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SUBCOHERENT ALGEBRAS

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Coherent algebras were introduced by D. Geiger [9]. The same author proved that any variety of coherent algebras is permutable and regular, see [3] and [11] for these concepts. Local versions of Geiger's results were formulated in [2] and [5]. Varieties of weakly regular algebras were investigated in [7]. The notion of weakly coherent algebras comes from I. Chajda [1]; it was shown that any variety of weakly coherent algebras is permutable and weakly regular. The concept of subregular algebras is due to J. Timm [12]. In this paper subcoherent algebras are introduced and their relationships to permutable and subregular algebras are studied.

Notation 1. Let A be an algebra, B a nonvoid subset of A and Θ a congruence on A. The symbol $[B]\Theta$ denotes the set union $\bigcup\{[b]\Theta;\, b \in B\}$.

Definition 1. Let A be an algebra. A subalgebra B of A is called subcoherent with a congruence Θ on A whenever the assumption $[C]\Theta \subseteq B$ for some subalgebra C of B implies $[B]\Theta = B$.

An algebra A is called subcoherent whenever every subalgebra of A is subcoherent with each congruence on A.

Theorem 1. *For a variety V, the following conditions are equivalent:*

(1) V is a variety of subcoherent algebras;

(2) there are unary terms $u_1, \ldots, u_n$, ternary terms $t_1, \ldots, t_n$, and a $(1+n)$-ary term s such that

$$y = s(x, t_1(x,y,z), \ldots, t_n(x,y,z))$$

and

$$u_i(z) = t_i(x,x,z), 1 \leqslant i \leqslant n,$$

are identities in V.

P r o o f. (1) $\Rightarrow$ (2): Let $A = F_V(x,y,z)$ be the V-free algebra with free generators x, y and z. Further, choose $C = F_V(z)$ and let $\Theta = \Theta(x,y)$ be the principal congruence on A collapsing x and y. Denote by B the subalgebra of A generated by the subset $\{x\}\cup[C]\Theta$. Since B is subcoherent with Θ the equality $[B]\Theta = B$ holds. In particular we have $y \in [x]\Theta \subseteq [B]\Theta = B$ and thus $y = s(x, t_1(x,y,z), \dots, t_n(x,y,z))$ where s is a $(1+n)$-ary term and $t_i(x,y,z) \in [C]\Theta$, $1 \leqslant i \leqslant n$. The last argument gives unary terms $u_1, \dots, u_n$ such that $\langle t_i(x,y,z), u_i(z)\rangle \in \Theta(x,y)$, $1 \leqslant i \leqslant n$. The remaining identities of (2) follow.

(2) $\Rightarrow$ (1): Let B be a subalgebra of an algebra $A \in V$, Θ a congruence on A. Suppose further that $[C]\Theta \subseteq B$ for some subalgebra C of B. We have to verify the inclusion $[B]\Theta \subseteq B$.

Take an element $d \in [B]\Theta$. Then $d \in [b]\Theta$ for some $b \in B$. In other words, $\langle b, d\rangle \in \Theta$. Choose an arbitrary element $c \in C$. Then $\langle t_i(b,d,c), u_i(c)\rangle = \langle t_i(b,d,c), t_i(b,b,c)\rangle \in \Theta$, $1 \leqslant i \leqslant n$, i.e. $t_i(b,d,c) \in [u_i(c)]\Theta \subseteq [C]\Theta \subseteq B$, $1 \leqslant i \leqslant n$. Consequently, $d = s\big(b, t_1(b,d,c), \dots, t_n(b,d,c)\big) \in B$ as required. The proof is complete. $\square$

Corollary 1. *Any variety of subcoherent algebras is permutable.*

Proof. We use the identities from Theorem 1(2). Let us introduce a ternary term p via $p(x,y,z) = s\big(z, t_1(y,x,z), \dots, t_n(y,x,z)\big)$. Then

$$\begin{aligned} p(x,x,z) &= s\big(z, t_1(x,x,z), \dots, t_n(x,x,z)\big) \\ &= s\big(z, u_1(z), \dots, u_n(z)\big) = s\big(z, t_1(z,z,z), \dots, t_n(z,z,z)\big) = z \end{aligned}$$

and

$$p(x,z,z) = s\big(z, t_1(z,x,z), \dots, t_n(z,x,z)\big) = x,$$

which means that p is a Mal'cev term. The permutability of V is verified, see [11]. $\square$

Definition 2. An algebra A is called subregular whenever every congruence Θ on A is uniquely determined by its blocks $[b]\Theta$, $b \in B$, for each subalgebra B of A.

A variety V is called subregular whenever any V-algebra has this property.

Corollary 2. *Any variety of subcoherent algebras is subregular.*

Proof. The identities $t_i(x,x,z) = u_i(z), 1 \leqslant i \leqslant n$, were shown in Theorem 1(2). Further suppose that $t_i(x,y,z) = u_i(z)$, $1 \leqslant i \leqslant n$. Then

$$\begin{aligned} y &= s\big(x, t_1(x,y,z), \dots, t_n(x,y,z)\big) = s\big(x, u_1(z), \dots, u_n(z)\big) \\ &= s\big(x, t_1(x,x,z), \dots, t_n(x,x,z)\big) = x. \end{aligned}$$

Altogether $\big(t_i(x,y,z) = u_i(z), 1 \leqslant i \leqslant n\big)$ iff $x = y$, i.e. V satisfies the criterion for subregularity, see [6; Theorem 1(3)]. $\square$

Notation 2. Let A be an algebra. The symbol ω_A denotes the diagonal on A, i.e. $\omega_A = \{ \langle a, a \rangle \; ; \; a \in A \}$.

Stronger (local) versions of the preceding corollaries follow.

Proposition 1. *Let A be an algebra. Then $A \times A$ subcoherent implies A permutable.*

P r o o f. Let Ψ, Φ be congruences on A. Then $T = \Psi \circ \Phi \cap \Phi \circ \Psi$ is a subalgebra of $A \times A$. Moreover, for a subalgebra ω_A of T we have $[\omega_A]\Psi \times \Psi = \Psi \circ \omega_A \circ \Psi = \Psi \subseteq T$. Hence $[T]\Psi \times \Psi = T$, by hypothesis. In the same way we obtain the equality $[T]\Phi \times \Phi = T$. Consequently $[T](\Psi \times \Psi) \vee (\Phi \times \Phi) = T$. However, $(\Psi \times \Psi) \vee (\Phi \times \Phi) = (\Psi \vee \Phi) \times (\Psi \vee \Phi)$, see [8], and so $\Psi \vee \Phi \subseteq (\Psi \vee \Phi) \circ T \circ (\Psi \vee \Phi) = [T](\Psi \vee \Phi) \times (\Psi \vee \Phi) = T$, which establishes the permutability of A. $\square$

Proposition 2. *Let A be an algebra. Then $A \times A$ subcoherent implies A subregular.*

P r o o f. Let Ψ, Φ be congruences on A, let B be a subalgebra of A. Suppose that $[b]\Psi = [b]\Phi$ for every $b \in B$. Then $[\omega_B]\Psi \times \Psi = [\omega_B]\Phi \times \Phi \subseteq [\omega_A]\Phi \times \Phi = \Phi \circ \omega_A \circ \Phi = \Phi$ and thus also $[\Phi]\Psi \times \Psi = \Phi$, by hypothesis. In other words, we have $\Psi \subseteq \Psi \circ \Phi \circ \Psi = [\Phi]\Psi \times \Psi = \Phi$. The opposite inclusion follows by symmetrical arguments. Altogether $\Psi = \Phi$, which proves the subregularity of A. $\square$

Definition 3. Let A be an algebra. A subalgebra B of $A \times A$ is called a diagonal subalgebra whenever the inclusion $\omega_A \subseteq B$ holds.

Definition 4. Let A be an algebra. A congruence Θ on $A \times A$ is called factorable whenever $\Theta = \Psi \times \Phi$ for some congruences Ψ, Φ on A.

Now we are ready to show the relationships between subcoherence, permutability and subregularity.

Theorem 2. *For a variety V, the following conditions are equivalent:*

(1) *any diagonal subalgebra of $A \times A$ is subcoherent with factorable congruences on $A \times A$, $A \in V$;*

(2) *any diagonal symmetric subalgebra of $A \times A$ is subcoherent with factorable congruences on $A \times A$, $A \in V$;*

(3) *V is permutable and subregular.*

P r o o f. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3): Use proofs of Proposition 1 and Proposition 2. (3) $\Rightarrow$ (1): Let S be a diagonal subalgebra of $A \times A$. By [13], permutability of V yields that S is a congruence on A, say $S = \Theta$. Further, let B be a subalgebra of Θ such that $[B]\Psi \times \Phi \subseteq \Theta$ for congruences Ψ, Φ on A. Consider a congruence block $[\langle b,c\rangle]\Psi\times\Phi$ for an arbitrary $\langle b,c\rangle\in B$. Take $\langle u,v\rangle\in[\langle b,c\rangle]\Psi\times\Phi$. Since $[\langle b,c\rangle]\Psi\times\Phi=[b]\Psi\times[c]\Phi$ we have also $\langle u,c\rangle\in[\langle b,d\rangle]\Psi\times\Phi\subseteq\Theta$. Now $\langle b,c\rangle\in\Theta$ and $\langle u,c\rangle\in\Theta$ give $\langle u,b\rangle\in\Theta$, by transitivity of Θ. Analogously $\langle v,c\rangle\in\Theta$ can be obtained. Altogether $\langle u,v\rangle\in[b]\Theta\times[c]\Theta=[\langle b,c\rangle]\Theta\times\Theta$, which proves the inclusion $[\langle b,c\rangle]\Psi\times\Phi\subseteq[\langle b,c\rangle]\Theta\times\Theta$. Then $\Psi\times\Phi\subseteq\Theta\times\Theta$, by subregularity. Consequently, $[\Theta]\Psi\times\Phi\subseteq[\Theta]\Theta\times\Theta=\Theta\circ\Theta\circ\Theta=\Theta$ as required. □

Theorem 3. *For a variety V, the following conditions are equivalent:*

(1) *any diagonal transitive subalgebra of $A\times A$ is subcoherent with factorable congruences on $A\times A$, $A\in V$;*

(2) *any congruence on A is subcoherent with factorable congruences on $A\times A$, $A\in V$;*

(3) *V is subregular.*

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3): See the proof of Proposition 2. (3) $\Rightarrow$ (1): By [10], any subregular variety is n-permutable for an integer $n>1$. Then any diagonal transitive subalgebra of the square is a congruence, see [10] again. The rest of the proof is the same as in the previous Theorem 2. □

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