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Emília Halušková; Danica Jakubíková-Studenovská

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# PARTIAL MONOUNARY ALGEBRAS WITH COMMON CLOSED QUASI-ENDOMORPHISMS

Emília Halušková, Danica Studenovská,[3] Košice

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The concept of a closed quasi-homomorphism in partial algebras was thoroughly studied by P. Burmeister and B. Wojdylo [1]. This concept is one of possible generalizations of the notion of homomorphism if we deal with partial algebras instead of complete algebras.

Homomorphisms of unary algebras were investigated e.g. in [6], [7] and [3]. In [3] pairs of monounary algebras $(A, f)$ and $(A, g)$ such that $(A, f)$ and $(A, g)$ have common systems of endomorphisms were studied. Analogous questions concerning endomorphisms have been investigated in [4] and [5].

Let $(A, f)$ be a partial monounary algebra. We denote by the symbols $EQ(f)$ and $EQ_c(f)$ the system of all partial mappings $g$ of $A$ into $A$ such that the partial algebras $(A, f)$ and $(A, g)$ have common sets of quasi-endomorphisms or common sets of closed quasi-endomorphisms, respectively. The system $EQ(f)$ was investigated in [2].

The present paper deals with systems of partial monounary algebras which have the same underlying set and common sets of closed quasi-endomorphisms. The main purpose consists in giving a constructive description of all partial mappings belonging to $EQ_c(f)$. It turns out that card $EQ_c(f) \leqslant c$. Next it will be proved that either $EQ_c(f) \subset EQ(f)$ or $EQ(f) \subset EQ_c(f)$ is valid (in fact, both these cases can occur).

## 1. Preliminaries

Let $\mathbf{N}$ be the set of all positive integers, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{Z}$ the set of all integers.

The system of all monounary algebras will be denoted by $\mathcal{U}$ and for the notation of the system of all partial monounary algebras we will use the symbol $\mathcal{U}_p$.

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Consider $(A, f) \in \mathscr{U}_p$. Let $B \subset A$. Put $f_B = \{[x, f(x)]: x \in B \cap \operatorname{dom} f\}$. If $\operatorname{rng} f_B \subset B$, then the partial algebra $(B, f_B)$ is called a subalgebra of $(A, f)$. A partial algebra $(A, f)$ is said to be connected, if for each $x, y \in A$ there exist $m, n \in \mathbf{N}_0$ such that $f^m(x) = f^n(y)$. If $(B, f_B)$ is a maximal connected subalgebra of $(A, f)$, then $B$ is said to be a component of $(A, f)$. We will say that partial algebras $(A, f)$ and $(A, g)$ have the same component partitions, if $B$ is a component of $(A, g)$ for each component $B$ of $(A, f)$ and conversely.

The system of all connected algebras belonging to $\mathscr{U}$ will be denoted by the symbol $\mathscr{U}_c$. The component of a partial monounary algebra $(A, f)$ containing an element $x \in A$ will be denoted by $K_f(x)$.

A nonempty set $C \subset A$ is called a cycle of $(A, f) \in \mathscr{U}_p$, if $C \subset K_f(x)$ for $x \in C$ and there exists $k \in \mathbf{N}$ with $f^k(y) = y$ for each $y \in C$.

A set $R \subset A$ is said to be a chain of $(A, f)$, if $(R, f_R)$ is a subalgebra of $(A, f)$ and one of the following conditions is satisfied:

1. $R = \{a_1, a_2, \ldots, a_n\}$, $n \in \mathbf{N}$, $n > 1$ and $f(a_i) = a_{i+1}$ for $i = 1, 2, \ldots, n-1$, $a_n \notin \operatorname{dom} f$;

2. $R = \{a_i, i \in \mathbf{N}\}$ and $f(a_i) = a_{i+1}$ for each $i \in \mathbf{N}$;

3. $R = \{a_i, i \in \mathbf{Z}\}$ and $f(a_i) = a_{i+1}$ for each $i \in \mathbf{Z}$;

4. $R = \{a_i; i \in \mathbf{Z}, i \leqslant 1\}$ and $f(a_i) = a_{i+1}$ for each $i \in \mathbf{Z}$, $i \leqslant 0$, $a_1 \notin \operatorname{dom} f$.

(In the above conditions we assume that $a_i \neq a_j$ for $i \neq j$.)

Put $F(A) = \{g: g \text{ is a partial mapping of } A \text{ into } A\}$.

A mapping $g \in F(A)$ is called an endomorphism of a partial monounary algebra $(A, f)$ if $\operatorname{dom} g = A$ and $x \in \operatorname{dom} f$ implies $g(x) \in \operatorname{dom} f$ and $g(f(x)) = f(g(x))$. Further, $g \in F(A)$ is said to be a quasi-endomorphism of $(A, f)$ if $x \in \operatorname{dom} f$ and $x$, $f(x) \in \operatorname{dom} g$ yield $g(x) \in \operatorname{dom} f$ and $g(f(x)) = f(g(x))$. If $g$ is a quasi-endomorphism and there is no $x \in A$ such that $x \in \operatorname{dom} f$ and $x$, $f(x) \in \operatorname{dom} g$, then we will say that $g$ is a trivial quasi-endomorphism of $(A, f)$.

For $(A, f) \in \mathscr{U}_p$ put

$$H(f) = \{g \in F(A): g \text{ is an endomorphism of } (A, f)\},$$
$$Q(f) = \{g \in F(A): g \text{ is a quasi-endomorphism of } (A, f)\},$$
$$Q_c(f) = \{g \in F(A): g \in Q(f) \text{ and } f \in Q(g)\}.$$

Following [1], an element of the set $Q_c(f)$ is called a closed quasi-endomorphism of $(A, f)$.

We will use the following notation:

$$EH(f) = \{g \in F(A) \colon H(f) = H(g)\},$$
$$EQ(f) = \{g \in F(A) \colon Q(f) = Q(g)\},$$
$$EQ_c(f) = \{g \in F(A) \colon Q_c(f) = Q_c(g)\},$$
$$EH_0(f) = EH(f) \cap H(f).$$

**Remark.** Let $(A,f) \in U_p$.
(A1) $H(f) \subset Q_c(f) \subset Q(f)$.
(A2) If $g \in EQ_c(f)$, then $f \in EQ_c(g)$.
(A3) If $f \notin Q(g)$, then $g \notin Q_c(f)$.
These facts follow immediately from the definition and will be sometimes used without quotation.

Further we put

$$K_d = \{a \in \operatorname{dom} f \colon \{a\} \text{ is a component of } (A,f)\},$$
$$K_n = \{a \notin \operatorname{dom} f \colon \{a\} \text{ is a component of } (A,f)\},$$
$$K = K_d \cup K_n.$$

We will say that $(A,f)$ is of type $\alpha$, $\tau$, $\pi$, $\gamma$ or $\delta$ if it fulfils the following condition ($\alpha$), ($\tau$), ($\pi$), ($\gamma$) or ($\delta$), respectively (cf. Fig. 1):
($\alpha$) $K \neq A$ and each component $B$ of $(A,f)$ such that $\|B\| > 1$ is a cycle or a chain;
($\tau$) $K \neq A$, $\operatorname{dom} f = A$ and there is $a \in A$ with $f(x) = a$ for each $x \in A$;
($\pi$) $K = A$, $\|K_d\| = 1$ and $\|A\| > 1$;
($\gamma$) $K_n = A$;
($\delta$) $K_d = A$.
Let us mention the following theorem from [2] which will be used in some proofs of this paper.

**Theorem 4.10/[2].** *Let* $(A,f) \in \mathcal{U}_p$.

1° *If* $(A,f)$ *is of type* $\alpha$, *then* $EQ(f) = \{f,g\}$, *where* $\operatorname{dom} g = \operatorname{rng} f$ *and* $g(f(a)) = a$ *for each* $a \in \operatorname{dom} f$.
2° *If* $(A,f)$ *is of type* $\tau$ *with* $a \in A$ *such that* $f(a) = a$, *then* $EQ(f) = \{f,g\}$, *where* $(A,g)$ *is of type* $\pi$ *with* $g(a) = a$.
3° *If* $(A,f)$ *is of type* $\pi$ *with* $a \in A$ *such that* $f(a) = a$, *then* $EQ(f) = \{f,g\}$, *where* $(A,f)$ *is of type* $\tau$ *with* $g(a) = a$.
4° *If* $(A,f)$ *is of type* $\delta$, *then* $EQ(f) = \{f,g\}$, *where* $(A,g)$ *is of type* $\gamma$.

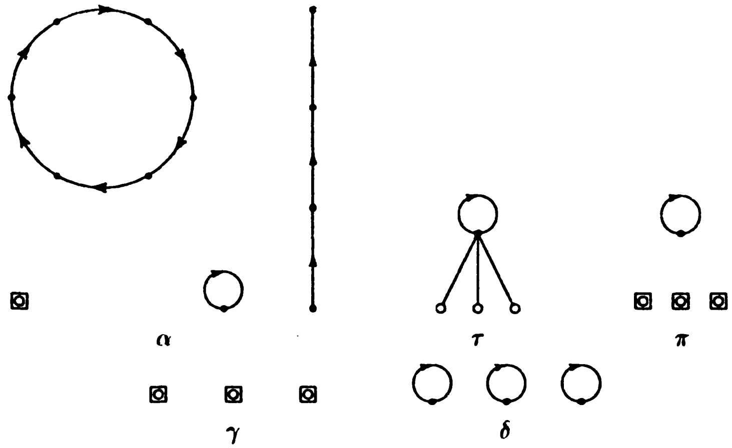


Figure 1.

5° *If* $(A, f)$ *is of type* $\gamma$, *then* $EQ(f) = \{f, g\}$, *where* $(A, g)$ *is of type* $\delta$.

6° *Otherwise* $EQ(f) = \{f\}$.

The following assertions can be proved quite analogously as the assertions 1.1–1.4 and 1.7–1.9 of [2].

**1.1. Lemma.** *Let* $(A, f) \in \mathcal{U}_p$. *Then* $EQ_c(f) = \{g \in Q_c(f) : Q_c(f) = Q_c(g)\}$.

**1.2. Lemma.** *Let* $(A, f) \in \mathcal{U}_p$. *If* $g \in EQ_c(f)$, *then* $EQ_c(f) = EQ_c(g)$.

**1.3. Lemma.** *Let* $(A, f) \in \mathcal{U}_p$. *Then* $EQ_c(f) \subset EH(f)$.

**1.4. Corollary.** *Let* $(A, f) \in \mathcal{U}_p$. *Then* $EQ_c(f) \cap H(f) \subset EH_0(f)$.

**1.5. Lemma.** *Let* $(A, f) \in \mathcal{U}_p$ *be neither of type* $\tau$ *nor of type* $\pi$, *and let* $g \in F(A)$. *If* $g \in EQ_c(f)$, *then* $(A, f)$ *and* $(A, g)$ *have the same component partitions.*

**1.6. Lemma.** *Let* $(A, f) \in \mathcal{U}_p$ *be neither of type* $\tau$ *nor of type* $\pi$, *let* $B$ *be a component of* $(A, f)$ *and* $g \in EQ_c(f)$. *Then* $Q_c(g_B) = Q_c(f_B)$.

**1.7. Lemma.** *Let* $(A, f) \in \mathcal{U}_p$. *Then* $\|EQ_c(f)\| \leqslant c$.

We will also apply some results of [3]. The notions used in the present paper differ from those in [3] only in the point that now we write $EH(f)$ instead of $Eq(f)$ (used in [3]).

## 2. Some auxiliary results

In this section we assume that $(A,f) \in U_p$.

**2.1. Lemma.** *Suppose that $(A,f)$ contains a cycle $C$ with $\|C\| = p > 1$ and let $h \in Q_c(f)$. Further suppose that $x \in C \cap \operatorname{dom} h$, $h(x) \in \operatorname{dom} f$ and $f^k(h(x)) \in \operatorname{dom} f$ for $k = 0, 1, \ldots, p-2$. Then $C \subset \operatorname{dom} h$ and $h(C)$ is a cycle such that $\|h(C)\|$ divides $p$.*

Proof. Let us show that $C \subset \operatorname{dom} h$. We have $f(x) \in \operatorname{dom} h$, because $f \in Q(h)$ (cf. (A3)). Further $h(f(x)) = f(h(x)) \in \operatorname{dom} f$ by assumption. Since $f \in Q(h)$, we obtain $f^2(x) \in \operatorname{dom} h$ and $h(f^2(x)) = f(h(f(x))) = f^2(h(x))$. By induction we get that $f^k(x) \in \operatorname{dom} h$ and $h(f^k(x)) = f(h(f^{k-1}(x))) = \ldots = f^k(h(x))$ for $k = 3, \ldots, p-1$ in the same way.

To complete the proof we will prove that $f^{p-1}(h(x)) \in \operatorname{dom} f$ and $f^p(h(x)) = h(x)$. The relations $h(f^{p-1}(x)) \in \operatorname{dom} f$ and $h(f^p(x)) = f(h(f^{p-1}(x)))$ are valid, because $h \in Q(f)$ and $f^{p-1}(x) \in \operatorname{dom} f$ and $f^{p-1}(x), f^p(x) \in \operatorname{dom} h$. Further $h(f^{p-1}(x)) = f^{p-1}(h(x)) \in \operatorname{dom} f$ and $f^p(h(x)) = f(f^{p-1}(h(x))) = f(h(f^{p-1}(x))) = h(f^p(x)) = h(x)$, as desired. □

**2.2. Lemma.** *Let $h \in Q_c(f)$ and let $B$ be a component of $(A,f)$ possessing a cycle $C$ with a period $p$, $p > 1$. Further let $x \in B \cap \operatorname{dom} h$ be such that $h(x) \in \operatorname{dom} f$ and $f^k(h(x)) \in \operatorname{dom} f$ for each $k \in \mathbb{N}$. Then $\{f^k(x) : k \in \mathbb{N}_0\} \subset \operatorname{dom} h$ and $h(C) = C'$, where $C'$ is a cycle belonging to a component $B'$ of $(A,f)$, $\|C'\|$ divides $\|C\|$ and $\{h(f^k(x)) : k \in \mathbb{N}_0\} \subset B'$.*

Proof. Let $m$ be the least non negative integer such that $f^m(x) \in C$. We can show that $f^k(x) \in \operatorname{dom} h$ and $h(f^k(x)) = f^k(h(x))$ for $k = 1, \ldots, m$ in the same way as in the proof of the assertion above. Therefore $\{x, f(x), \ldots, f^m(x)\} \subset \operatorname{dom} h$ and the $h$-image of $\{x, f(x), \ldots, f^m(x)\}$ belongs to one component of $(A,f)$. Further $f^m(x) \in C$ and the previous lemma implies $C \subset \operatorname{dom} h$ and $h(C) = C'$, where $C'$ is a cycle of $(A,f)$ with a period $q$, $q$ divides $p$. Assume that $C' \subset B'$, where $B'$ is a component of $(A,f)$. We conclude now $\{f^k(x) : k \in \mathbb{N}_0\} \cup C \subset \operatorname{dom} h$ and $h(\{f^k(x) : k \in \mathbb{N}_0\}) = h(\{x, f(x), \ldots, f^m(x)\} \cup C) \subset B'$. □

**2.3. Lemma.** *Let $(B, f_B)$ be a subalgebra of $(A,f)$ and $h \in Q_c(f)$. Then*

$$\{[x, h(x)] : x \in B, h(x) \in B\} \in Q_c(f_B).$$

Proof. Let $\bar{h} = \{[x, h(x)] : x \in B, h(x) \in B\}$. It is easy to see that $\bar{h} \in Q(f_B)$. Consider $x \in \operatorname{dom} \bar{h}$ and $x$, $\bar{h}(x) \in \operatorname{dom} f$. Since $f \in Q(h)$ (due to (A3)), we have $f(x) \in \operatorname{dom} h$ and $h(f(x)) = f(h(x))$. Further $h(x) \in B$ yields $f(h(x)) \in B$. Therefore $h(f(x)) \in B$ and $f(x) \in \operatorname{dom} \bar{h}$. □

**2.4. Lemma.** *Let $(A, f)$ be not of type $\pi$ and let each component of $(A, f)$ have only one element. Then $EQ_c(f) = EQ(f)$.*

Proof. First notice that $Q(h) = Q_c(h)$ if $h$ is the identity on some $B \subset A$, because $g \in Q(h)$ implies $h \in Q(g)$.

Suppose that $g \in EQ(f)$. The algebra $(A, g)$ consists of one-element components according to 4.10/[2]. Thus $EQ(f) = \{g \in Q(f): Q(f) = Q(g)\} = \{g \in Q_c(f): Q_c(f) = Q(g)\} = \{g \in Q_c(f): Q_c(f) = Q_c(g)\} = EQ_c(f)$, as desired. □

**2.5. Lemma.** *Suppose that $(A, f) \in \mathcal{U}$ and $(A, f)$ contains a component with more than one element. Then $EQ_c(f) \subset EH_0(f)$.*

Proof. Consider $h \in Q_c(f) - H(f)$. We will show that $h \notin EQ_c(f)$.

Choose $y_0 \notin \operatorname{dom} h$. First assume that $f(y_0) \neq y_0$. To argue the desired conclusion $Q_c(h) \neq Q_c(f)$ define $\varphi \in F(A)$ as $\varphi = \{[y_0, y_0]\}$. Then $\varphi \in Q_c(h)$, because the conditions $\varphi \in Q(h)$ and $h \in Q(\varphi)$ are trivially satisfied. But $y_0 \in \operatorname{dom} \varphi$, $\operatorname{dom} f = A$ and $f(y_o) \notin \operatorname{dom} \varphi$. Therefore $f \notin Q(\varphi)$, which implies that $\varphi \notin Q_c(f)$ by (A3).

Now let $f(y_0) = y_0$. There exists $y \in A$ such that $f(y) \neq y$ by the assumption. Consider $y \in \operatorname{dom} h$. (If $y \notin \operatorname{dom} h$ we can use the previous part of this proof for $y_0 = y$.) Let us define $\psi \in F(A)$ such that $\operatorname{dom} \psi = \{f^k(y): k \in \mathbb{N}_0\} \cup \{f^k(h(y)): k \in \mathbb{N}_0\}$ and $\psi(z) = y_0$ for each $z \in \operatorname{dom} \psi$. We have $\psi \notin Q(h)$, because $y \in \operatorname{dom} h$ and $y, h(y) \in \operatorname{dom} \psi$ and $\psi(y) \notin \operatorname{dom} h$. If $x \in \operatorname{dom} \psi$, then $f(x) \in \operatorname{dom} \psi$ and $\psi(f(x)) = y_0 = f(\psi(x))$. Thus we can conclude $\psi \in Q_c(f)$. □

**2.6. Lemma.** *Suppose that $(A, f)$ is of none of the types $\tau$, $\pi$ and $\delta$ and $a \in A$ is such that $f(a) = a$. If $g \in EQ_c(f)$, then $a \in \operatorname{dom} g$ and $g(a) = a$.*

Proof. Let $B = K_f(a)$. The algebra $(A, g)$ has the same partition into components according to 1.5. Thus $B$ is a component of $(A, g)$. We have $Q_c(f_B) = Q_c(g_B)$ by 1.6. First we will show that $a \in \operatorname{dom} g$.

Assume that $\|B\| > 1$. The relation $EQ_c(f_B) \subset EH_0(f_B)$ is valid by 2.5. We get $g_B \in H(f_B)$ and $a \in \operatorname{dom} g$.

Further let $B = \{a\}$. Suppose that $a \notin \operatorname{dom} g$. Since $(A, f)$ is not of type $\delta$, we can choose $y \in A$ such that either $y \notin \operatorname{dom} f$ or $f(y) \neq y$. Define $\varphi \in F(A)$ such that $\varphi = \{[a, y]\}$. Then $\varphi \in Q_c(g) - Q(f) \subset Q_c(g) - Q_c(f)$ according to (A1), which contradicts the hypothesis $g \in EQ_c(f)$.

It remains to show that $g(a) = a$. Consider $\psi = \{[a, a]\}$. Then $\psi \in Q_c(f) = Q_c(g)$. This implies $g(a) = a$, because $a \in \operatorname{dom} \psi$ and $a, \psi(a) \in \operatorname{dom} g$. □

**2.7. Lemma.** *Let $(A, f)$ be of type $\pi$. Then $EQ_c(f) = \{f\}$.*

Proof. Assume that $g \in EQ_c(f)$.

If $\operatorname{dom} g = \emptyset$, then 2.4 and 4.10/[2] imply $EQ_c(g) = EQ(g) = \{g, g'\}$, where $\operatorname{dom} g' = A, g'(y) = y$ for each $y \in A$. This contradicts the relation $f \in EQ_c(g)$ (cf. (A2)).

If $(A, g)$ contains a component with more than one element, then $EQ_c(g) \subset EH_0(g)$, therefore $f \in EH_0(g)$, thus $(A, f) \in \mathscr{U}$, a contradiction.

We have $\operatorname{dom} g \neq \emptyset$ and $g(y) = y$ for each $y \in \operatorname{dom} g$. Denote by $a$ an element of $A$, such that $\operatorname{dom} f = \{a\}$ ($(A, f)$ is of type $\pi$). Let $g \neq f$. Then we can choose $x \in A$ such that $x \in \operatorname{dom} g - \operatorname{dom} f$. Define $\varphi = \{[a, x]\}$. Then $\varphi \in Q_c(g) - Q_c(f)$, a contradiction.

We conclude $g = f$ as desired. □

**2.8. Lemma.** *Let $(A, f)$ be of none of the types $\tau$, $\pi$, $\gamma$ and $\delta$. If $g \in EQ_c(f)$, then $(A, g)$ is of none of the types $\tau$, $\pi$, $\gamma$ and $\delta$.*

Proof. If $(A, g)$ is of type $\delta$, then 2.4 implies $EQ_c(g) = EQ(g)$. Thus $f \in EQ(g)$ (by (A2)) and $(A, f)$ is of type $\gamma$ by 4.10/[2], which is a contradiction. If $(A, g)$ is of type $\gamma$, then 2.4 implies $EQ_c(g) = EQ(g)$, thus $f \in EQ(g)$ and $(A, f)$ is of type $\delta$ by 4.10/[2], a contradiction. If $(A, g)$ is of type $\tau$, then $EQ_c(g) \subset EH_0(g) = \{g\}$ by 2.5 and Th. 3/[3]. Hence $(A, f)$ is of type $\tau$, because $f \in EQ_c(g)$. If $(A, g)$ is of type $\pi$, then $(A, f)$ is of type $\pi$ by 2.7 and this completes the proof. □

**2.9. Lemma.** *Assume that $(A, f)$ is of none of the types $\pi, \gamma$ and $\delta$ and $a \in A$ with $K_f(a) = \{a\}$. Further let $g \in EQ_c(f)$.*

a) *If $f(a) = a$, then $g(a) = a$.*

b) *If $a \notin \operatorname{dom} f$, then $a \notin \operatorname{dom} g$.*

Proof. It follows from 1.5 that $(A, f)$ and $(A, g)$ have the same component partitions, thus $\{a\}$ is a component of $(A, g)$. Therefore either $a \notin \operatorname{dom} g$ or $g(a) = a$.

If $f(a) = a$, then $g(a) = a$ by virtue of 2.6. Let $a \notin \operatorname{dom} f$. Then $(A, f)$ is not of type $\tau$. Assume that $g(a) = a$. According to the assumption and in view of 2.8 we obtain that $(A, g)$ is of none of the types $\tau$, $\pi$, $\gamma$ and $\delta$. We can use 2.6 with $f$ and $g$ interchanged; this implies $a \in \operatorname{dom} f$, a contradiction. □

## 3. Algebras with a cycle in each component

For $p \in \mathbb{N}$ let $\mathcal{O}(p)$ be the system of all connected monounary algebras $(A, f)$ such that $(A, f)$ contains a cycle $C$ with $\|C\| = p$ and $f(x) \in C$ for each $x \in A$.

In this section we assume that $(A, f) \in \mathcal{U}$ and each component of $(A, f)$ has a cycle.

**3.1. Lemma.** *Let $(A, f) \in \mathcal{O}(p)$ for some $p \in \mathbb{N}$ and let $t \in \mathbb{N}$ be such that $(t, p) = 1, 0 < t < p$. Then $Q_c(f^t) = Q_c(f)$.*

Proof. Suppose that $C$ is a cycle of $(A, f)$. Since $(t, p) = 1$ the set $C$ is a cycle of $(A, f^t)$ and there exists $k \in \mathbb{N}$ such that $f^{kt}(x) = f(x)$ for each $x \in A$.

First we prove that $Q_c(f^t) \subset Q_c(f)$. Let $\varphi \in Q_c(f^t)$. If $\operatorname{dom}\varphi = \emptyset$, then $\varphi \in Q_c(f)$. Consider $\operatorname{dom}\varphi \neq \emptyset$. Then $x \in \operatorname{dom}\varphi$ implies $f^t(x) \in \operatorname{dom}\varphi$ and $f^{nt}(x) \in \operatorname{dom}\varphi$ for each $n \in \mathbb{N}$ by induction. We have $f(x) = f^{kt}(x) \in \operatorname{dom}\varphi$. Further $f(\varphi(x)) = f^{kt}(\varphi(x)) = f^{(k-1)t}(\varphi(f^t(x))) = \ldots = \varphi(f^{kt}(x)) = \varphi(f(x))$ We conclude $\varphi \in Q_c(f)$.

To complete the proof let us show that $Q_c(f) \subset Q_c(f^t)$. Assume that $\varphi \in Q_c(f)$, $\operatorname{dom}\varphi \neq \emptyset$. We have $C \subset \operatorname{dom}\varphi$ and thus $f^n(y) \in \operatorname{dom}\varphi$ for any $y \in A$ and $n \in \mathbb{N}$. Choose $x \in \operatorname{dom}\varphi$. We get $\varphi(f^t(x)) = f(\varphi(f^{t-1}(x))) = \ldots = f^t(\varphi(x))$. □

**3.2. Lemma.** *Let $(A, f) \in \mathcal{U}_c$ and $\|A\| > 1$. If $(A, f) \in \mathcal{O}(p)$ for some $p \in \mathbb{N}$, $p > 2$, then $EQ_c(f) = \{f^t : 0 < t < p, (t, p) = 1\}$; otherwise $EQ_c(f) = \{f\}$.*

Proof. If there exists $p \in \mathbb{N}, p > 2$ such that $(A, f) \in \mathcal{O}(p)$ then the inclusion $\{f^t : 0 < t < p, (t, p) = 1\} \subset EQ_c(f)$ follows, and the converse inclusion is obtained by 2.5 and Th.2/[3]. In the other case we have $EH_0(f) = \{f\}$ according to Th.3/[3] and $EQ_c(f) = \{f\}$ according to 2.5. □

**3.3. Notation.** *Let $B$ and $C$ be components of $(A, f)$ which have cycles with the period $p$ or $q$, respectively. Further let $g \in F(A)$ be such that $B$, $C$ are connected components of $(A, g)$. Consider the following conditions:*

*($\alpha$1) If $(B, f_B) \notin \mathcal{O}(p)$ and $q/p$, then $g_C = f_C$.*

*($\alpha$2) If $(B, f_B) \notin \mathcal{O}(p)$, $q > 1$ and $p/q$, then there exists $n \in \mathbb{N}$ such that $0 < n < q$, $(n, q) = 1$, $n \equiv 1 \pmod p$ and $g_C = f_C^n$.*

*($\beta$) If $(B, f_B) \in \mathcal{O}(p)$, $(C, f_C) \in \mathcal{O}(q)$, $p > 1$, $q > 1$ and $q/p$, then there exists $n \in \mathbb{N}$ such that $0 < n < p$, $(n, p) = 1$ and $g_C = f_C^n$, $g_B = f_B^n$.*

**3.4. Theorem.** *Suppose that $(A, f)$ fails to contain only one-element components. Let $g \in F(A)$. Then $g \in EQ_c(f)$ if and only if*

(i) *$(A, f)$ and $(A, g)$ have the same partition into components,*

(ii) *if* $B$ *is a component of* $(A,f)$, *then* $Q_c(f_B) = Q_c(g_B)$,

(iii) *if* $B, C$ *are components of* $(A,f)$, *then the conditions* $(\alpha 1), (\alpha 2)$ *and* $(\beta)$ *are satisfied,*

(iv) *if* $\{a\}$ *is a component of* $(A,f)$, *then* $a \in \operatorname{dom} g$.

Proof. Assume that $g \in EQ_c(f)$. Then 2.5 yields that $g \in EH_0(f)$. If $(A,f)$ is of type $\tau$, then Th.3/[3] implies $EQ_c(f) = EH_0(f) = \{f\}, g = f$, thus (i)–(iv) are trivially satisfied. Assume that $(A,f)$ is not of type $\tau$. The assertions (i) and (ii) are valid in view of 1.5 and 1.6. Since $g \in EH_0(f)$, we obtain that (iii) is satisfied according to Th.4/[3]. Further, $\operatorname{dom} g = A$, thus (iv) is valid.

On the other hand, suppose that $g \in F(A)$ is such that (i)–(iv) are satisfied. We will show that $Q_c(f) = Q_c(g)$.

Assume that $\varphi \in Q_c(f)$. Let $x \in \operatorname{dom}\varphi \cap B$, where $B$ is a component of $(A,f)$. Then $\{f^i(x) : i \in \mathbf{N}\} \subset \operatorname{dom}\varphi$. If $(B, f_B) \in \mathcal{O}(p)$ for $p \in \mathbf{N}, p > 1$, then $g_B = f_B^k$ for some $0 < k < p$, $(k,p) = 1$ by virtue of (ii) and 3.2. Thus $g_B(x) = g(x)$ belongs to the cycle of $(B, f_B), g_B(x) \in \operatorname{dom}\varphi$. If $(B, f_B) \notin \mathcal{O}(p)$ for any $p \in \mathbf{N} - \{1\}$, then $g_B = f_B$ according to 3.2 and thus $g(x) = f(x) \in \operatorname{dom}\varphi$.

Let us prove that $\varphi \in Q(g)$. Assume that $(B, f_B)$ contains a cycle with a period $p$ and let $x \in \operatorname{dom} g, g(x) \in \operatorname{dom}\varphi$.

a) Suppose that $p = 1$. If $\|B\| = 1$, then $g_B(x) = x = f_B(x)$, i.e., $g_B = f_B$, since $B$ is a component of $(A,g)$ by (i) and $x \in \operatorname{dom} g$. If $\|B\| > 1$, then (ii) and 3.2 imply $g_B = f_B$. Since $\varphi \in Q_c(f), \varphi(x)$ belongs to a component $C$ of $(A,f)$, which possesses a one-element cycle. As above, if $\|C\| > 1$, then $g_C = f_C$. If $\|C\| = 1$, then (iv) implies $\varphi \in \operatorname{dom} g$, thus $g_C = f_C$. We obtain $g(\varphi(x)) = g_C(\varphi(x)) = f_C(\varphi(x)) = f(\varphi(x)) = \varphi(f(x)) = \varphi(f_B(x)) = \varphi(g_B(x)) = \varphi(g(x))$, since $\varphi \in Q(f)$ (by (A1)).

b) Now let $p > 1$ and $(B, f_B) \notin \mathcal{O}(p)$. We obtain $g_B = f_B$ according to (ii) and 3.2. Further $\varphi(x) \in C$, where $C$ is a component of $(A,f)$ with a cycle with a period $q, q/p$ in view of 2.2. We get $g_C = f_C$ by $(\alpha 1)$. Then $g(\varphi(x)) = \varphi(g(x))$ similarly as in a).

c) Let $p > 1$ and $(B, f_B) \in \mathcal{O}(p)$. We have $\varphi(x) \in C$, where $C$ is a component of $(A,f)$ with a cycle with a period $q, q/p$.

Let $(C, f_C) \in \mathcal{O}(q)$. There exists $n \in \mathbf{N}$ such that $0 < n < p$, $(n,p) = 1$ and $g_B = f_B^n$, $g_C = f_C^n$ by virtue of $(\beta)$. Using $\varphi \in Q(f)$, we get $g(\varphi(x)) = g_C(\varphi(x)) = f_C^n(\varphi(x)) = f^n(\varphi(x)) = f^{n-1}(\varphi(f(x))) = \ldots = \varphi(f^n(x)) = \varphi(f_B^n(x)) = \varphi(g_B(x)) = \varphi(g(x))$.

Let $(C, f_C) \notin \mathcal{O}(q)$. By $(\alpha 2)$ we get $g_C = f_C$ and $g_B = f_B^k$ for some $k \in \mathbf{N}$, $0 < k < p$, $(k,p) = 1$, $k \equiv 1 \pmod q$. We get $\varphi(g(x)) = \varphi(f_B^k(x)) = f^k(\varphi(x)) = f_C^k(\varphi(x)) = f_C(\varphi(x)) = g_C(\varphi(x)) = g(\varphi(x))$. Therefore $\varphi \in Q(g)$. Since we have

$g(x) \in \operatorname{dom} \varphi$ and $\varphi(g(x)) = g(\varphi(x))$, we obtain $g \in Q(\varphi)$ and hence $\varphi \in Q_c(g)$, which completes the proof of the relation $Q_c(f) \subset Q_c(g)$.

Now let us prove the inclusion $Q_c(g) \subset Q_c(f)$. First assume that there is $a \in A - \operatorname{dom} g$. The element $a$ belongs to a component $B$ of $(A, f)$; $B$ is a component of $(A, g)$, since (i) is valid. Then $Q_c(f_B) = Q_c(g_B)$ in view of (ii). It follows from (iv) that $\|B\| > 1$. According to 2.5 we obtain $EQ_c(f_B) \subset EH_0(f_B)$, thus $g_B \in EH_0(f_B)$ and $(B, g_B)$ is a complete monounary algebra, which is a contradiction. Thus $A = \operatorname{dom} g$, $(A, g) \in \mathcal{U}$. The condition (i) implies that $(A, g)$ contains a component with more than one element. Let us denote by (i')–(iv') the conditions analogous to the conditions (i)–(iv), where $f$ and $g$ are interchanged. Then (i) and (i') are identical. Using (i) we obtain that (i) and (ii) are equivalent to (i') and (ii'), and (i) and (iii) are equivalent to (i') and (iii') (notice that if $g_C = f_C^n$, then there is $j$ with $f_C = g_C^j$). Further, $(A, f)$ is complete, thus $a \in \operatorname{dom} f$ for each $a \in A$ and obviously, (iv') is satisfied. Therefore (i')–(iv') are valid. Under these assumptions we obtain that $Q_c(g) \subset Q_c(f)$, using what we have proved above if we interchange $f$ and $g$.

Hence $Q_c(f) = Q_c(g)$. □

**3.5. Corollary.** *Assume that $(A, f)$ contains a component with more than one element. Then $EQ_c(f) = EH_0(f)$.*

P r o o f. If $g \in EQ_c(f)$, then 2.5 yields $g \in EH_0(f)$. Let $g \in EQ_c(f) - EH_0(f)$. Then $\operatorname{dom} g \neq A$ and there is $a \in A - \operatorname{dom} g$. It follows from 3.4 (iv) that $\{a\}$ is not a component of $(A, f)$ and the element $a$ belongs to a component $B$ with $\|B\| > 1$ ($B$ is a component of $(A, f)$ and of $(A, g)$ too, with respect to 3.4(i)). Then 3.4(ii) implies $Q_c(f_B) = Q_c(g_B)$ and by 2.5, $g_B \in EQ_c(g_B) \subset EH_0(f_B)$, a contradiction, since $(B, g_B)$ is not complete. □

**3.6. Corollary.** *There exists a countable set $A$ and a unary operation $f$ on $A$ such that $\|EQ_c(f)\| = c$.*

P r o o f. Let $\{p_n : n \in \mathbf{N}\}$ be a set of primes greater than 2. Define a monounary algebra $(A, f)$ such that $(A, f)$ consists of components $A_n, n \in \mathbf{N}$, which are $p_n$-element cycles of $(A, f)$. Then $EQ_c(f) = EH_0(f)$ by 3.5 and Th.5.2/[3] implies that $\|EH_0(f)\| = c$. □

In this section we suppose that $(A, f) \in \mathcal{U}_p$ and $(A, f)$ contains a component $B$ without a cycle such that $\|B\| > 1$, i.e., $(A, f)$ contains a chain as its subalgebra.

**4.1. Lemma.** *If* $\operatorname{dom} f_B \neq B$, *then* $EQ_c(f_B) = \{f_B\}$.

Proof. Let $g \in EQ_c(f_B), g \neq f_B$. The algebra $(B, g)$ is connected by 1.5 and thus $\|B - \operatorname{dom} g\| \leqslant 1$. If $\operatorname{dom} g = B$, then $EQ_c(g) \subset EH_0(g)$ by 2.5. But $f_B \notin H(g)$ and therefore $Q_c(g) \neq Q_c(f_B)$, a contradiction. Thus $\|B - \operatorname{dom} g\| = 1$.

Let us denote by $a, b$ such elements of $B$ that $a \notin \operatorname{dom} f_B, b \notin \operatorname{dom} g$. If $a = b$, then we will show that $g = f_B$ and if $a \neq b$, then we will show that $Q_c(f_B) \neq Q_c(g)$; this will complete the proof.

Let $a = b$. Since $(B, f_B)$ is connected and $\operatorname{dom} f_B \neq B$, for each $x \in B$ there is a uniquely determined number $k \in \mathbb{N}_0$ such that $f^k(x) = a$. Proceeding by induction with respect to $k$ we will prove that $g(x) = f_B(x)$ for each $x \in B$.

Let $k = 1$ and $x \in B$ be such that $f_B(x) = a$. Let $g(x) \neq a$. Define $\varphi = \{[a, a], [x, x]\}$. Then $\varphi \in Q_c(f_B)$. Further $\varphi(x) = x, x \in \operatorname{dom} g$ and $g(x) \notin \operatorname{dom} \varphi$, because $(B, g)$ contains no cycle and $g(x) \neq a$. Thus $g \notin Q(\varphi)$ and $\varphi \in Q_c(f_B) - Q_c(g)$ according to (A3), a contradiction. We conclude $g(x) = a$.

Now assume that for $0 < s < k, f_B^s(y) = a$ implies $f_B(y) = g(y)$ for $y \in B$. Let $x \in B$ be such that $f_B^k(x) = a$. We get $f_B(f_B(x)) = g(f_B(x)) = f_B(g(x))$ according to $f_B^k(x) = f_B^{k-1}(f_B(x)) = a, g \in Q(f_B)$ (cf. ((A1)) and the induction hypothesis. Further $f_B^{k-1}(g(x)) = f_B^{k-2}(f_B(g(x)) = f_B^k(x) = a$ and $f_B(g(x)) = g(g(x)) = g^2(x)$ by assumption. We have $f_B^2(x) = g^2(x) = f_B(g(x)) = g(f_B(x))$.

Let $f_B(x) \neq g(x)$. Define $\psi = \{[f^s(x), f^s(x)] : s = 0, 1, \ldots, k\}$. Then $\psi \in Q_c(f_B)$. Further $x \in \operatorname{dom} \psi \cap \operatorname{dom} g, \psi(x) \in \operatorname{dom} g$ and $g(x) \notin \operatorname{dom} \psi$, because $g(x) \neq f(x)$ and $f^{k-1}(g(x)) = a$. Thus $g \notin Q(\psi)$ and $\psi \notin Q_c(g)$ by (A3). This is a contradicton and consequently $g(x) = f_B(x)$.

Now consider $a \neq b$. Denote $V = \{x \in B : g^k(x) = a \text{ for some } k \in \mathbb{N}_0\}$. We have $a \in V, b \notin V$. Define $\zeta = \{[y, y] : y \in B - V\}$. Let $n$ be the least natural number such that $f_B^n(b) \in V$. Put $a_0 = f_B^n(b)$ and $b_0 = f_B^{n-1}(b)$. We obtain $f_B \notin Q(\zeta)$ and thus $\zeta \notin Q_c(f_B)$, because $b_0 \in \operatorname{dom} \zeta \cap \operatorname{dom} f_B, \zeta(b_0) = b_0$ and $f_B(b_0) = a_0, a_0 \notin \operatorname{dom} \zeta$.

If $x \in \operatorname{dom} g$ and $x, g(x) \in \operatorname{dom} \zeta$, then $\zeta(x) = x \in \operatorname{dom} g$ and $\zeta(g(x)) = g(x) = g(\zeta(x))$. Hence $\zeta \in Q(g)$. If $x \in \operatorname{dom} g \cap \operatorname{dom} \zeta$ and $\zeta(x) \in \operatorname{dom} g$, then $x \notin V$. That means that there exists no $k \in \mathbb{N}_0$ such that $g^k(x) = a$. This yields $g(x) \notin V$ and thus $g(x) \in \operatorname{dom} \zeta$.

We have shown $\zeta \in Q_c(g) - Q_c(f_B)$. □

**4.2. Lemma.** *Let* $\operatorname{dom} f_B = B$. *Then* $EQ_c(f_B) = \{f_B\}$.

Proof. Since $\operatorname{dom} f_B = B$, 2.5 implies $EQ_c(f_B) \subset EH_0(f_B)$. Thus it suffices to investigate the case when $EH_0(f_B) \neq \{f_B\}$. Th.2/[3] implies that $B = \bigcup_{j \in \mathbf{Z}} \{x_j\} \cup B_j, x_j \notin B_j$ and $f_B(b_j) = x_{j+1}$ for each $b_j \in \{x_j\} \cup B_j, j \in \mathbf{Z}$, where $x_i \neq x_j$ for $i \neq j, i, j \in \mathbf{Z}$. According to this theorem $EH_0(f_B) = \{f_B, g\}$, where $g(b_j) = x_{j-1}$ for each $b_j \in \{x_j\} \cup B_j, j \in \mathbf{Z}$.

We will show that $Q_c(f_B) \neq Q_c(g)$. Define $\varphi(b_j) = b_j$ for $j \in \mathbf{N}_0, b_j \in \{x_j\} \cup B_j$. It is obvious that $\varphi \in Q_c(f_B)$. We have $\varphi(x_0) = x_0, g(x_0) = x_{-1}$ and $g(x_0) \notin \operatorname{dom} \varphi$. Therefore $g \notin Q(\varphi)$ and $\varphi \notin Q_c(g)$ by (A3). Hence $g \notin EQ_c(f_B)$, i.e. $EQ_c(f_B) = \{f_B\}$. □

**4.3. Theorem.** *If $(A, f)$ contains a subalgebra which is a chain, then $EQ_c(f) = \{f\}$.*

Proof. Let $g \in EQ_c(f), g \neq f$. The algebra $(A, g)$ has the same partition into components by 1.5 and $Q_c(f_C) = Q_c(g_C)$ for each component $C$ of $(A, f)$. Let $(B, f_B)$ be a component of $(A, f)$ which contains a chain. Then $g_B = f_B$ in view of 4.1 and 4.2.

Suppose that $x \in A$ and either $g(x) \neq f(x)$ or $x \in (\operatorname{dom} g - \operatorname{dom} f) \cup (\operatorname{dom} f - \operatorname{dom} g)$. Let $C$ be a component of $(A, f)$ such that $x \in C$. If a component $C$ contains a chain of $(A, f)$, then $EQ_c(f_C) = \{f_C\}$ by 4.1 and 4.2 and $g_C = f_C$, a contradiction. Thus either $\|C\| = 1$ or $(C, f_C)$ has a cycle. According to the assumption, $(A, f)$ is of none of the types $\pi$, $\gamma$ and $\delta$ and then 2.9 yields that $\|C\| > 1$ (using the properties of the element $x$). Then 3.2 implies that $(C, f_C) \in \mathcal{O}(p)$ for some $p \in \mathbf{N}$, $p > 2$ and that $g = f^t$, $0 < t < p$, $(t, p) = 1$.

Choose $z \in B \cap \operatorname{dom} f$. Define $\varphi = \{[z, x]\} \cup \{[f^k(z), f^k(x)] \colon k \in \mathbf{N}, f^{k-1}(z) \in \operatorname{dom} f\}$. Clearly $\varphi \in Q_c(f)$. We have $z \in \operatorname{dom} g$ and $z, g(z) \in \operatorname{dom} \varphi$, because $z \in \operatorname{dom} f$ and $g_B = f_B$. But $\varphi(z) \in \operatorname{dom} g, g(\varphi(z)) = g(x) = f^t(x)$ and $\varphi(g(z)) = \varphi(f(z)) = f(x)$. Since $1 < t < p$ we see that $g(\varphi(z)) \neq \varphi(g(z))$. Thus $\varphi \notin Q(g)$ and $Q_c(f) \neq Q_c(g)$ (cf. (A1)), a contradiction. □

## 5. The remaining case

If $(A, f)$ is a complete monounary algebra, then either each component contains a cycle or some component contains a chain. The first possibility was investigated in Section 3, the second in Section 4. Thus we shall study $(A, f) \in \mathcal{U}_p - \mathcal{U}$. Further, if a component which has nonempty intersection with $A - \operatorname{dom} f$ has more than one element,then $(A, f)$ contains a component with a chain, which was investigated in Section 4. If $(A, f)$ contains only one-element components and is not of type $\pi$, it was studied in 2.4. If $(A, f)$ is of type $\pi$, it was studied in 2.7.

Let $(A,f) \in \mathscr{U}_p$. Put $A_2 = \operatorname{dom} f$, $A_1 = A - \operatorname{dom} f$. Therefore, the remaining case we ought to study is as follows:

(1) $A_1 \neq \emptyset$,
(2) if $B$ is a component of $(A,f)$ with $B \cap A_1 \neq \emptyset$, then $\|B\| = 1$,
(3) $A_2 \neq \emptyset$,
(4) if $B$ is a component of $(A,f)$ with $B \cap A_1 = \emptyset$, then $B$ contains a cycle of $(A,f)$,
(5) there exists a component $B$ of $(A,f)$ with $\|B\| > 1$.

In this section we will assume that (1)–(5) are valid.

According to the assumption (5), the assertions 1.5 and 1.6 yield that if $g \in EQ_c(f)$, then $(A,g)$ has the same partition into components as $(A,f)$ and each component has the same system of closed quasi-endomorphisms with respect to $f$ as with respect to $g$.

**5.1. Lemma.** *Let $g \in EQ_c(f)$. Then $\operatorname{dom} g = \operatorname{dom} f$ and if $x \in A_2$, then either $g(x) = f(x)$ or $g(f(x)) = x$.*

P r o o f. Consider $g \neq f$. The assumptions of 2.9 are satisfied and we see that $\operatorname{dom} g \cap A_1 = \emptyset$. Let $B$ be a component of $(A,f)$ (i.e., $B$ is a component of $(A,g)$). If $\|B\| > 1$, then $B \subset A_2$ and the relation $Q_c(f_B) = Q_c(g_B)$ and 2.5 imply that $\operatorname{dom} g_B = B$. If $B = \{a\}$ and $f(a) = a$, then 2.9 yields $g(a) = a$. Therefore $\operatorname{dom} g = A_2 = \operatorname{dom} f$.

Let $a \in A_2$ be such that $g(a) \neq f(a)$ and $g(f(a)) \neq a$. Suppose that $a$ belongs to a component $C$ of $(A,f)$. Then $\|C\| > 1$ and it follows from 3.2 that $(C, f_C) \in \mathscr{O}(p)$ for some $p \in \mathbb{N}, p > 2$ and $g_C = f_C^t$ for some $0 < t < p$, $(t,p) = 1$. Then $g(a) \neq a$. Choose $z \in A_1$. Define $\varphi = \{[a,z],[f(a),z]\}$. We have $\varphi \notin Q_c(f)$, because $a \in \operatorname{dom} f, a, f(a) \in \operatorname{dom}\varphi$ and $\varphi(a) = z, z \notin \operatorname{dom} f$. Assume that $x \in \operatorname{dom} g$ and $x, g(x) \in \operatorname{dom}\varphi$. Thus $x \neq a$, because $g(a) \neq a, g(a) \neq f(a)$, i.e. $g(a) \notin \operatorname{dom}\varphi$. Hence $x = f(a)$ and $g(f(a)) \in \operatorname{dom}\varphi$, then $g(f(a)) = f(a)$. This means that the component $C$ contains a one-element cycle, a contradiction. We have proved that $\varphi$ is a trivial element of $Q(g)$. Since $\varphi(y) = z \notin \operatorname{dom} g$ for any $y \in \operatorname{dom}\varphi$, we obtain the relation $g \in Q(\varphi)$, and hence $\varphi \in Q_c(g)$. This is a contradiction, $Q_c(f) \neq Q_c(g)$.

□

**5.2. Corollary.** *Let $B \subset A_2$ be a component of $(A,f)$, $g \in EQ_c(f)$.*

(i) *If $B$ is a cycle with $\|B\| = p$, then either $g_B = f_B$ or $g_B = f_B^{p-1}$.*
(ii) *If $B$ is not a cycle, then $g_B = f_B$.*

P r o o f. Suppose that $g_B \neq f_B$. Since $g_B \in EQ_c(f_B)$, it follows from 3.2 that $(B, f_B) \in \mathscr{O}(p)$ for some $p \in \mathbb{N}$, $p > 2$ and $g_B = f_B^t$ for some $0 < t < p$, $(t,p) = 1$.

This implies $(B, g_B) \in \mathcal{O}(p)$ and $a \in B$ belongs to the cycle of $(B, f_B)$ if and only if $a$ belongs to the cycle of $(B, g_B)$. Further $g(y) \neq f(y)$ for each $y \in B$. According to 5.1 we have $g(f(y)) = y$. Thus $t = p - 1$ and every element of $B$ belongs to the cycle of $(B, g_B)$. □

**5.3. Lemma.** *Let $g \in F(A)$. If*

(a) $\operatorname{dom} g = \operatorname{dom} f$,
(b) $g_{A_2} \in EQ_c(f_{A_2})$,
(c) *for $x \in A_2$ either $g(x) = f(x)$ or $g(f(x)) = x$,*

*then $g \in EQ_c(f)$.*

Proof. Notice that $A_2 = \operatorname{dom} f = \operatorname{dom} g$. First let us show that $Q_c(f) \subset Q_c(g)$. Consider $\varphi \in Q_c(f)$. Let $x \in \operatorname{dom} \varphi, x, \varphi(x) \in \operatorname{dom} g$. Put $\bar{\varphi} = \{[a, \varphi(a)] : a \in A_2, \varphi(a) \in A_2\}$. This mapping belongs to $Q_c(f_{A_2})$ by 2.3. Thus $\bar{\varphi} \in Q_c(g_{A_2})$ by (b). Since $x \in \operatorname{dom} \bar{\varphi}$, we obtain $g(x) \in \operatorname{dom} \bar{\varphi} \subset \operatorname{dom} \varphi$ and $g(\varphi(x)) = g(\bar{\varphi}(x)) = \bar{\varphi}(g(x)) = \varphi(g(x))$. Therefore $g \in Q(\varphi)$. Now let $y \in \operatorname{dom} g$ and $y, g(y) \in \operatorname{dom} \varphi$. By (c) we have either $g(y) = f(y)$ or $g(f(y)) = y$. If $g(y) = f(y)$, then the relation $\varphi \in Q(f)$ implies that $\varphi(y) \in \operatorname{dom} f$ and according to (b) we obtain that $\varphi(y) \in \operatorname{dom} g$ and $g(\varphi(y)) = \varphi(g(y))$. Let $g(y) \neq f(y)$, i.e. $g(f(y)) = y$. Assume that $B$ is a component of $(A, f)$ such that $y \in B$. Since (b) is valid, we have $g_B \in EQ_c(f_B)$ and then 3.2 yields that $g_B = f_B^k$, $1 < k < p$, where $p > 2$ is a period of a cycle in $B$. In view of the fact that $g_B(f_B(y)) = y$ we conclude that $k = p - 1$ and hence $f_B(g_B(y)) = f_B(f_B^{p-1}(y)) = y$. Put $g(y) = a$. Then $a \in \operatorname{dom} f$, $a$, $f(a) \in \operatorname{dom} \varphi$, thus the relation $\varphi \in Q(f)$ (cf. (A1)) implies that $\varphi(y) \in \operatorname{dom} f$ and in view of (b) we obtain that $\varphi(y) \in \operatorname{dom} g, g(\varphi(y)) = \varphi(g(y))$. Therefore $\varphi \in Q(g)$ and hence $\varphi \in Q_c(g)$.

The proof of the inclusion $Q_c(g) \subset Q_c(f)$ is analogous. □

**5.4. Theorem.** *Let $g \in F(A)$. Then $g \in EQ_c(f)$ if and only if*

(a) $\operatorname{dom} g = \operatorname{dom} f$,
(b) $g_{A_2} \in EQ_c(f_{A_2})$,
(c) *for $x \in A_2$ either $g(x) = f(x)$ or $g(f(x)) = x$.*

Proof. According to 5.1 and 5.3 we have to prove only that $g \in EQ_c(f)$ implies $g_{A_2} \in EQ_c(f_{A_2})$. We will show (i)–(iv) from 3.4 (with $A$ replaced by $A_2$).

As we have remarked before 5.1, the conditions (i) and (ii) are satisfied. The condition (iv) follows by 2.9. Let $g \in EQ_c(f)$.

Suppose that $B$ and $C$ are components of $(A_2, f_{A_2})$ which have cycles with the period $p$ or $q$, respectively. To prove the condition ($\alpha$2) suppose that $(B, f_B) \notin$

$\mathcal{O}(p), q > 1$ and $p/q$. If $g_C \neq f_C$, then $C$ is a cycle and $g_C = f_C^{q-1}$ by virtue of 5.2. We will show that $g_C = f_C^{q-1}$ implies $q - 1 \equiv 1 \bmod p$. Choose $x \in B$, $y \in C$ such that they belong to cycles. Define $h' = \{[g^k(y), g^k(x)] : k = 0, 1, \ldots, q-1\}$. Then $h' \in Q_c(g)$. Using $Q_c(f) = Q_c(g)$ we have $g_B(x) = g(x) = h'(g(y)) = h'(g_C(y)) = h'(f_C^{q-1}(y)) = h'(f^{q-1}(y)) = f(h'(f^{q-2}(y))) = \ldots = f^{q-1}(h'(y)) = f_B^{q-1}(x)$. Since $g_B = f_B$, we obtain $q - 1 \equiv 1 \bmod p$.

Now let us show that if $g_B = f_B$ and $q/p$, then $g_C = f_C$. Suppose that $g_B = f_B$ and $q/p$. Choose $x \in B, y \in C$ such that $x$ and $y$ belong to the corresponding cycles. Define $h = \{[f^k(x), f^k(y)] : k = 0, \ldots, p-1\}$. Then $h \in Q_c(f)$. Using $Q_c(f) = Q_c(g)$ we obtain $f_C(y) = f(y) = h(f(x)) = h(f_B(x)) = h(g_B(x)) = g_C(h(x)) = g_C(y)$. In view of (ii) and 3.2 we conclude that $g_C = f_C$.

If $(B, f_B) \notin \mathcal{O}(p)$ and $q/p$, then $g_B = f_B$ by (ii) and 3.2. Thus $g_C = f_C$ and this gives ($\alpha$1).

Finally, let $(B, f_B) \in \mathcal{O}(p), (C, f_C) \in \mathcal{O}(q), p > 1, q > 1$ and $q/p$. The relation $p - 1 \equiv q - 1 \bmod q$ holds. We have shown that $g_B = f_B$ implies $g_C = f_C$. We need to show that if $g_B = f_B^{p-1}$ and $f_C^{p-1} = f_C^{q-1}$ then $g_C = f_C^{p-1}$. Using the mapping $h \in Q_c(f)$ from this proof we have $g_C(y) = g_C(h(x)) = h(g(x)) = h(g_B(x)) = h(f_B^{p-1}(x)) = f(h(f^{p-1}(x))) = \ldots = f^{p-1}(h(x)) = f_C^{p-1}(y)$. This completes the proof of ($\beta$) and of the theorem, too. □

## 6. The relationship between $EQ(f)$ and $EQ_c(f)$

**6.1. Lemma.** *Suppose that $(A, f)$ is of type $\alpha$ and $(A, f)$ contains no chain as its subalgebra. Further let $g \in F(A)$ be such that $\operatorname{dom} g = \operatorname{dom} f$ and $g(f(x)) = x$ for each $x \in \operatorname{dom} f$. Then $g \in EQ_c(f)$.*

Proof. We have $\operatorname{dom} f \neq \emptyset$ by the definition of an algebra of type $\alpha$. Denote $A_2 = \operatorname{dom} f$. We will verify (a)–(c) from 5.3. To prove (b) we will show (i)–(iv) from 3.4 for the algebra $(A_2, f_{A_2})$ and the mapping $g_{A_2}$. By the assumptions of the lemma (a), (c), (i), (ii) and (iv) are valid. We need to show ($\alpha$1), ($\alpha$2) and ($\beta$) for $(A_2, f_{A_2})$ and $g_{A_2}$.

Assume that $B$ and $C$ are components of $(A, f)$ which have cycles with the period $p$ or $q$, respectively. The conditions ($\alpha$1), ($\alpha$2) are trivially satisfied, because each component of $(A, f)$ is an element of $\mathcal{O}(r)$ for some $r \in \mathbb{N}$. Suppose that $p > 1, q > 1$ and $q/p$. We know that $g_B(f_B(x)) = x$ for each $x \in B$ and therefore $g_B = f_B^{p-1}$. In the same way $g_C = f_C^{q-1}$. Since $q/p$, we get $q - 1 \equiv p - 1 \bmod q$. Analogously as in the previous proof we conclude $g_B = f_B^{p-1}$ and $g_C = f_C^{p-1}$. □

**6.2. Proposition.**

(1) *Let* $(A, f)$ *be of type* $\alpha$ *and contain a chain. Then* $EQ_c(f) = \{f\}$ *according to 4.3 and* $\|EQ(f)\| = 2$ *by 4.10/[2].*

(2) *If* $(A, f)$ *is of type* $\alpha$ *and contains no chain which is its subalgebra, then* $EQ(f) = \{f, g\}$, *where* $\operatorname{dom} f = \operatorname{dom} g$ *and* $g(f(x)) = x$ *for each* $x \in \operatorname{dom} f$ *in view of 4.10/[2], and 6.1 yields that* $EQ(f) \subset EQ_c(f)$.

(3) *If* $(A, f)$ *is either of type* $\tau$ *or of type* $\pi$, *then* $EQ_c(f) = \{f\}$ *by virtue of 2.5 and 2.7.*

(4) *If* $(A, f)$ *is either of type* $\delta$ *or of type* $\gamma$, *then* $EQ_c(f) = EQ(f)$ *by 2.4.*

(5) *Let* $(A, f)$ *be of none of the types* $\alpha$, $\pi$, $\tau$, $\gamma$, $\delta$. *Then* $EQ(f) = \{f\}$ *according to 4.10/[2] and* $f \in EQ_c(f)$.

These considerations imply

**6.3. Theorem.** *Suppose that* $(A, f) \in \mathscr{U}_p$.
*Then either* $EQ(f) \subset EQ_c(f)$ *or* $EQ_c(f) \subset EQ(f)$.

*References*

[1] *P. Burmeister, B. Wojdylo*: Properties of homomorphisms and quomorphisms between partial algebras, Contributions to general algebra 5, Proc. of Salzburg Conference, May 29–June 1, 1986, Verlag Hölder-Pichler-Tempsky, Wien, Verlag B. G. Teubner, Stuttgart, 1987, pp. 69–90.

[2] *E. Halušková, D. Studenovská*: Partial monounary algebras with common quasi-endomorphisms, Czech. Math. J. 42(1992), 59–72.

[3] *D. Jakubíková-Studenovská*: Systems of unary algebras with common endomorphisms I, II, Czech. Math. J. *29 (104)* (1979), 406–420, 421–429.

[4] *D. Jakubíková-Studenovská*: Endomorphisms and connected components of partial mono-unary algebras, Czech. Math. J. *35 (110)* (1985), 467–489.

[5] *D. Jakubíková-Studenovská*: Endomorphisms of partial mono-unary algebras, Czech. Math. J. *36 (111)* (1986), 467–489.

[6] *O. Kopeček, M. Novotný*: On some invariants of unary algebras, Czech. Math. J. *24 (99)* (1974), 219–246.

[7] *M. Novotný*: Über Abbildungen von Mengen, Pacific J. Math. *13* (1963), 1359–1369.

*Authors' addresses*: E. Halušková, Matematický ústav SAV, Grešákova 6, CS-040 01 Košice, Slovakia, musavke@csearn.bitnet; D. Studenovská, Prírodovedecká fakulta UPJŠ, Jesenná 5, CS-041 54 Košice, Slovakia.