Czechoslovak Mathematical Journal

Teklehaimanot Retta

Some cardinal generalizations of pseudocompactness

Czechoslovak Mathematical Journal, Vol. 43 (1993), No. 3, 385–390

Persistent URL: http://dml.cz/dmlcz/128417

Terms of use:

© Institute of Mathematics AS CR, 1993

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 43 (118) 1993, Praha

SOME CARDINAL GENERALIZATIONS OF PSEUDOCOMPACTNESS

TEKLEHAIMANOT RETTA, Addis Ababa

(Received September 3, 1986, in revised form September 10, 1991)

1. INTRODUCTION

All spaces considered in this paper are assumed to be Tychonoff. A space X is said to be ***initially m-compact*** if every open cover of X of cardinality $\leqslant m$ has a finite subcover. Equivalently, X is initially m-compact if every filterbase of cardinality $\leqslant m$ has a nonvoid adherence. X is said to be ***weakly initially m-compact*** if every open cover of cardinality $\leqslant m$ has a finite subset with a dense union. X is called ***m-pseudocompact*** if every continuous image of X in $\mathbf{R}^m$ is compact. X is said to be ***m-pseudocompact in the sense of complete accumulation points*** (***mpcap*** for short) if every family of $\leqslant m$ open sets in X has a complete accumulation point, i.e., a point each neighbourhood of which meets k members of the family where k is the cardinality of the family.

When m is countable each of the properties of being weakly initially m-compact, m-pseudocompact, and mpcap is equivalent to being pseudocompact. See [7], [6], [2], and [5] for a discussion of initially m-compact, weakly initially m-compact, and m-pseudocompact spaces.

It is shown that if $m \geqslant c$ then the product of any collection of initially m-compact spaces is m-pseudocompact; that a regular closed set in an m-pseudocompact space and a perfect irreducible preimage of an m-pseudocompact space may fail to be m-pseudocompact. These statements are false in case m is countable. We also show that a weakly initially m-compact space is m-pseudocompact but that in general the converse is false. Further we show that the properties m-pseudocompactness and m-pseudocompactness in the sense of complete accumulation points are in general incomparable.

All undefined notation and terminology is as in [3].

2. m-PSEUDOCOMPACT SPACES

A G_m*-set* in X is an intersection of $\leqslant m$ open sets. A subset A of X is to be G_m*-dense* in X if every nonvoid G_m-set in X meets A. It is clear that A is G_m-dense in X if and only if it meets every nonvoid intersection of $\leqslant m$ zero sets in X. A family of sets is said to have the *m-intersection property* (*m.i.p.* for short) if every subset of $\leqslant m$ members has a nonempty intersection.

In the following theorem we collect some conditions equivalent to m-pseudocompactness. The equivalence of the conditions (a) and (d) is noted in [5]. The proof is left to the reader.

Theorem 1. *The following conditions on a space X are equivalent:*
(a) *Every zero set filter in X has the m.i.p.;*
(b) *every cozero cover of X of cardinality $\leqslant m$ has a finite subcover;*
(c) *every continuous image of X in a space of weight $\leqslant m$ is compact;*
(d) *X is m-pseudocompact;*
(e) *X is G_m-dense in $\beta(X)$.*

Corollary 1. *The product of any collection of m-pseudocompact spaces is m-pseudocompact iff it is pseudocompact.*

Proof. Necessity is obvious. To prove sufficiency, let $X = \pi X_i$, where X_i is m-pseudocompact for each i. Then X_i is G_m-dense in $\beta(X_i)$ for each i which implies that πX_i is G_m-dense in $\pi\beta(X_i)$. But $\pi\beta(X_i) = \beta(\pi X_i)$ by Glickberg's Theorem (see [4]) since πX_i is pseudocompact by assumption. Hence X is m-pseudocompact by Theorem 1 (e). □

Corollary 2. *If $m \geqslant c$ then the product of any collection of initially m-compact spaces is m-pseudocompact.*

Proof. Since an initially m-compact space is obviously m-pseudocompact it suffices to show, by Corollary 1, that the product is pseudocompact. It follows from Theorem 5 of [6] that an initially m-compact space is totally bounded if $m \geqslant c$. (Recall that a space is said to be *totally bounded* if the closure of every countable set is compact.) Since the product of any collection of totally bounded spaces is totally bounded and a totally bounded space is pseudocompact the assertion follows. □

Remark. It is well known that there are countably compact, i.e., initially ω-compact, spaces whose product is not pseudocompact. (See, for example, [3]). The above result shows that the corresponding result is not valid for $m \geqslant c$. The familiar examples of pseudocompact spaces whose product is not pseudocompact are subspaces of βD containing D where D is a discrete space. The result below shows that examples of this kind do not exist for $m \geqslant c$.

Corollary 3. *Let $m \geqslant c$, let $\{D_i : i \in I\}$, be a collection of discrete spaces, and let $\{X_i : i \in I\}$ be a collection of m-pseudocompact spaces such that $D_i \subseteq X_i \subseteq \beta D_i$ for each i. Then πX_i is m-pseudocompact.*

Proof. It is clear from the proof of Corollary 2 that the product of m-pseudocompact spaces each containing a dense totally bounded subspace is m-pseudocompact. Let $A_i = \{p \in \beta D_i : p$ is in the closure of some countable subset of $D_i\}$. Then A_i is totally bounded for each i. Since each singleton set in A_i is a G_m-set in βD_i and X_i is G_m-dense in βX_i it follows that $A_i \subseteq X_i$. This concludes the proof. □

Example. Let m be uncountable. Then the space $X = [0,1]^m - \{p\}$ where p is any point of $[0,1]^m$ is k-pseudocompact for any $k < m$ but not k-pseudocompact for any $k \geqslant m$. Thus k-pseudocompactness is in general weaker than m-pseudocompactness if $k < m$.

3. Weakly initially m-compact spaces

Recall that X is weakly initially m-compact if every open cover of X of cardinality $\leqslant m$ has a finite subset with a dense union. Equivalently X is weakly initially m-compact if every open filter base in X of cardinality $\leqslant m$ has an adherence point.

Theorem 2.

(a) *A regular closed set in a weakly initially m-compact space is weakly initially m-compact.*

(b) *The preimage under a perfect irreducible map of a weakly initially m-compact space is weakly initially m-compact.*

(c) *A weakly initially m-compact space is m-pseudocompact.*

(d) *An extremally disconnected m-pseudocompact space is weakly initially m-compact.*

Proof. (a) The interiors in X of the members of an open filter base in a regular closed set form a filter base in X with the same adherence as the original filter base.

(b) Let $f: X \to Y$ be a perfect irreducible map from X onto a weakly initially m-compact space Y. Let $\mathbf{U}$ be an open filter base in X of cardinality $\leqslant m$. Then it follows from the closedness and irreducibility of f that $\mathbf{V} := \{\operatorname{int} f[U] : U \in \mathbf{U}\}$ is an open filter base in Y. Let y be an adherent point of $\mathbf{V}$ and let $K = f^{-1}[p]$. Then an easy compactness argument shows that K contains an adherence point of $\mathbf{U}$.

(c) Let X be a weakly initially m-compact space and let $\mathbf{F}$ be a zero set filter and let $\mathbf{E}$ be a subset $\mathbf{F}$ of cardinality $\leqslant m$. For each Z in $\mathbf{E}$ there is a countable family of cozero sets $\{C_n(Z) \colon n \in \mathbf{N}\}$ such that $\overline{C_{n+1}}(Z) \subseteq C_n(Z)$ for $n \in \mathbf{N}$ and $\bigcap C_n(Z) = Z$. Then the family $\{C_n(Z) \colon n \in \mathbf{N},\ Z \in \mathbf{E}\}$ is an open filter base in X of cardinality $\leqslant m$ whose adherence is the intersection of $\mathbf{E}$. Hence X is m-pseudocompact by Theorem 1.

(d) Let X be an extremally disconnected m-pseudocompact space annd let $\mathbf{U}$ be an open filter base in X of cardinality $\leqslant m$. Let $\mathbf{V} = \{\overline{U} \colon U \in \mathbf{U}\}$. Then $\mathbf{V}$ is a family of zero (in fact clopen) sets in X whose intersection is nonvoid since X is m-pseudocompact. □

Examples. We now show that for $m \geqslant c$

(1) a regular closed set in an m-pseudocompact space need not be m-pseudocompact;

(2) an m-pseudocompact space need not be weakly initially m-compact;

(3) a perfect irreducible preimage of an m-pseudocompact space need not be m-pseudocompact.

R. M. Stephenson, Jr. and J. E. Vaughan [8], show that for each m and each discrete space D of cardinality m, there are weakly initially m-compact subspaces X and Y of βD containing D whose intersection (and so $X \times Y$) is not weakly initially m-compact. By Corollary 3, $X \times Y$ is m-pseudocompact. Thus an m-pseudocompact space need not be weakly initially m-compact. Let Z be the diagonal of $X \times Y$. Then Z is extremally disconnected and not weakly initially m-compact and hence not m-pseudocompact by Theorem 2. Since Z is a regular closed set in $X \times Y$ this shows that a regular closed set in an m-pseudocompact space need not be one. Finally let E be the Gleason cover of $X \times Y$, i.e., an extremally disconnected space which is mapped onto $X \times Y$ by a perfect irreducible map. Then E is not m-pseudocompact, since otherwise, it would be weakly initially m-compact, by Theorem 2(d), which is impossible. Hence a perfect irreducible preimage of an m-pseudocompact space need not be one.

4. m-PSEUDOCOMPACTNESS IN THE SENCE OF COMPLETE ACCUMULATION POINTS

W. W. Comfort and S. Negrepontis [1] define a space X to be *pseudo-(k,k)-compact*, where k is an infinite cardinal number, if for each family $\{U_i \colon i < k\}$ of nonvoid open sets indexed by ordinals less than k, there is $x \in X$ such that for each neighbourhood V of x $|\{i < k \colon U_i \cap V \neq 0\}| = k$. Recall that mpcap stands for m-pseudocompact in the sense of complete accumulation points.

Theorem 3.

(a) *A regular closed set in a mpcap space is mpcap.*

(b) *A perfect irreducible preimage of an mpcap space is mpcap.*

(c) *An initially m-compact space is mpcap.*

(d) *A space is mpcap iff it is pseudo-(k,k)-compact for each $k \leqslant m$.*

(e) *Let $D \subseteq X \subseteq \beta D$, where D is discrete. Then X is mpcap iff every infinite subset of D of cardinality $\leqslant m$ has a complete accumulation point in X.*

Proof. The proofs of parts (a) and (b) are similar to those of the corresponding parts of Theorem 2.

(c) Recall that a space X is intially m-compact iff each infinite subset of cardinality $\leqslant m$ has a complete accumulation point. (See for example, [7].) Let X be an initially m-compact space and let $\mathbf{U}$ be an infinite family of open sets of cardinality $k \leqslant m$. Let $\mathbf{U} = \{U_i : i < k\}$ be a one to one indexing of $\mathbf{U}$. Let $f: k \to D$ be such that $f(i) \in U_i$, for each $i \in k$. Let $A = \{f(i): i \in k\}$ and for each $a \in A$ let $c(a) = |f^{-1}[a]|$. If $|A| = k$ or if $c(a) = k$ for some $a \in A$ then $\mathbf{U}$ has a complete accumulation point. So assume that $|A| < k$ and $c(a) < k$ for each $a \in A$. We may assume that $|A| = \operatorname{cf} k$, the cofinality of k. We can define a function g: cf $k \to A$ such that $c(g(i)) < c(g(j))$ whenever $i < j < k$ and such that $\{c(g(i)) : i < \operatorname{cf} k\}$ is cofinal with k. We may assume that g is onto. Let x be a complete accumulation point of A, let V be a neighbourhood of x and let $B = V \cap A$. Then $|B| = \operatorname{cf} k$. Hence $\sum\{c(b): b \in B\} = k$. Hence $|\{i < k : V \cap U_i \neq 0\}| = k$. Hence x is complete accumulation point of $\mathbf{U}$.

(d) The proof is similar to that of part (c) and is left to the reader.

(e) If X is mpcap and A is an infinite subset of D of cardinality $\leqslant m$ then it must have a complete accumulation point since A is a union of singleton open sets of cardinality $\leqslant m$. Conversely let $\mathbf{U}$ be an infinite family of nonvoid open sets of cardinality $\leqslant m$. Let $\mathbf{U} = \{U_i : i < k\}$ be a one to one indexing of $\mathbf{U}$ where $k \leqslant m$. Let $f : k \to D$ be such that $f(i) \in U_i$ for $i < k$. Proceed as in part (c) above. □

Examples. Let $m \geqslant c$. We give examples to show that

(1) an mpcap space need not be m-pseudocompact (and hence not weakly initially m-compact);

(2) an m-pseudocompact space need not be mpcap;

(3) the product of two mpcap spaces need not be mpcap.

Let D be a discrete space and let $p \in \beta D$. The type of p, $T(p) := \{\overline{f}(p): \overline{f}$ is a mapping from βD to βD whose restriction to D is a permutation of $D\}$. Let $n(p) = \min\{|A|: A \in p\}$. For each infinite cardinal $k \leqslant |D|$, let $p(k)$ be an ultrafilter on D such that $n(p(k)) = k$. Let $X = D \cup \bigcup\{T(p(k)): k \leqslant m$ and $k \leqslant |D|\}$. Then any infinite subset of D of cardinality of k has a complete accumulation point in $T(p(k))$. Hence X is mpcap by Theorem 3.

In particular if D is countable and p is any free ultrafilter on D then $D \cup T(p)$ is mpcap for any m. If $m \geqslant c$ then the space is not m-pseudocompact since any m-pseudocompact subset of βD containing D is βD itself.

Let $m \geqslant c$ and let D be a discrete space of cardinality m. There exist weakly initially m-pseudocompact spaces X and Y of βD containing D such that $X \cap Y$ contains no uniform ultrafilter (an ultrafilter each member of which has cardinality m). (See [8].) Then $X \cap Y$ is not mpcap since D has no complete accumulation point in $X \cap Y$. Hence $X \times Y$ is not mpcap since the diagonal of $X \times Y$, which is a regular closed set in $X \times Y$, is not mpcap. Hence the product of two mpcap spaces need not be mpcap. We also see that an m-pseudocompact space need not be mpcap.

I conclude this discussion with two questions:

(1) Are there m-pseudocompact spaces whose product is not m-pseudocompact where $m > \omega$?

(2) Are weakly initially m-compact spaces necessarily mpcap?

In connection with question 2, we note that if D is a discrete space and $D \subseteq X \subseteq \beta D$ and X is m-pseudocompact (and hence weakly initially m-compact) then it is mpcap by Theorem 4.

References

[1] *W. W. Comfort and S. Negrepontis*: Chain conditions in topology, Cambridge University press, 1982.

[2] *Z. Frolík*: Generalizations of compact and Lindelöf spaces, Czechoslovak Math. J. *84* (1959), 174–217.

[3] *L. Gilman and M. Jerison*: Rings of continuous functions, Van Nostrand, Princeton, N.J., 1960.

[4] *I. Glicksberg*: Stone-Čech compactifications of products, Trans. Amer. Math. Soc. *90* (1959), 369–382.

[5] *J. F. Kennison*: m-pseudocompactness, Trans. Amer. Math. Soc. *104* (1962), 436–442.

[6] *V. Saks and R. M. Stephenson*: Initially m-compact spaces, Proc. Amer. Math. Soc. *28* (1971), 279–288.

[7] *R. M. Stephenson, Jr.*: Initially m-compact spaces, Handbook of set theoretical topology, K. Kunen and J. E. Vaughan, Ed., North Holland, 1984.

[8] *R. M. Stephenson, Jr. and J. E. Vaughan*: Products of initially m-compact spaces, Trans. Amer. Math. Soc. *196* (1974), 177–189.

Author's address: Mathematics Department, Addis Ababa University, P. O. Box 1176, Addis Ababa, Ethiopia.