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ASYMPTOTIC INTERTWINING AND SPECTRAL INCLUSIONS ON BANACH SPACES

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INTRODUCTION

The question of the extent to which the spectrum of a continuous linear operator is an invariant under reasonable equivalence relations has received an enormous amount of attention, particularly within the scope of Hilbert spaces and for relations such as quasi-similarity. In this note, we shall develop some general results on spectral inclusions of the form $\sigma(T) \subseteq \sigma(S)$ or $\sigma(S) \subseteq \sigma(T)$, where T and S are continuous linear operators on Banach spaces X, resp. Y, linked by a continuous linear mapping A from X to Y. A typical ancestor of this theory is the following result of Colojoară and Foiaş [8]. They show that $\sigma(T) = \sigma(S)$ if both S and T are decomposable and A is injective, has dense range, and intertwines S and T in the sense that $SA = AT$.

Here we shall obtain theorems of this type for considerably more general classes of operators S and T and for a very weak notion of intertwining which also dates back to Foiaş, cf. [8]. This provides a unified approach to a variety of situations, involving quasi-nilpotent equivalence, quasi-similarity, and related notions. Our results will cover operators with Dunford's property (C), Bishop's property (β), or certain rather weak spectral decomposition properties. Our principal tools are from local spectral theory, and we shall make essential and frequent use of the recent results on restrictions and quotients of decomposable operators due to Albrecht and Eschmeier [3].

The relevant definitions and background material will be collected in Section 1. Section 2 contains the basic results on asymptotic intertwining. The development is mainly in the spirit of [8], but here we have to contend with two distinct classes of analytic spectral subspaces. Asymptotically intertwined operators will necessarily have overlapping spectra. In fact, the approximate point spectrum of one will touch the surjectivity spectrum of the other. In Section 3, the emphasis will be on surjective intertwiners and the permanence of various spectral properties under asymptotic similarity. The main results on inclusions for the finer spectral structure will be obtained in Section 4 under the appropriate weak assumptions on the intertwiner. Among the consequences we shall record spectral inclusions for hyponormal operators and some of their generalizations. Even more significant examples of the applicability of the theory come from harmonic analysis where the Fourier transformation acts as an intertwiner for convolution operators and the corresponding multiplication operators.

1. Preliminaries from local spectral theory

We first recall some basic notions and results from spectral theory; the monographs [8] and [24] contain further information. Given a complex Banach space X and the Banach algebra $L(X)$ of all bounded linear operators on X, an operator $T \in L(X)$ is called *decomposable* if, for every open covering $\{U_1, U_2\}$ of the complex plane $\mathbf{C}$, there are T-invariant closed linear subspaces Y_1 and Y_2 of X such that $Y_1 + Y_2 = X$ and $\sigma(T|Y_k) \subseteq U_k$ for $k = 1, 2$ where σ denotes the spectrum, cf. [2]. If it is only required that the sum $Y_1 + Y_2$ be dense in X, one obtains the definition of the *weak 2-spectral decomposition property* (weak 2-SDP), cf. [10]. It follows from the example given by Albrecht [1] that, in general, this property is strictly weaker than decomposability.

We shall also need some closely related notions. An operator $T \in L(X)$ is said to have *Bishop's property* (β) if, for every open subset U of $\mathbf{C}$ and for every sequence of analytic functions $f_n : U \to X$ for which $(T-\lambda)f_n(\lambda)$ converges uniformly to zero on each compact subset of U, it follows that also $f_n(\lambda) \to 0$ as $n \to \infty$, locally uniformly on U, cf. [6]. Obviously, property (β) implies that T has the *single valued extension property*, which means that, for every open $U \subseteq \mathbf{C}$, the only analytic solution f: $U \to X$ of the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the constant $f \equiv 0$, cf. [8]. Finally, an operator $T \in L(X)$ is said to have the *decomposition property* (δ) if, given an arbitrary open covering $\{U_1, U_2\}$ of $\mathbf{C}$, every $x \in X$ has a decomposition $x = u_1 + u_2$ where $u_1, u_2 \in X$ satisfy $u_k = (T-\lambda)f_k(\lambda)$ for all $\lambda \in \mathbf{C} \setminus \overline{U}_k$ and some analytic function $f_k : \mathbf{C} \setminus \overline{U}_k \to X$ for $k = 1, 2$; cf. [3]. Note that it follows from the example given in [1] and Theorem I.4.5 of [11] that operators with the weak 2-SDP need not have property (δ). Conversely, we shall see at the end of this section that there are operators with property (δ) which do not have the weak 2-SDP.

It has been observed in [4] that an operator $T \in L(X)$ is decomposable if and only if it has both properties (β) and (δ). More significantly, Albrecht and Eschmeier [3] have recently completed the duality program for linear operators on Banach spaces initiated by Bishop [6]. They prove in [3] that the properties (β) and (δ) are dual to each other in the sense that an operator $T \in L(X)$ satisfies (β) if and only if the adjoint operator T^* on the dual space X^* satisfies (δ) and that the corresponding statement remains valid if both properties are interchanged. It has also been shown in [3] that an operator $T \in L(X)$ has property (β) if and only if T is similar to the restriction of a decomposable operator to one of its closed invariant subspaces and that T has property (δ) if and only if T is similar to a quotient of a decomposable operator. These results have been very useful in recent work on the invariant subspace problem for operators on Banach spaces, see for instance [12].

Finally, given an arbitrary operator $T \in L(X)$ and a closed subset F of $\mathbb{C}$, let $X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}$ denote the corresponding ***analytic spectral subspace***, where $\sigma_T(x) \subseteq \mathbb{C}$ is the *local spectrum* of T at the point $x \in X$, i.e. the complement of the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ in $\mathbb{C}$ and an analytic function $f\colon U \to X$ such that $(T-\mu)f(\mu) = x$ for all $\mu \in U$, cf. [8]. Similarly, for each closed $F \subseteq \mathbb{C}$, let $\mathfrak{X}_T(F)$ denote the space of all $x \in X$ for which there exists some analytic function $f\colon \mathbb{C} \setminus F \to X$ with $(T-\mu)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus F$, cf. [3]. Obviously, property (δ) means precisely that $X = \mathfrak{X}_T(\overline{U}) + \mathfrak{X}_T(\overline{V})$ for every open covering $\{U, V\}$ of $\mathbb{C}$. In the next proposition, the equivalence of (a) and (b) was observed by T. V. Petersen.

Proposition 1.1. *$\mathfrak{X}_T(\emptyset) = \{0\}$, $\mathfrak{X}_T(F) = \mathfrak{X}_T(\sigma(T) \cap F)$ and $\mathfrak{X}_T(F) \subseteq X_T(F)$ for all closed $F \subseteq \mathbb{C}$. Moreover, the following assertions are equivalent:*

(a) *T has the single valued extension property.*
(b) *$\mathfrak{X}_T(F) = X_T(F)$ for all closed $F \subseteq \mathbb{C}$.*
(c) *$X_T(\emptyset)$ is closed.*
(d) *$X_T(\emptyset) = \{0\}$.*

Proof. The first identity follows from Liouville's theorem, the second identity can be easily verified, and the inclusion $\mathfrak{X}_T(F) \subseteq X_T(F)$ is obvious. The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are trivial, and (d) $\Rightarrow$ (a) has been obtained in Proposition IV.3.6 of [24]. Finally note that, by Proposition IV.3.4 of [24], we have $(T-\lambda)X_T(\emptyset) = X_T(\emptyset)$ for all $\lambda \in \mathbb{C}$. Hence, if $X_T(\emptyset)$ is a Banach space, then it follows from elementary spectral theory that $X_T(\emptyset) = \{0\}$, see for instance Proposition 1.3 below. Thus (c) implies (d). □

Recall that an operator $T \in L(X)$ is said to have ***Dunford's property*** (C) if $X_T(F)$ is closed for each closed $F \subseteq \mathbb{C}$. It is an intriguing open problem whether the properties (β) and (C) are equivalent. The following implications hold in general.

Proposition 1.2. *Bishop's property (β) implies Dunford's property (C), and property (C) implies the single valued extension property.*

The first implication is well known and easily seen, and the second implication is clear from Proposition 1.1. In particular, by Proposition 1.3.8 of [8], it follows that an operator is decomposable if and only if it has both properties (δ) and (C).

As usual, let $\sigma_p(T)$ and $\sigma_{ap}(T)$ denote the *point spectrum* and the *approximate point spectrum* of an operator $T \in L(X)$. Thus $\sigma_{ap}(T)$ consists of all $\lambda \in \mathbf{C}$ for which there exists a sequence of unit vectors $x_n \in X$ such that $(T-\lambda)x_n \to 0$ as $n \to \infty$. Further, let $\sigma_{su}(T) := \{\lambda \in \mathbf{C} \colon (T-\lambda)X \neq X\}$ denote the *surjectivity spectrum of* T. The following properties of the surjectivity spectrum will be useful.

Proposition 1.3. *For each $T \in L(X)$, $\sigma_{su}(T) = \sigma_{ap}(T^*)$ and $\sigma_{su}(T^*) = \sigma_{ap}(T)$. Moreover, $\sigma_{su}(T)$ is compact with $\partial\sigma(T) \subseteq \sigma_{su}(T) \subseteq \sigma(T) = \sigma_{su}(T) \cup \sigma_p(T)$ and*

$$\sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x).$$

Finally, if T has the single valued extension property, then $\sigma(T) = \sigma_{su}(T)$, and if T^* has the single valued extension property, then $\sigma(T) = \sigma_{ap}(T)$.

Proof. The first two identities are standard, see for instance Corollary 57.17 and Theorem 57.18 of [5]. The remaining assertions follow easily from these identities and from Lemmas 1 and 2 of [19], see also [25]. □

As a consequence we obtain the following generalization of a classical result on decomposable operators, cf. Corollary 2.1.4 of [8].

Proposition 1.4. *If the operator $T \in L(X)$ has either property (δ) or the weak 2-SDP, then $\sigma(T) = \sigma_{ap}(T)$.*

Proof. By Proposition 1.3 it suffices to show that T^* has the single valued extension property. If T has the weak 2-SDP, this follows from Corollary I.2.8 of [11], and if T has (δ), we know from [3] that T^* has property (β). □

Example 1.5. The left shift L on the Hilbert space $\ell^2(\mathbf{N})$ has property (δ), but not the weak 2-SDP.

Proof. Since the right shift R on $\ell^2(\mathbf{N})$ is subnormal as the restriction of the bilateral right shift on $\ell^2(\mathbf{Z})$, it is clear that R has property (β). Since L is the adjoint of R, it follows from [3] that L has property (δ). Now suppose that L has the weak 2-SDP. Then, since $\sigma(L)$ is the unit disc, there exist non-trivial and proper closed L-invariant subspaces Y and Z of $\ell^2(\mathbf{N})$ for which $Y + Z$ is dense in $\ell^2(\mathbf{N})$. But then $Y^\perp \cap Z^\perp = \{0\}$. Since $Y^\perp$ and $Z^\perp$ are non-zero R-invariant subspaces, this contradicts Beurling's characterization of these latter spaces, cf. Corollary 2 of Problem 126 in [14]. □

In the following, let X and Y be complex Banach spaces, and let $L(X,Y)$ denote the space of all continuous linear operators from X to Y. For given operators $T \in L(X)$ and $S \in L(Y)$, we consider the corresponding commutator $C(S,T)$: $L(X,Y) \to L(X,Y)$ defined by $C(S,T)(A) := SA - AT$ for all $A \in L(X,Y)$. Clearly, for all $n \in \mathbf{N}$ and all $A \in L(X,Y)$ we have

$$C(S,T)^n(A) := C(S,T)^{n-1}(SA - AT) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A\, T^k.$$

An operator $A \in L(X,Y)$ is said to *intertwine S and T asymptotically* if

$$\|C(S,T)^n(A)\|^{1/n} \to 0 \quad \text{as } n \to \infty.$$

This condition has been investigated by Colojoară and Foiaş [8] and Vasilescu [24] in the context of decomposable operators. Here we shall extend some of their results to more general classes of operators.

Lemma 2.1. *Assume that the operator $A \in L(X,Y)$ intertwines S and T asymptotically, let $x \in X$, and consider an analytic function $f: U \to X$ on an open subset U of $\mathbf{C}$ such that $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. Then the infinite series*

$$g(\lambda) := \sum_{n=0}^{\infty} (-1)^n C(S,T)^n(A) \frac{f^{(n)}(\lambda)}{n!} \qquad \text{for all } \lambda \in U$$

converges locally uniformly on U and hence defines an analytic function $g: U \to Y$. Moreover, we have $(S - \lambda)g(\lambda) = Ax$ for all $\lambda \in U$.

Proof. We follow the line of reasoning in the proof of Theorem 2.3.3 in [8]. Consider a pair of concentric closed discs $E \subset D \subset U$ with radii $0 < s < r$ and choose a constant $K \geqslant 0$ such that $\|f(\lambda)\| \leqslant K$ for all $\lambda \in D$. Then, for each $\lambda \in E$, we obtain from Cauchy's integral formula

$$\left\|\frac{f^{(n)}(\lambda)}{n!}\right\| = \left\|\frac{1}{2\pi i}\int_{\partial D} (\zeta - \lambda)^{-n-1} f(\zeta)\, d\zeta\right\| \leqslant K\, r(r-s)^{-n-1} \qquad \text{for all } n \geqslant 0.$$

Also, by assumption, for $\varepsilon := \frac{1}{2}(r - s)$ there exists some constant $L \geqslant 0$ such that $\|C(S,T)^n(A)\| \leqslant L\varepsilon^n$ for all $n \geqslant 0$. An obvious combination of these estimates yields

$$\left\|C(S,T)^n(A)\frac{f^{(n)}(\lambda)}{n!}\right\| \leqslant K\, L\, r(r-s)^{-1} 2^{-n} \qquad \text{for all } \lambda \in E \text{ and } n \geqslant 0.$$

We conclude that the infinite series defining $g(\lambda)$ converges uniformly on E and hence locally uniformly on U. To prove the last assertion, we first observe that $S\ C(S,T)^n(A) = C(S,T)^{n+1}(A) + C(S,T)^n(A)T$ for all $n \geqslant 0$. Also, from $(T-\lambda)f(\lambda) = x$ for all $\lambda \in U$ we obtain by induction that $(T-\lambda)f^{(n)}(\lambda) = n\ f^{(n-1)}(\lambda)$ for all $\lambda \in U$ and $n \geqslant 1$. Consequently, for each $\lambda \in U$, we have

$$
\begin{aligned}
(S-\lambda)g(\lambda) &= \sum_{n=0}^{\infty}(-1)^n (S-\lambda)C(S,T)^n(A)\frac{f^{(n)}(\lambda)}{n!} \\
&= \sum_{n=0}^{\infty}(-1)^n \left(C(S,T)^{n+1}(A) + C(S,T)^n(A)(T-\lambda)\right)\frac{f^{(n)}(\lambda)}{n!} \\
&= \sum_{n=1}^{\infty}(-1)^n \left(C(S,T)^{n+1}(A)\frac{f^{(n)}(\lambda)}{n!} + C(S,T)^n(A)\frac{f^{(n-1)}(\lambda)}{(n-1)!}\right) \\
&\qquad + C(S,T)(A)f(\lambda) + A(T-\lambda)f(\lambda)
\end{aligned}
$$

and hence $(S-\lambda)g(\lambda) = A(T-\lambda)f(\lambda) = Ax$, which completes the proof of the lemma. □

As an immediate corollary we obtain that asymptotic intertwining implies an inclusion for local spectra, see also Theorem 2.3.3 of [8]. In terms of spectral subspaces, this result reads as follows.

Proposition 2.2. *If $A \in L(X,Y)$ intertwines S and T asymptotically, then the inclusions $AX_T(F) \subseteq Y_S(F)$ and $A\mathfrak{X}_T(F) \subseteq \mathfrak{Y}_S(F)$ hold for all closed $F \subseteq \mathbf{C}$.*

An easy consequence is the following generalization of Rosenblum's theorem, cf. [21] and [15]: if $\sigma(T) \cap \sigma(S) = \emptyset$, then the zero operator is the only operator A which intertwines S and T asymptotically. Indeed, it follows from Propositions 1.1 and 2.2 that $A\ X = A\ \mathfrak{X}_T(\sigma(T)) \subseteq \mathfrak{Y}_S(\sigma(T)) = \mathfrak{Y}_S(\sigma(T)\cap\sigma(S)) = \mathfrak{Y}_S(\emptyset) = \{0\}$ and hence $A = 0$. Actually we can do better:

Proposition 2.3. *If there exists a non-zero operator $A \in L(X,Y)$ which intertwines S and T asymptotically, then $\sigma_{su}(T) \cap \sigma_{ap}(S) \neq \emptyset$.*

Proof. By Theorem 4 of [9], if $\sigma_{su}(T) \cap \sigma_{ap}(S) = \emptyset$, then $0 \notin \sigma_{ap}(C(S,T))$ and hence there exists some constant $M > 0$ such that $||C(S,T)(A)|| \geqslant M||A||$ for every $A \in L(X,Y)$. If A intertwines S and T asymptotically, then $||A|| = 0$ since

$$||C(S,T)^n(A)||^{1/n} \geqslant M||A||^{1/n} \quad \text{for all } n \in \mathbf{N}.$$

□

With additional assumptions on the operators S and T, we obtain the converse of Proposition 2.2. This is included in the following theorem which generalizes Proposition IV.6.2 of [24] and Theorem 2.3.3 of [8].

Theorem 2.4. *Assume that $T \in L(X)$ has property (δ) and that $S \in L(Y)$ has property (C). Then the operator $C(S,T)$ has the single valued extension property, and for each $A \in L(X,Y)$ the following statements are equivalent:*

(a) $\|C(S,T)^n(A)\|^{1/n} \to 0$ *as* $n \to \infty$.

(b) $AX_T(F) \subseteq Y_S(F)$ *for all closed* $F \subseteq \mathbb{C}$.

(c) $A\mathfrak{X}_T(F) \subseteq \mathfrak{Y}_S(F)$ *for all closed* $F \subseteq \mathbb{C}$.

(d) $\sigma_{C(S,T)}(A) = \{0\}$.

Proof. To verify that $C(S,T)$ has the single valued extension property, we adopt some techniques from the proof of Proposition IV.6.2 in [24]. First note that, if $H \in L(X,Y)$ and $\lambda \in \mathbb{C}$ satisfy $(C(S,T)-\lambda)H = 0$, then $S\,H = H(T+\lambda)$ from which it is immediate that $\sigma_S(Hx) \subseteq \sigma_{T+\lambda}(x) = \sigma_T(x)+\lambda$ and hence $Hx \in Y_S(\sigma_T(x)+\lambda))$ for all $x \in X$. Now let $U \subseteq \mathbb{C}$ be open and connected and $H: U \to L(X,Y)$ be analytic such that $(C(S,T)-\lambda)H(\lambda) = 0$ for all $\lambda \in U$. Then $H(\lambda)x \in Y_S(\sigma_T(x)+\lambda)$ for all $\lambda \in U$ and $x \in X$. We next choose a pair of non-trivial closed discs D_1, $D_2 \subseteq U$ with positive distance $\varepsilon > 0$. For $k = 1,2$ and arbitrary $x \in X$ we conclude that $H(\lambda)x \in Y_S(\sigma_T(x) + D_k)$ for all $\lambda \in D_k$ and therefore, by analytic continuation, for all $\lambda \in U$, since $Y_S(\sigma_T(x) + D_k)$ is a Banach space by property (C). Hence for all $\lambda \in U$ we obtain

$$H(\lambda)x \in Y_S(\sigma_T(x) + D_1) \cap Y_S(\sigma_T(x) + D_2) = Y_S\left((\sigma_T(x) + D_1) \cap (\sigma_T(x) + D_2)\right).$$

But $(\sigma_T(x)+D_1)\cap(\sigma_T(x)+D_2) = \emptyset$ whenever $\operatorname{diam}\sigma_T(x) < \varepsilon$. Since Proposition 1.2 shows that S has the single valued extension property, we conclude from $Y_S(\emptyset) = \{0\}$ that $H(\lambda)x = 0$ for all $\lambda \in U$ and all $x \in X$ with $\operatorname{diam}\sigma_T(x) < \varepsilon$. Since T has property (δ), every element in X can be written as a finite sum of elements $x \in X$ with $\operatorname{diam}\sigma_T(x) < \varepsilon$. Therefore $H(\lambda) \equiv 0$ for all $\lambda \in U$, which proves that $C(S,T)$ has the single valued extension property. By Corollary 2.4 of [16], this implies that, for each $A \in L(X,Y)$, the conditions (a) and (d) are equivalent. The implication (a) $\Rightarrow$ (b) is clear from Proposition 2.2, and (b) $\Rightarrow$ (c) follows immediately from Proposition 1.1 and the inclusions $A\mathfrak{X}_T(F) \subseteq A\ X_T(F) \subseteq Y_S(F) = \mathfrak{Y}_S(F)$ for all closed $F \subseteq \mathbb{C}$. Finally, assume that condition (c) holds. Since T has property (δ), we know from [3] that there exists a decomposable operator $R \in L(Z)$ on some Banach space Z and a continuous linear surjection $Q \in L(Z,X)$ such that $TQ = QR$. From (c) we conclude that

$$(AQ)Z_R(F) = (AQ)\mathfrak{Z}_R(F) \subseteq A\ \mathfrak{X}_T(F) \subseteq \mathfrak{Y}_S(F) = Y_S(F) \quad \text{for all closed } F \subseteq \mathbb{C}.$$

Since R is decomposable and S has property (C), the proof of Theorem 2.3.3 in [8] now shows that $\|C(S,R)^n(AQ)\|^{1/n} \to 0$ as $n \to \infty$. But $C(S,R)^n(AQ) = C(S,T)^n(A)Q$ for all $n \in \mathbf{N}$. Since, by the open mapping theorem for the surjection Q, there exists some constant $M > 0$ such that $M\|B\| \leqslant \|BQ\|$ for all $B \in L(X,Y)$, we conclude that $\|C(S,T)^n(A)\|^{1/n} \to 0$ as $n \to \infty$, which completes the proof. □

3. Spectral consequences of asymptotic similarity

Again, let X and Y be complex Banach spaces. In this section, we shall investigate the case of surjective and injective operators $A \in L(X,Y)$ which intertwine asymptotically two given operators $S \in L(Y)$ and $T \in L(X)$.

Proposition 3.1. *Assume that $A \in L(X,Y)$ satisfies $\|C(S,T)^n(A)\|^{1/n} \to 0$ as $n \to \infty$. If A is injective, then $\sigma_p(T) \subseteq \sigma_{ap}(S)$. If A is surjective, then $\sigma_{su}(S) \subseteq \sigma_{su}(T)$; in particular, $\sigma(S) \subseteq \sigma(T)$ when S has the single valued extension property.*

Proof. First assume that A is injective. Since $C(S-\lambda, T-\lambda) = C(S,T)$ for all $\lambda \in \mathbf{C}$, it suffices to show that $0 \in \sigma_p(T)$ implies that $0 \in \sigma_{ap}(S)$. Choose a non-zero $x \in X$ so that $Tx = 0$ and suppose that $0 \notin \sigma_{ap}(S)$. Then there exists a constant $M > 0$ such that $\|Sy\| \geqslant M\|y\|$ for all $y \in Y$. From

$$C(S,T)^n(A)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A\, T^k x = S^n A x \qquad \text{for all } n \in \mathbf{N}$$

it follows that $\|C(S,T)^n(A)x\|^{1/n} \geqslant M\|Ax\|^{1/n}$ for all $n \in \mathbf{N}$. Since A intertwines S and T asymptotically and since $Ax \neq 0$ by the injectivity of A, we conclude that $M = 0$. This contradiction shows that $\sigma_p(T) \subseteq \sigma_{ap}(S)$ whenever A is one-to-one. Now assume that A is surjective. Then it follows from Propositions 1.3 and 2.2 that

$$\sigma_{su}(S) = \bigcup_{y \in Y} \sigma_S(y) = \bigcup_{x \in X} \sigma_S(Ax) \subseteq \bigcup_{x \in X} \sigma_T(x) = \sigma_{su}(T).$$

The final assertion is also a consequence of Proposition 1.3. □

Corollary 3.2. *Assume that there exist a surjective operator $A \in L(X,Y)$ such that $\|C(S,T)^n(A)\|^{1/n} \to 0$ as $n \to \infty$ and an injective operator $B \in L(Y,X)$ such that $\|C(T,S)^n(B)\|^{1/n} \to 0$ as $n \to \infty$. Then $\sigma(S) \subseteq \sigma(T)$.*

Proof. From Proposition 3.1 we obtain $\sigma_{su}(S) \subseteq \sigma_{su}(T)$ and $\sigma_p(S) \subseteq \sigma_{ap}(T)$ and therefore $\sigma(S) \subseteq \sigma(T)$ by Proposition 1.3. □

Remark 3.3. (i) If $A \in L(X,Y)$ is an injective operator with $C(S,T)^n(A) = 0$ for some $n \in \mathbf{N}$, it follows that $\sigma_p(T)$ is actually contained in $\sigma_p(S)$. Indeed, if $x \in X$ is an eigenvector for the eigenvalue λ of T, then we obtain as before that

$$0 = C(S,T)^n(A)x = C(S-\lambda, T-\lambda)^n(A)x = (S-\lambda)^n Ax$$

and therefore $\lambda \in \sigma_p(S)$. Similarly it can be shown that, under this stronger assumption on A, the single valued extension property carries over from S to T.

(ii) Classical examples of quasi-similar operators with different spectra show that, even under the assumption that $C(S,T)(A) = 0$, the results of Proposition 3.1 cannot be improved in general. For instance by [13] or [15], there exist bounded linear operators A, S, T on a Hilbert space such that A and A^* are injective, $SA = AT$, S is quasi-nilpotent, and the spectrum of T is the unit disc. Clearly, in this case $\sigma_{su}(T^*) \not\subseteq \sigma_{su}(S^*)$, which shows that surjectivity of the intertwiner in Proposition 3.1 cannot be relaxed to an assumption of dense range. Also $\sigma_{ap}(T) \not\subseteq \sigma_{ap}(S)$, which shows that the first part of Proposition 3.1 cannot be improved in general.

(iii) Note, however, that in certain special cases the inclusion of approximate point spectra does hold. Indeed, if some surjective operator $A \in L(X,Y)$ intertwines S and T asymptotically, then $\sigma_{ap}(T^*) \subseteq \sigma_{ap}(S^*)$ by Propositions 1.3 and 3.1. In this case, A^* is injective and intertwines T^* and S^* by the next lemma, which follows immediately from

$$[C(S,T)^n(A)]^* = (-1)^n C(T^*, S^*)^n(A^*) \quad \text{for all } n \in \mathbf{N}.$$

Lemma 3.4. *An operator $A \in L(X,Y)$ intertwines S and T asymptotically if and only if its adjoint $A^* \in L(Y^*, X^*)$ intertwines T^* and S^* asymptotically.*

We shall call the operators $T \in L(X)$ and $S \in L(Y)$ *asymptotically similar* if there exists a bijection $A \in L(X,Y)$ such that A intertwines S and T asymptotically and its inverse A^{-1} intertwines T and S asymptotically. Asymptotic similarity generalizes slightly the notion of quasi-nilpotent equivalence where, in the above definition, $X = Y$ and $A = I$ is the identity operator on X, cf. [8]. It is easily seen that T and S are asymptotically similar if and only if T and $A^{-1}SA$ are quasi-nilpotent equivalent. In particular, it follows that asymptotic similarity is an equivalence relation. Moreover, if T and S are both decomposable, then Theorem 2.4 shows that T and S are asymptotically similar if and only if there exists an invertible operator $A \in L(X,Y)$ such that the identity $AX_T(F) = Y_S(F)$ holds for all closed $F \subseteq \mathbf{C}$, see also Chapter 2 of [8].

Theorem 3.5. *The following are preserved under asymptotic similarity: spectrum, surjectivity spectrum, approximate point spectrum, single valued extension* *property, Dunford's property (C), property (δ), Bishop's property (β), and decomposability.*

Proof. Assume that T and S are asymptotically similar and choose a corresponding bijection $A \in L(X,Y)$ for the asymptotic intertwining of (S,T) and (T,S). Clearly $\sigma(T) = \sigma(S)$ by Corollary 3.2 and $\sigma_{su}(T) = \sigma_{su}(S)$ by Proposition 3.1. It then follows from Proposition 1.3 and Lemma 3.4 that $\sigma_{ap}(T) = \sigma_{su}(T^*) = \sigma_{su}(S^*) = \sigma_{ap}(S)$. Moreover, by Proposition 2.2 we have $AX_T(F) = Y_S(F)$ for all closed $F \subseteq \mathbb{C}$. This shows that property (C) carries over from T to S and, by Proposition 1.1, the same is true for the single valued extension property. By Proposition 2.2, we also have $A\mathfrak{X}_T(F) = \mathfrak{Y}_S(F)$ for all closed $F \subseteq \mathbb{C}$, which implies that property (δ) is preserved. Since, by the results of [3], the properties (β) and (δ) are dual to each other and since T^* and S^* are asymptotically similar by Lemma 3.4, it follows that property (β) is also retained by asymptotic similarity. Finally, since both properties (C) and (δ) are preserved under asymptotic similarity, the same holds for decomposability. □

The preceding theorem subsumes and unifies several classical theorems on quasinilpotent equivalence, cf. Chapters 1 and 2 of [8]. The results involving (β), (δ), and the finer structure of the spectrum appear to be new.

4. Quasi-affine transformations and spectral inclusions

We now come to spectral consequences of asymptotic intertwining under very mild assumptions on the intertwiner. Again, let $T \in L(X)$ and $S \in L(Y)$ be given operators on complex Banach spaces X and Y. The next theorem is the main result of this note.

Theorem 4.1. *Suppose that $A \in L(X,Y)$ intertwines S and T asymptotically.*

(a) *If A has dense range and S has property (C), then $\sigma(S) \subseteq \sigma_{su}(T)$.*

(b) *If A is injective and T has the weak 2-SDP, then $\sigma(T) \subseteq \sigma(S)$.*

(c) *If A is injective and T has property (δ), then $\sigma(T) \subseteq \sigma_{ap}(S)$.*

Proof. (a) From Proposition 1.3 we know that $\sigma_{su}(T)$ is closed and equal to the union of the local spectra $\sigma_T(x)$ over all $x \in X$. This implies that $X = X_T(\sigma_{su}(T))$ and therefore $Y = (AX)^- = (A\ X_T(\sigma_{su}(T)))^- \subseteq Y_S(\sigma_{su}(T))^- = Y_S(\sigma_{su}(T))$ by Proposition 2.2. From Propositions 1.2 and 1.3 we conclude that $\sigma(S) = \sigma_{su}(S) \subseteq \sigma_{su}(T)$.

(b) Let $U \subseteq \mathbb{C}$ be an arbitrary open neighborhood of $\sigma(S)$ and choose an open set $V \subseteq \mathbb{C}$ such that $U \cup V = \mathbb{C}$ and $\sigma(S) \cap \overline{V} = \emptyset$. By the weak 2-SDP, there exist T-invariant closed linear subspaces Y and Z of X such that $\sigma(T|Y) \subseteq U$, $\sigma(T|Z) \subseteq V$, and $Y + Z$ is dense in X. Then obviously $Z \subseteq \mathfrak{X}_T(\overline{V})$ and therefore $AZ \subseteq A\mathfrak{X}_T(\overline{V}) \subseteq \mathfrak{Y}_S(\overline{V}) = \mathfrak{Y}_S(\sigma(S) \cap \overline{V}) = \mathfrak{Y}_S(\emptyset) = \{0\}$ by Propositions 1.1 and 2.2. Since A is injective, it follows that $Z = \{0\}$ and hence $Y = X$. We conclude that $\sigma(T) \subseteq U$ for every open neighborhood U of $\sigma(S)$ and therefore $\sigma(T) \subseteq \sigma(S)$.

(c) Since T has property (δ), we obtain from [3] a decomposable operator $R \in L(Z)$ on some Banach space Z and a surjection $Q \in L(Z, X)$ such that $TQ = QR$. In analogy with the preceding argument, we first show that $\sigma_{su}(T) \subseteq \overline{U}$, where U denotes an arbitrary open neighborhood of $\sigma_{ap}(S)$. Choose an open set $V \subseteq \mathbf{C}$ such that $\sigma_{ap}(S) \cap \overline{V} = \emptyset$ and $U \cup V = \mathbf{C}$. By the decomposability of R, we have $Z = Z_R(\overline{U}) + Z_R(\overline{V})$. It is clear that the restrictions $Q_V := Q|Z_R(\overline{V})$ and $R_V := R|Z_R(\overline{V})$ satisfy

$$C(S, R_V)^n(AQ_V) = C(S,T)^n(A)Q_V \quad \text{for all } n \in \mathbf{N},$$

which implies that AQ_V intertwines S and R_V asymptotically. Since $\sigma(R_V) \subseteq \overline{V}$ and $\sigma_{ap}(S) \cap \overline{V} = \emptyset$, it follows from Proposition 2.3 that $AQ_V \equiv 0$ and hence $Q_V \equiv 0$, by the injectivity of A. We conclude that $Q(Z_R(\overline{V})) = \{0\}$ and consequently

$$X = Q(Z) = Q(Z_R(\overline{U})) \subseteq X_T(\overline{U}).$$

This implies that $\sigma_{su}(T) \subseteq \overline{U}$ for every open neighborhood U of $\sigma_{ap}(S)$ and therefore $\sigma_{su}(T) \subseteq \sigma_{ap}(S)$. But from Proposition 3.1 we also know that $\sigma_p(T) \subseteq \sigma_{ap}(S)$. The desired conclusion follows from the observation in Proposition 1.3 that $\sigma(T) = \sigma_{su}(T) \cup \sigma_p(T)$. □

Part (c) of the preceding result is, in several respects, an improvement of Lemma 1 of [18], and part (b) generalizes Lemma 1 of [10]. These results have been very useful in the spectral theory of convolution operators and multipliers, cf. [17] and [18]. Both (b) and (c) contain Corollary 2.12 in [13] as a special case. We also have the following generalization of Theorem 2.4.4 in [8], which is immediate from Theorem 4.1. Recall that an operator $A \in L(X, Y)$ is a *quasi-affinity* if A is injective and has dense range.

Corollary 4.2. *Assume that the quasi-affinity $A \in L(X,Y)$ intertwines S and T asymptotically. If S has property (C) and T has either property (δ) or the weak 2-SDP, then $\sigma(T) = \sigma(S)$.*

Remark 4.3. (i) It is interesting to compare the statements in Theorem 4.1 by dualizing. If we assume in part (a) that S has property (β), and not just (C), then this weaker result follows easily from part (c) by duality: in this case, A^* is injective and, by [3], the adjoint S^* has property (δ), hence from Lemma 3.4 and part (c) we obtain $\sigma(S^*) \subseteq \sigma_{ap}(T^*)$ and therefore $\sigma(S) \subseteq \sigma_{su}(T)$ by Proposition 1.3. An attempt to derive, analogously, part (c) from (a) immediately hits the obstacle that the adjoint A^* of an injective operator A will have only weak $*$-dense, but not necessarily norm-dense range. Fortunately, the proof of assertion (a) shows that the result remains valid for operators on a dual space if the assumptions of dense range and closed spectral subspaces are fulfilled with respect to the weak $*$-topology. Now, if $T \in L(X)$ satisfies condition (δ), then $T^* \in L(X^*)$ has property (β) and hence norm-closed spectral subspaces. By Proposition I.4.4 of [11], it follows that these spaces are also weak $*$-closed, but this is not easily established. Thus, using duality as above, it is possible to give a proof of part (c) based on assertion (a), but the details involve quite some additional machinery. However, we exhaust our luck when trying to prove also part (b) by this approach, since there are examples of operators with the weak 2-SDP for which the adjoint does not have norm-closed and hence not weak $*$-closed spectral subspaces, cf. Remark I.4.6 of [11].

(ii) Any attempt to obtain general spectral inclusions by swapping the assumptions on S and T in Theorem 4.1 is doomed to fail. In Remark 3.3 (ii) we have mentioned an example where $SA = AT$, A is a quasi-affinity, $\sigma(T)$ is the unit disc, and $\sigma(S) = \{0\}$. Thus $\sigma(T) \not\subseteq \sigma(S)$ although S, as a quasi-nilpotent operator, is decomposable and hence has both property (δ) and the weak 2-SDP. Therefore the analog of (b) or (c) in Theorem 4.1 with decomposability assumptions on S instead of T does not hold. Moreover, a glance at the dual operators in the same example shows that the spectral inclusion from part (a) of Theorem 4.1 need not hold if T is assumed to be decomposable and no particular assumptions are made on S.

Remark 4.4. There are interesting situations in harmonic analysis where the spectral inclusions from part (b) or (c) of Theorem 4.1 actually characterize the decomposability of the operator T. Indeed, let $X := L_1(G)$ denote the group algebra of a locally compact abelian group G, and consider the Banach algebra $Y := C_o(\Gamma)$ of all continuous complex-valued functions on the dual group Γ which vanish at infinity. Then the Fourier transformation $A\colon X \to Y$ given by $Af := \widehat{f}$ for all $f \in X$ is injective and has dense range, cf. [22]. Now, for a regular Borel measure μ on G, let $T_\mu \in L(X)$ denote the corresponding convolution operator given by $T_\mu f := \mu * f$ for all $f \in X$, and let $S_\mu \in L(Y)$ be the operator of multiplication by the Fourier-Stieltjes transform $\widehat{\mu}$ on Y. Then obviously $S_\mu A = AT_\mu$. Also, using the regularity of the Banach algebra $C_o(\Gamma)$, it is easily seen that the operator S_μ is decomposable. Moreover, as an elementary fact, we always have $\overline{\widehat{\mu}(\Gamma)} = \sigma(S_\mu) \subseteq \sigma(T_\mu)$. Note that this containment also follows from Theorem 4.1. If the group G is non-discrete, then classical results from Fourier analysis show that the spectral inclusion will be strict for certain measures μ on G, cf. [26]. On the other hand, Theorem 4.1 proves the identity

$\sigma(T_\mu) = \overline{\widehat{\mu}(\Gamma)}$ whenever T_μ has the weak 2-SDP or property (δ). The converse holds if G is compact and $\widehat{\mu}$ vanishes at infinity. In this case, it has been shown in [17] and [18] that T_μ is decomposable if and only if the measure μ has a natural spectrum in the sense that $\sigma(T_\mu) = \overline{\widehat{\mu}(\Gamma)}$ and that this property is also equivalent to property (δ), to the weak 2-SDP, and to the countability of the spectrum of T_μ. Actually, these results hold in the more general context of multipliers on semi-simple commutative Banach algebras with scattered maximal ideal space, see [17] and [18]. In this case, the injective intertwiner is, of course, the Gelfand transformation.

If the operators $T \in L(X)$ and $S \in L(Y)$ are intertwined asymptotically by quasi-affinities $A \in L(X,Y)$ and $B \in L(Y,X)$, *i.e.* if A and B are quasi-affinities for which $\|C(S,T)^n(A)\|^{1/n} \to 0$ and $\|C(T,S)^n(B)\|^{1/n} \to 0$ as $n \to \infty$, we say that T and S are ***asymptotically quasi-similar***. In this situation, the invariance of the spectrum is valid under quite mild assumptions on the operators. Note that this covers the case of quasi-similar operators T and S where it is required that $C(S,T)(A) = 0$ and $C(T,S)(B) = 0$.

Corollary 4.5. *Suppose that T and S are asymptotically quasi-similar. If T and S have any of the properties (δ), (C) or the weak 2-SDP, then $\sigma(T) = \sigma(S)$*

Proof. By symmetry, it suffices to show that $\sigma(T) \subseteq \sigma(S)$. If T has either the weak 2-SDP or property (δ), this follows from Theorem 4.1 applied to the quasi-affinity $A \in L(X,Y)$. And if T has property (C), the inclusion follows from Theorem 4.1 applied to the quasi-affinity $B \in L(Y,X)$. □

In particular, Corollary 4.5 gives the spectral invariance for quasi-similar operators with Dunford's property (C), a fact noted, for Hilbert space, by Stampfli [23]. Evidently, Corollary 4.5 covers not only the classical case of quasi-similar decomposable operators from [8], but also the case of quotients and restrictions of these, since such operators are characterized by property (δ), resp. property (β), cf. [3]. In particular, quasi-similar subdecomposable operators have identical spectra, which generalizes a recently announced result of L. Yang in the Hilbert space setting. Note that it is shown in [20] that hyponormal operators are subscalar and hence subdecomposable, and small changes of Putinar's argument will yield the same result for M-hyponormal operators. Thus we also capture, as a special case, a generalization of Clary's result [7] that, for quasi-similar M-hyponormal operators, the spectrum is invariant. Finally, [16] studies another class of Banach space operators, the totally paranormal operators, to which the present theory applies. It is shown that these operators have property (C) and that all hyponormal operators belong to this class.

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