsilon}{p},p}(M)$$

such that

$$Tu = u\big|_M$$

for all $u \in C^\infty(\overline{\Omega})$.

3. Inverse theorems

Let $N \geqslant 2$ and let k be fixed, $1 \leqslant k \leqslant N - 1$. For $x \in (x_1, \ldots, x_N) \in \mathbf{R}^N$ we shall write $x = (x', x'')$, $x' = (x_1, \ldots, x_k)$, $x'' = (x_{k+1}, \ldots, x_N)$. Let Φ be a function of $C^\infty(\mathbf{R}^k)$ such that $\Phi(x') = 0$ for $|x'| \geqslant 1$, $\Phi(x') > 0$ for $|x'| < 1$ and $\int_{|x'|\leqslant 1} \Phi(x') \, dx' = 1$. We define an operator R by

$$Ru(x', x'') = |x''|^{-k} \int\limits_{|x'-t'|\leqslant|x''|} \Phi\Big(\frac{x' - t'}{|x''|}\Big) u(t') \, dt', \quad u \in L^1_{\text{loc}}(\mathbf{R}^k). \tag{3.1}$$

Lemma 3.1. *Let $k - N < \varepsilon < p + k - N$ and $K > 0$. Let $Q^k = (-K, K)^k$,*

$$D = \{x \in \mathbf{R}^N : x = (x', x''), x' \in Q^k, |x''| \leqslant K - \max_{i=1,\ldots,k} |x_i|, x_j \geqslant 0 \text{ for } k+1, \ldots, N\}$$

and $M = \overline{Q^k}$.

Then the operator R is a bounded linear mapping of $W^{1-\frac{N-k+\varepsilon}{p},p}(Q^k)$ in $W^{1,p}(D, d^\varepsilon)$.

P r o o f. The linearity of the operator R is obvious. Now, we shall follow the idea of the proof of Lemma 6.9.1 in [3]. Without loss of generality we may assume $K = 1$. From (3.1) we get

$$D_i Ru(x', x'') = \frac{1}{|x''|^{k+1}} \int\limits_{|x'-t'|<|x''|} D_i\Phi\Big(\frac{x' - t'}{|x''|}\Big) u(t') \, dt', \; i = 1, 2, \ldots, k, \tag{3.2}$$

(3.3) $$D_i Ru(x',x'') = \frac{1}{|x''|^{k+1}} \int\limits_{|x'-t'|<|x''|} \Big(-\frac{kx_i}{|x''|}\Phi\Big(\frac{x'-t'}{|x''|}\Big) - \sum_{j=1}^{k} D_j\Phi\Big(\frac{x'-t'}{|x''|}\Big)\frac{(x_j-t_j)x_i}{|x''|^2}\Big)u(t')\,\mathrm{d}t',$$
$$i = k+1,\dots,N.$$

If we take $u \equiv 1$ in (3.1) we get $Ru(x',x'') = 1$ and from (3.2) and (3.3) we obtain

$$0 = \frac{1}{|x''|^{k+1}} \int\limits_{|x'-t'|\leqslant|x''|} D_i\Phi\Big(\frac{x'-t'}{|x''|}\Big)\,\mathrm{d}t', \quad i = 1,2,\dots,k,$$
$$0 = \frac{1}{|x''|^{k+1}} \int\limits_{|x'-t'|\leqslant|x''|} \Big(-\frac{kx_i}{|x''|}\Phi\Big(\frac{x'-t'}{|x''|}\Big) - \sum_{j=1}^{k} D_j\Phi\Big(\frac{x'-t'}{|x''|}\Big)\frac{(x_j-t_j)x_i}{|x''|^2}\Big)\,\mathrm{d}t',$$
$$i = k+1,\dots,N.$$

Thus, (3.2) and (3.3) can be rewritten in the following way:

$$D_i Ru(x',x'') = \frac{1}{|x''|^{k+1}} \int\limits_{|x'-t'|\leqslant|x''|} D_i\Phi\Big(\frac{x'-t'}{|x''|}\Big)\big(u(t')-u(x')\big)\,\mathrm{d}t',$$
$$i = 1,\dots,k,$$
$$D_i Ru(x',x'') = \frac{1}{|x''|^{k+1}} \int\limits_{|x'-t'|\leqslant|x''|} \Big(-\frac{kx_i}{|x''|}\Phi\Big(\frac{x'-t'}{|x''|}\Big) - \sum_{j=1}^{k} D_j\Phi\Big(\frac{x'-t'}{|x''|}\Big)\frac{(x_j-t_j)x_i}{|x''|^2}\Big)\big(u(t')-u(x')\big)\,\mathrm{d}t',$$
$$i = k+1,\dots,N.$$

We shall estimate Ru and $D_i Ru$ in the norm of $L^p(D,d^\varepsilon)$. Note that $d(x) = |x''|$ for $x \in D$.

Let us estimate $I_0 = \|Ru|L^p(D,d^\varepsilon)\|^p$:

We put $b(x') = 1 - \max\limits_{i=1,\dots,k} |x_i|$. According to (3.1), we have

$$I_0 = \int\limits_{Q^k} \int\limits_{|x''|<b(x')} \Big|\frac{1}{|x''|^k} \int\limits_{|x'-t'|\leqslant|x''|} \Phi\Big(\frac{x'-t'}{|x''|}\Big)u(t')\,\mathrm{d}t'\Big|^p |x''|^\varepsilon\,\mathrm{d}x''\,\mathrm{d}x'.$$

Using the substitution $s' = \frac{x'-t'}{|x''|}$, the Hölder inequality, the spherical coordinates for x'' and the Fubini theorem we obtain

$$I_0 = \int\limits_{Q^k} \Bigg(\int\limits_{|x''|<b(x')} \Bigg| \int\limits_{|s'|\leqslant 1} \Phi(s')u(x' - s'|x''|)\,\mathrm{d}s' \Bigg|^p |x''|^\varepsilon\,\mathrm{d}x'' \Bigg)\,\mathrm{d}x'$$

$$\leqslant c_1 \int\limits_{Q^k} \int\limits_{0<r<b(x')} \int\limits_{|s'|\leqslant 1} |u(x' - s'r)|^p\,\mathrm{d}s' r^{N-k-1+\varepsilon}\mathrm{d}r\,\mathrm{d}x'$$

$$= c_1 \int\limits_{|s'|\leqslant 1} \int\limits_0^1 \int\limits_{(r-1,1-r)^k} |u(x' - s'r)|^p\,\mathrm{d}x' r^{N-k-1+\varepsilon}\,\mathrm{d}r\,\mathrm{d}s'.$$

Now we use the substitution $y' = x' - s'r$. Obviously, $|y'| - |s'|r \leqslant |y' + s'r| = |x'| \leqslant 1 - r$, and so $|y'| \leqslant 1 - r(1 - |s'|) \leqslant 1$.

Increasing the integration domain we obtain

$$I_0 \leqslant c_1 \int\limits_{|s'|\leqslant 1} \int\limits_0^1 \int\limits_{|y'|\leqslant 1} |u(y')|^p\,\mathrm{d}y' r^{N-k-1+\varepsilon}\,\mathrm{d}r\,\mathrm{d}s' \leqslant c_2 \|u|L^p(Q^k)\|^p,$$

where $c_2 = c_1 \int\limits_{|s'|\leqslant 1} \mathrm{d}s' \int\limits_0^1 r^{N-k-1+\varepsilon}\mathrm{d}r$.

To estimate $I_i = \|D_i Ru|L^p(D, d^\varepsilon)\|^p$ for $i = 1, 2, \ldots, N$ we proceed in the same way as in the previous case. We obtain

$$I_i \leqslant c_3 \int\limits_{Q^k} \int\limits_{0<r<b(x')} \int\limits_{|s'|\leqslant 1} \frac{|u(x' - s'r) - u(x')|^p}{r^p} r^{N-k-1+\varepsilon}\mathrm{d}s'\,\mathrm{d}r\,\mathrm{d}x'$$

$$\leqslant c_3 \int\limits_{Q^k} \int\limits_{|t'-x'|\leqslant b(x')} \frac{|u(t') - u(x')|^p}{|t' - x'|^{p+2k-N-\varepsilon}} \int\limits_{|t'-x'|}^{b(x')} \frac{|t' - x'|^{p+2k-N-\varepsilon}}{r^{p+2k-N-\varepsilon-1}}\,\mathrm{d}r\,\mathrm{d}t'\,\mathrm{d}x'.$$

Direct calculation yields

$$\int\limits_{|t'-x'|}^{b(x')} \frac{|t' - x'|^{p+2k-N-\varepsilon}}{r^{p+2k-N-\varepsilon-1}}\,\mathrm{d}r \leqslant \frac{1}{p + 2k - N - \varepsilon}.$$

Since $\{t' \colon |x' - t'| \leqslant 1 - \max\limits_{i=1,\ldots,k} |x_i|\} \subset Q^k$, we have

$$I_i \leqslant c_4 \int\limits_{Q^k} \int\limits_{Q^k} \frac{|u(t') - u(x')|^p}{|t' - x'|^{p+2k-N-\varepsilon}}\,\mathrm{d}t'\,\mathrm{d}x' \leqslant c_4 \|u|W^{1-\frac{N-k-\varepsilon}{p},p}(Q^k)\|^p,$$

which completes the proof. □

Lemma 3.2. *Let $G \subset \mathbf{R}^k$ have a Lipschitz boundary, i.e. $G \in C^{0,1}$ in the sense of Definition 5.5.6 in [3], $s \in \mathbf{R}$, $0 < s < 1$. Then there exists a bounded linear operator*

$$S\colon W^{s,p}(G) \to W^{s,p}(\mathbf{R}^k)$$

such that

$$Su = u \quad \text{on} \quad \Omega.$$

Proof. We shall write $x = (x', x_k)$ where $x' = (x_1, \dots, x_{k-1})$. Let $Q_k^- = \{x = (x', x_k),\ x' \in Q_{k-1},\ -1 < x_k < 0\}$. Define an extension operator S_0 in the following way:

$$(S_0 u)(x', x_k) = \begin{cases} u(x', x_k) & \text{for} \quad x \in Q_k \\ u(x', -x_k) & \text{for} \quad x \in Q_k^- . \end{cases}$$

It is not difficult to prove that S_0 is a bounded linear operator from $W^{s,p}(Q_k)$ into $W^{s,p}(Q_k \cup Q_k^-)$.

Using the local description of ∂G, the corresponding partition of unity and the standard technique we obtain the assertion of the lemma. □

Lemma 3.3. *Let $N \geqslant 2$, $k - N < \varepsilon < p + k - N$ and let $M = [0,1]^k$. Then there exists a bounded linear operator*

$$R_0\colon W^{1-\frac{N-k+\varepsilon}{p},p}(M) \to W^{1,p}(Q_N, d^\varepsilon)$$

such that

$$T(R_0 u) = u.$$

Proof. Take K sufficiently large so that $D \supset Q_N$ where D is the domain defined in Lemma 3.1. Set $Q^k = [-K, K]^k$. Let S be the operator from Lemma 3.2 and R the operator from Lemma 3.1 (with $G = Q^k$ and $s = 1 - \frac{N-k+\varepsilon}{p}$). Let $u \in W^{1-\frac{N-k+\varepsilon}{p},p}(M)$. Obviously,

$$\|RSu | W^{1,p}(Q_N, d^\varepsilon)\| \leqslant c_1 \|u | W^{1-\frac{N-k+\varepsilon}{p},p}(M)\|.$$

It remains to prove that $T(RSu) = u$. Note that $Su \in L^p(Q^k)$ and $Su = u$ on M. Put

$$I(x'') = \int_M |(RSu)(x', x'') - u(x')|^p \, dx'.$$

Then

$$I(x'') = \int\limits_M \left| \int\limits_{|s'|<1} \Phi(s')\big((Su)(x' - s'|x''|) - u(x')\big)\, \mathrm{d}s' \right|^p \mathrm{d}x'$$

$$\leqslant c_2 \int\limits_{|s'|<1} \int\limits_M |(Su)(x' - s'|x''|) - (Su)(x')|^p \, \mathrm{d}x'.$$

The p-mean continuity of Su yields

$$\lim_{|x''|\to 0} I(x'') = 0.$$

By virtue of the trivial imbedding

$$L^p(M) \hookrightarrow L^1(M)$$

we have

$$\lim_{|x''|\to 0} \|(RSu)(x', x'') - u(x')|L^1(M)\| = 0. \tag{3.4}$$

Since $RSu \in W^{1,p}(D, d^\varepsilon)$, by Theorem 2.11 the trace of RSu on M exists. Denote this trace by v. We shall prove that

$$\lim_{|x''|\to 0} \|(RSu)(x', x'') - v(x'')|L^1(M)\| = 0.$$

Let $u_n \in C^\infty(\overline{Q}_N)$, $u_n \to RSu$ in $W^{1,p}(Q_N, d^\varepsilon)$. We can write

$$\begin{aligned}
\int\limits_M |(RSu)(x', x'') - v(x')|\, \mathrm{d}x' &\leqslant \int\limits_M |(RSu)(x', x'') - (RSu_n)(x', x'')|\, \mathrm{d}x' \\
&+ \int\limits_M |(RSu_n)(x', x'') - u_n(x', 0)|\, \mathrm{d}x' + \int\limits_M |u_n(x', 0) - v(x')|\, \mathrm{d}x' \\
&= J_1(n, x'') + J_2(n, x'') + J_3(n).
\end{aligned} \tag{3.5}$$

Set $\sigma_{N-k} = \frac{1}{N-k} s_{N-k}$ where s_{N-k} is the $(N-k-1)$-dimensional Hausdorff measure of the $(N-k)$-dimensional unit sphere. For simplicity we write f_n instead of $RSu - RSu_n$. From the definition of the operator RS we have $f_n(x', x_1'') = f_n(x', x_2'')$ for $|x_1''| = |x_2''|$. Thus, we can rewrite the integral $J_1(u, x'')$ in the following way:

$$\begin{aligned}
J_1(u, x'') &= \int\limits_M |f_n(x', x'')|\, \mathrm{d}x' \\
&= \frac{1}{\sigma_{N-k}|x''|^{N-k-1}} \int\limits_M \int\limits_{|y''|=|x''|} |f_n(x', y'')|\, \mathrm{d}x'\, \mathrm{d}y''.
\end{aligned} \tag{3.6}$$

Put

$$M(|x''|) = \{y'' \colon |y''| < |x''|, y_i > 0 \text{ for } i = k+1, \dots, N\}.$$

Since the traces of $W^{1,1}(\Omega)$ are in $L^1(\partial\Omega)$, there exists a constant $c_3 > 0$ independent of $|x''|$ and such that

$$J_1(u, x'') \leqslant \frac{c_3}{\sigma_{N-k}|x''|^{N-k-1}} \int\limits_M \int\limits_{M(|x''|)} \Big(|f_n(x', y'')| + \sum_{i=1}^{N} |D_i f_n(x', y'')| \Big) \, dx' \, dy''.$$

We use (3.5), (3.6), the Hölder inequality and the general cylindrical coordinates to obtain

$$\begin{aligned} J_1(n, x'') &\leqslant c_4 \int\limits_M \int\limits_{M(|x''|)} \Big(\big(|f_n(x', y'')|^p + \sum_{i=1}^{N} |D_i f_n(x', y'')|^p\big) |y''|^{\varepsilon} \, dx' \, dy'' \Big)^{1/p} \cdot \Big(\int\limits_0^{|x''|} r^{N-k-1-\frac{\varepsilon p'}{p}} \, dr \Big)^{1/p'} \\ &\leqslant c_5 |x''|^{1-\frac{N-k+\varepsilon}{p}} \| f_n | W^{1,p}(Q_N, d^{\varepsilon}) \|^p. \end{aligned}$$

Consequently,

$$J_1(n, x'') \leqslant c_5 |x''|^{1-\frac{N-k+\varepsilon}{p}} \| RSu - RSu_n | W^{1,p}(Q_N, d^{\varepsilon}) \|. \tag{3.7}$$

Note that evidently $1 - \frac{N-k+\varepsilon}{p} > 0$.

Let $\delta > 0$ be a fixed real number. Then for every n there exists $r(n) > 0$ such that

$$J_2(n, x'') = \int\limits_M |(RSu_n)(x', x'') - u_n(x', 0)| \, dx' < \delta/3 \quad \text{for} \quad |x''| < r(n). \tag{3.8}$$

The proof is analogious to the estimate of $I(x'')$.

Finally, from the trivial imbedding

$$W^{1,p}(Q_N, d^{\varepsilon}) \hookrightarrow W^{1,1}(Q_N)$$

and from Lemma 2.2 we have

$$\lim_{n \to \infty} J_3(n) = 0.$$

Now, fix n_0 such that $J_3(n_0) < \delta/3$. From (3.8) we can find a corresponding number $r(n_0)$ such that $J_2(n_0, x'') < \delta/3$ for $|x''| < r(n_0)$. According to (3.7) there exists a number $r_1 > 0$ such that

$$J_1(n_0, x'') < \delta/3 \quad \text{for} \quad |x''| < r_1.$$

Let $r = \min(r_1, r(n_0))$.

The estimate (3.5) implies

$$J(x'') < \delta \quad \text{for} \quad |x''| < r.$$

Therefore,

$$\lim_{|x''| \to 0} J(x'') = 0. \tag{3.9}$$

By (3.4) and (3.9) we have $u = v$ in $L^1(M)$. Setting $R_0 = RS$ we complete the proof. □

Theorem 3.4. *Let $N \geqslant 2$, $1 \leqslant k \leqslant N-1$, $\varepsilon \in \mathbf{R}$, $k - N < \varepsilon < p + k - N$ and let $(\Omega, M) \in B(k, N)$. Then there exists a bounded linear operator*

$$R\colon W^{1-\frac{N-k+\varepsilon}{p},p}(M) \to W^{1,p}(\Omega, d^\varepsilon)$$

such that

$$T(Ru) = u.$$

P r o o f. The theorem is an easy consequence of Lemma 3.3. □

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Author's address: Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 166 29 Praha 6, Czech Republic.