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METRIZATION OF UNIFORM LATTICES

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0. INTRODUCTION

In [W2] we have studied uniform lattices as generalization of Boolean rings endowed with an FN-topology and of Riesz spaces endowed with a locally solid linear topology. In these two special cases the uniformity (induced by an FN-topology or a locally solid linear topology) is generated by a system $(d_\alpha)_{\alpha\in A}$ of pseudo-metrics with the property

$$(*)\qquad d_\alpha(x\vee z, y\vee z)\leqslant d_\alpha(x,y),\qquad d_\alpha(x\wedge z, y\wedge z)\leqslant d_\alpha(x,y).$$

More generally, I. Fleischer and T. Traynor [FT] have proved that any uniformity on a lattice induced by a modular function with values in a commutative topological groups is generated by a system (d_α) of pseudo-metrics with the property $(*)$. It is natural question whether that also holds for an arbitrary uniform lattice, i.e. for a uniformity on a lattice such that the lattice operations $\vee$ and $\wedge$ are uniformly continuous. The answer is no in general (see section 2), but yes in the case that the lattice is distributive (see section 1, (1.6)). The setting of section 1 is more general. There we study uniform spaces with one or more operations. In particular, section 1 contains a simple proof of the known fact that the uniformity of a uniform semigroup $(X,+)$ is induced by a system of pseudo-metrics (d_α) such that

$$d_\alpha(x+y, x'+y')\leqslant d_\alpha(x,x')+d_\alpha(y,y').$$

Hereby uniform semigroup is defined as a semigroup endowed with a uniformity such that the semigroup operation is uniformly continuous.

In the following let (X, u) be a uniform space. We denote by Δ the diagonal $\Delta := \{(x, x) : x \in X\}$.

Proposition 1.1. *Let $+ : X \times X \to X$ be an operation on X.*

(a) *$+$ is uniformly continuous iff for every $U \in u$ there is a $V \in u$ such that $V + \Delta \subset U$ and $\Delta + V \subset U$.*

(b) *If $+$ is associative, then $+$ is uniformly continuous iff u has a base of sets U with $U + \Delta \subset U$ and $\Delta + U \subset U$.*

Proof. (a) Since $+$ is uniformly continuous iff for every $U \in u$ there is a $V \in u$ with $V + V \subset U$, one implication ($\Rightarrow$) is obvious. To prove the other implication ($\Leftarrow$), let $U \in u$ and $V, W \in u$ with $V \circ V \subset U$ and $\Delta + W, W + \Delta \subset V$. We show that $W + W \subset U$. If $(x, x'), (y, y') \in W$, then $(x+y, x'+y) = (x, x') + (y, y) \in W + \Delta \subset V$ and similarly $(x' + y, x' + y') \in V$, hence $(x + y, x' + y') \in V \circ V \subset U$.

(b) $\Leftarrow$ follows from (a).

$\Rightarrow$: Let $+$ be associative and $W \in u$. We show that W contains an $U \in u$ with $U + \Delta$, $\Delta + U \subset U$. Choose $V \in u$ with $V + V + V \subset W$, $V + V \subset W$ and put

$$U := \{(x, y) \in W : (x, y) + \Delta, \Delta + (x, y), \Delta + (x, y) + \Delta \subset W\}.$$

By definition, $U \subset W$. Since $\Delta \subset V$ and $V + V$, $V + V + V \subset W$, one gets $V \subset U$, hence $U \in u$. We show that $U + \Delta \subset U$; analogously one obtains $\Delta + U \subset U$. To prove $U + \Delta \subset U$ we have to check that $U + \Delta \subset W$ and $(U + \Delta) + \Delta$, $\Delta + (U + \Delta)$, $\Delta + (U + \Delta) + \Delta \subset W$. But this holds obviously, since $+$ is associative, $\Delta + \Delta \subset \Delta$ and $U + \Delta$, $\Delta + U$, $\Delta + U + \Delta \subset W$ by the definition of U. $\square$

If we write in (1.1) (b) $f(x, y)$ instead of $x + y$, then the inclusions $U + \Delta \subset U$ and $\Delta + U \subset U$ mean that $\big(f(x, y), f(x', y)\big) \in U$ and $\big(f(y, x), f(y, x')\big) \in U$ hold for any $(x, x') \in U$ and $y \in X$. This formulation is used in the next proposition.

Proposition 1.2. *Let $f_i : X \times X \to X$ be operations on X for $i \in I$ and I_0 a finite subset of I; ($I_0 = \emptyset$ or $I \setminus I_0 = \emptyset$ are admitted). Further, let q be a real number, $q > 1$. Then there is a system D of pseudo-metrics on X, which generates the uniformity u, such that for any $d \in D$ and x, x', y, $y' \in X$*

$$d(f_i(x, y), f_i(x', y')) \leqslant q(d(x, x') + d(y, y')) \quad \text{for} \quad i \in I_0$$

and

$$d(f_i(x,y), f_i(x',y')) \leqslant d(x,x') + d(y,y') \quad \text{for} \quad i \in I \setminus I_0$$

iff f_i are uniformly continuous for $i \in I_0$ and u has a base of sets U such that

$$(f_i(x,y), f_i(x',y)) \in U \quad \text{and} \quad (f_i(y,x), f_i(y,x')) \in U$$

for any $(x,x') \in U$, $y \in X$ and $i \in I \setminus I_0$. Moreover, if u has a countable base, then one can replace in this equivalence the system D by a single pseudo-metric d.

Proof. One implication ($\Rightarrow$) is obvious. Suppose now that f_i is uniformly continuous for $i \in I_0$ and u has a base of sets U such that $\big(f_i(x,y), f_i(x',y)\big)$, $\big(f_i(y,x), f_i(y,x')\big) \in U$ for $(x,x') \in U$, $y \in X$, $i \in I \setminus I_0$.

Let A be the system of all sequence $(U_n)_{n \in \mathbf{N}}$ of symmetric sets of u with the property that for any $n \in \mathbf{N}$ $U_n \circ U_n \circ U_n \subset U_{n-1}$ (with $U_0 := X \times X$) and that for $(x,x') \in U_n$ and $y \in X$ the pairs $(f_i(x,y), f_i(x',y))$ and $(f_i(y,x), f_i(y,x'))$ belong to U_{n-1} for $i \in I_0$ and belong to U_n for $i \in I \setminus I_0$.

For $\alpha = (U_n) \in A$ define g_α by $g_\alpha(x,y) = 2^{-n}$ iff $(x,y) \in U_{n-1} \setminus U_n$ and $g_\alpha(x,y) = 0$ iff (x,y) belongs to each U_n.

Now define $d_\alpha \colon X \times X \to [0,1]$ by

$$d_\alpha(x,y) := \inf\Big\{ \sum_{j=0}^{n} g_\alpha(x_j, x_{j+1}) \colon n \in \mathbf{N}, x_j \in X, x_0 = x, x_{n+1} = y \Big\}.$$

On p. 185 of [K] it is proved that d_α is a pseudo-metric and $U_n \subset \{(x,y) \in X \times X : d_\alpha(x,y) < 2^{-n}\} \subset U_{n-1}$ Therefore $(d_\alpha)_{\alpha \in A}$ generates u, since every $U \in u$ contains a sequence of A.

Let x, x', y, $y' \in X$. Obviously $g_\alpha(f_i(x,y), f_i(x',y)) \leqslant 2g_\alpha(x,x')$ for $i \in I_0$ and $g_\alpha(f_i(x,y), f_i(x',y)) \leqslant g_\alpha(x,x')$ for $i \in I \setminus I_0$, hence

$$\begin{aligned} d_\alpha(f_i(x,y), f_i(x',y)) &\leqslant \\ &\leqslant \inf\Big\{ \sum_{j=0}^{n} g_\alpha(f_i(x_j,y), f_i(x_{j+1}y)) \colon n \in \mathbf{N}, x_j \in X, x_0 = x, x_{n+1} = x' \Big\} \\ &\leqslant \inf\Big\{ \sum_{j=0}^{n} 2g_\alpha(x_j, x_{j+1}) \colon n \in \mathbf{N}, x_j \in X, x_0 = x, x_{n+1} = x' \Big\} = 2d_\alpha(x,x') \\ &\quad \text{for } i \in I_0 \end{aligned}$$

and similarly

$$d_\alpha(f_i(x,y), f_i(x',y)) \leqslant d(x,x') \quad \text{for } i \in I \setminus I_0.$$

Analogously one gets

$$d_\alpha(f_i(x',y),f_i(x',y')) \leqslant 2d_\alpha(y,y') \quad \text{for } i \in I_0$$

and

$$d_\alpha(f_i(x',y),f_i(x',y')) \leqslant d_\alpha(y,y') \quad \text{for } i \in I \setminus I_0.$$

Finally

$$\begin{aligned} d_\alpha(f_i(x,y),f_i(x',y')) &\leqslant d_\alpha(f_i(x,y),f_i(x',y)) + d_\alpha(f_i(x',y),f_i(x',y')) \\ &\leqslant 2(d_\alpha(x,x') + d_\alpha(y,y')) \quad \text{for } i \in I_0 \end{aligned}$$

and analogously

$$d_\alpha(f_i(x,y),f_i(x',y')) \leqslant d_\alpha(x,x') + d_\alpha(y,y') \quad \text{for } i \in I \setminus I_0.$$

Now choose $n \in \mathbf{N}$ with $2^{1/n} \leqslant q$. Then the family $D := \{d_\alpha^{1/n} : \alpha \in A\}$ has the desired properties.

If u has a countable base, A contains one sequence $\gamma = (U_n)$, which is a base of u. In this case we can take $D = \{d_\gamma^{1/n}\}$. □

Some remarks to (1.2) are given in (2.1) and (2.2).

The next two corollaries immediately follow from (1.1) (b) and (1.2) (applied with $|I| = 1$, $I_0 = \emptyset$ or $|I| = 2$, $|I_0| = 1$, respectively).

Corollary 1.3. *If $(X, u, +)$ is a uniform semigroup, then u us generated by a system D of pseudo-metrics on X such that $d(x+y, x'+y') \leqslant d(x,x') + d(y,y')$ for all x, x', y, $y' \in X$ and $d \in D$.*

Note that in the commutative case in (1.3) the condition "$d(x+y,x'+y') \leqslant d(x,x') + d(y,y')$ for all x, x', y, $y' \in X$" is equivalent to the condition "$d(x+z,y+z) \leqslant d(x,y)$ for all x, y, $z \in X$".

(1.3) was first given in [W1, Hilfssatz (1.1)]. The proof, the idea of which was given in [W1, p. 414], was elaborated in detail in [FM, p. 3–7] and [P, p. 8–11] and is quite long. In the proof given here, however, we can at once apply with the help of (1.1)(b) the metrization lemma [K, p. 185], which leads to an essentially simpler proof.

Corollary 1.4. *Let $(X, u, \vee, \wedge)$ be a uniform lattice and $q > 1$. Then u is generated by a system D of pseudo-metrics on X such that $d(x \vee z, y \vee z) \leqslant d(x,y)$ and $d(x \wedge z, y \wedge z) \leqslant q \cdot d(x,y)$ for all x, y, $z \in X$ and $d \in D$.*

In general one cannot replace in (1.4) q by 1 (see (2.3)), but that is possible in the distributive case; more general holds:

Theorem 1.5. *Assume that* $+, \cdot : X \times X \to X$ *are two uniformly continuous associative operations on* (X, u), *which satisfy the distributive laws*

$$(x + y) \cdot z = (x \cdot z) + (y \cdot z)$$

and

$$z \cdot (x + y) = (z \cdot x) + (z \cdot y) \quad \text{for all } x, y, z \in X.$$

Then u *is generated by a system* D *of pseudo-metrics on* X *such that*

$$d(x + y, x' + y') \leqslant d(x, x') + d(y, y'),$$
$$d(x \cdot y, x' \cdot y') \leqslant d(x, x') + d(y, y')$$

for all $x, x', y, y' \in X$ *and* $d \in D$.

Proof. Let $W \in u$. By (1.2) it is enough to prove that W contains a $U \in u$ with $U + \Delta$, $\Delta + U$, $U \cdot \Delta$, $\Delta \cdot U \subset U$.

By (1.1) (b), there is a $V \in u$ such that $V + \Delta$, $\Delta + V \subset V \subset W$. Put

$$U := \{(x, y) \in V : (x, y) \cdot \Delta, \Delta \cdot (x, y), \Delta \cdot (x, y) \cdot \Delta \subset V\}.$$

Of course $U \subset W$. As in the proof of (1.1) (b) one gets that $U \in u$ and $U \cdot \Delta, \Delta \cdot U \subset U \subset V$. Now we prove that $U + \Delta \subset U$; analogously one gets $\Delta + U \subset U$. To prove $U + \Delta \subset U$ we have to check that $U + \Delta \subset V$ and $(U + \Delta) \cdot \Delta, \Delta \cdot (U + \Delta), \Delta \cdot (U + \Delta) \cdot \Delta \subset V$. First we have $U + \Delta \subset V + \Delta \subset V$. Further $(U + \Delta) \cdot \Delta \subset U \cdot \Delta + \Delta \cdot \Delta \subset U + \Delta \subset V + \Delta \subset V$, analogously $\Delta \cdot (U + \Delta) \subset U + \Delta \subset V$. Finally $[\Delta \cdot (U + \Delta)] \cdot \Delta \subset (U + \Delta) \cdot \Delta \subset V$. □

Corollary 1.6. *If* $(X, u, \vee, \wedge)$ *is a distributive uniform lattice, then* u *is generated by a system* D *of pseudo-metrics on* X *such that* $d(x \vee z, y \vee z) \leqslant d(x, y)$ *and* $d(x \wedge z, y \wedge z) \leqslant d(x, y)$ *for all* x, y, $z \in X$ *and* $d \in D$.

I. Fleischer and T. Traynor [FT] have proved that the uniformity on a lattice induced by a modular function with values in a quasinormed group is generated by a pseudo-metric d such that

(i) $d(x \vee z, y \vee z) \leqslant d(x, y), d(x \wedge z, y \wedge z) \leqslant d(x, y)$,

(ii) $d(u, v) \leqslant d(x, y)$ if $x \leqslant u \leqslant v \leqslant y$,

(iii) $d(x \wedge y, x) = d(y, x \vee y)$

(iv) $d(x,y) = d(x \wedge y, x \vee y)$.

In brief, we examine these properties in a more general setting.

Proposition 1.7. *Let d be a pseudo-metric on a lattice X such that for all x, y, $z \in X$ hold*

$$d(x \vee z, y \vee z) \leqslant d(x,y) \quad \text{and} \quad d(x \wedge z, y \wedge z) \leqslant d(x,y).$$

Then

(a) $x \leqslant u \leqslant v \leqslant y$ *implies* $d(u,v) \leqslant d(x,y)$,

(b) $d(x \wedge y, x) = d(y, x \vee y) \leqslant d(x,y)$,

(c) $\frac{1}{2}d(x,y) \leqslant d(x \wedge y, x \vee y) \leqslant 2d(x,y)$ *for all* x, y, u, $v \in X$.

Proof. (a) $d(u,v) = d(u \wedge v, y \wedge v) \leqslant d(u,y) = d(x \vee u, y \vee u) \leqslant d(x,y)$.

(b) $d(x \wedge y, x) = d(y \wedge x, (x \vee y) \wedge x) \leqslant d(y, x \vee y)$, dually $d(x \vee y, y) = d(x \vee y, (x \wedge y) \vee y) \leqslant d(x, x \wedge y)$. Hence $d(x \wedge y, x) = d(x \vee y, y) = d(x \vee y, y \vee y) \leqslant d(x,y)$. (c) $d(x,y) \leqslant d(x, x \wedge y) + d(x \wedge y, y) \leqslant 2d(x \wedge y, x \vee y)$ by (a). $d(x \wedge y, x \vee y) \leqslant d(x \wedge y, x) + d(x, x \vee y) \leqslant 2d(x,y)$ by (b).

The inequalities in (1.7) (c) are sharp: Define on the free lattice $\{0, a, b, 1\}$ with generators a, b a metric by $d(0,a) = d(0,b) = d(1,a) = d(1,b) = 1$ and $d(a,b) = 2 \cdot d(0,1) = 2$ or $d(0,1) = 2 \cdot d(a,b) = 2$; in the first case we have $\frac{1}{2}d(a,b) = d(a \wedge b, a \vee b)$, in the second case $d(a \wedge b, a \vee b) = 2 \cdot d(a,b)$.

Given (X,d) as in (1.7). I don't know whether there exists another pseudo-metric on X with the properties (i) to (iv), which generates the same uniformity as d. If we define $d_1(x,y) := d(x \wedge y, x) + d(x \wedge y, y)$, then d_1 is a pseudo-metric with $d \leqslant d_1 \leqslant 2d$, with the properties (i) to (iii) and at least $d_1(x,y) \geqslant d_1(x \wedge y, x \vee y)$ for all x, $y \in X$; but in the example given before ($X = \{0, a, b, 1\}$, d with $d(0,1) = 1$) we have $d_1(a,b) > d_1(a \wedge b, a \vee b)$. It would be near at hand to take as distance function

$$d_2(x,y) := d(x \wedge y, x \vee y)$$

or

$$d_3(x,y) := \sup\{d(u,v) \colon x \wedge y \leqslant u, v = x \vee y\}.$$

But neither d_2 nor d_3 satisfies, in general, the triangular inequality, as shows the following example: Define on the lattice $X = \{x_1, \ldots, x_9\}$ of figure 1 a metric by $d(x_i, x_{i+1}) = 1 (i = 1, \ldots, 8)$, $d(x_i, x_9) = 1$ $(i = 2, 3, 4, 6, 7)$, $d(x_1, x_8) = d(x_2, x_4) = d(x_6, x_8) = 1$, $d(x_1, x_5) = 3$ and $d(x_i, x_j) = 2$ for all other pairs with $i < j$. Then

$$d_2(x_3, x_7) = d_3(x_3, x_7) = 3$$

but

$$d_2(x_3, x_9) = d_2(x_9, x_7) = d_3(x_3, x_9) = d_3(x_9, x_7) = 1.$$

□

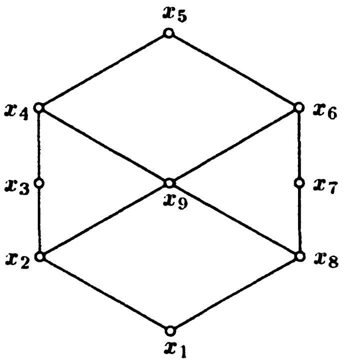

Figure 1

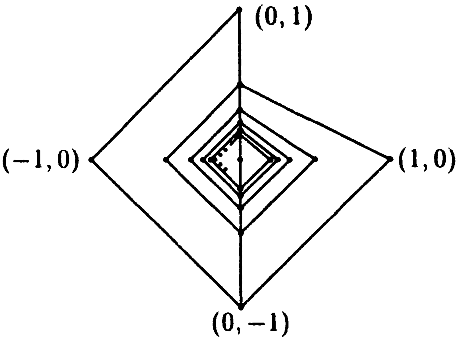

Figure 2

2. Counterexamples

Remark 2.1. In (1.2), the assumption that I_0 is finite is not superfluous.

Proof. Take $(X, u) = \mathbf{R}$ the reals with the usual uniformity, $I = I_0 = \mathbf{N}$ and $f_n(x, y) = nxy$ ($n \in \mathbf{N}$; $x, y \in \mathbf{R}$). Suppose that u is generated by a metric d such that

$$d(f_n(x, y), f_n(x', y)) \leqslant d(x, x') \text{ for all } n \in \mathbf{N} \text{ and } x, x', y \in \mathbf{R}.$$

Then

$$d(1, 0) = d\left(f_n\left(\frac{1}{n}, 1\right), f_n(0, 1)\right) \leqslant d\left(\frac{1}{n}, 0\right) \to 0 \quad (n \to \infty),$$

a contradiction. □

Remark 2.2. In (1.3), the assumption that the addition is associative is not superfluous.

Proof. Take $(X, u) = [0, 2]$ with the usual uniformity, $x \oplus y := \min\{2, xy\}$ for x, $y \in [0, 2]$. Suppose that u is generated by a metric d such that $d(x \oplus y, x' \oplus y) \leqslant d(x, x')$ for all x, x', $y \in [0, 2]$. Then $d(x, 0) = d((\frac{1}{2}x) \oplus 2, 0 \oplus 2) \leqslant d(\frac{1}{2}x, 0)$ and by induction $d(x, 0) \leqslant d(2^{-n}x, 0)$, hence $d(1, 0) = d(2^{-n}, 0) \to 0$ ($n \to \infty$), a contradiction. □

The examples given in (2.1) and (2.2) also show that in (1.2) one cannot replace $q > 1$ by $q = 1$.

The example (2.3) shows that in (1.6) we cannot dispense with the distributivity.

Example 2.3. (cf. Figure 2). Let $K_0 := \{x \in \mathbf{R}: x = 0 \text{ or } |x| = \frac{1}{n} \text{ for some } n \in \mathbf{N}\}$. Define on $L := (\{0\} \times K_0) \cup (K_0 \times \{0\}) \subset \mathbf{R}^2$ two real-valued functions f and g by $f(0,y) = g(0,y) = y$, $f(x,0) = -|x|$; $g(x,0) = |x|$ if $x \leqslant 0$ and $g(x,0) = \frac{1}{n+1}$ if $x = \frac{1}{n}$ for an $n \in \mathbf{N}$.

For a, $b \in L$ define

$$a \leqslant b \text{ iff } g(a) \leqslant f(b) \quad \text{or} \quad a = b.$$

Let u be the uniformity induced on L by the usual uniformity of $\mathbf{R}^2$.

(a) Then $(L, \leqslant)$ is a lattice, u is a compact metrizable uniformity and (L, u) is a uniform lattice.

(b) If d is any continuous pseudo-metric on L such that for all x, y, $z \in L$

$$d(x \vee z, y \vee z) \leqslant d(x,y) \qquad \text{and} \qquad d(x \vee z, y \vee z) \leqslant d(x,y),$$

then $d((0,-1),(1,0)) = 0$.

In particular, u is not generated by a metric satisfying (*).

Proof. (a) $(L, \leqslant)$ is a lattice by the next lemma (2.4), applied for $K = \{0\} \times K_0$ with its natural order, $s(a) = (0, g(a))$ and $i(a) = (0, f(a))$ $(a \in L)$.

By definition, u is metrizable. L is a closed, bounded subset of $\mathbf{R}^2$, hence (L, u) is compact.

We prove now that $\vee$ and $\wedge$ are continuous. From that it follows that $\vee$, $\wedge$ are uniformly continuous since (L, u) is compact. Since $(0,0)$ is the only accumulation point of L, it is enough to show that

(i) $(a,b) \mapsto a \vee b$ and $(a,b) \mapsto a \wedge b$ are continuous in $((0,0),(0,0))$ and that

(ii) $a \mapsto a \vee b$ and $a \mapsto a \wedge b$ are continuous in $(0,0)$ for every $b \in L$, $b \neq (0,0)$.

(i) By (2.4), $a \vee b$ and $a \wedge b$ belong to $\{s(a), s(b), i(a), i(b), a, b\}$. Hence $\|a \vee b\|_\infty$, $\|a \wedge b\|_\infty \leqslant \max\{\|a\|_\infty, \|b\|_\infty\}$. This implies (i).

(ii) Let $b \in L$, $b \neq (0,0)$. Put $U := \{a \in L: \|a\|_\infty < \frac{1}{3}\|b\|_\infty\}$. If $b = (0,y) \in K$ with $y > 0$, then $a \wedge b = a$ for $a \in U$, hence $a \mapsto a \wedge b$ is continuous in $(0,0)$. Similarly, if $b = (0,y) \in K$ with $y < 0$, then $a \vee b = a$ for $a \in U$, hence $a \mapsto a \vee b$ is continuous in $(0,0)$. In all other cases (for b) the functions $a \mapsto a \vee b$ and $a \mapsto a \wedge b$ are constant on U and therefore continuous in $(0,0)$.

(b) Suppose that d is a pseudo-metric on L, which is continuous in $(0,0)$ and satisfies (*). For $n \in \mathbf{N}$, put

$$r_n = \left(\frac{1}{n}, 0\right), \quad l_n = \left(-\frac{1}{n}, 0\right), \quad a_n = \left(0, \frac{1}{n}\right), \quad b_n = \left(0, -\frac{1}{n}\right).$$

Then $d(b_n, r_n) = d(l_{n+1} \wedge r_n, a_{n+1} \wedge r_n) \leqslant d(l_{n+1}, a_{n+1}) = d(b_{n+1} \vee l_{n+1}, r_{n+1} \vee l_{n+1}) \leqslant d(b_{n+1}, r_{n+1})$, hence by induction $d(b_1, r_1) \leqslant d(b_n, r_n)$ for $n \in \mathbb{N}$. Since $d(b_n, r_n) \to 0$ $(n \to \infty)$, it follows that $d((0,-1),(1,0)) = d(b_1, r_1) = 0$. □

Lemma 2.4. *Let K be a lattice, L a set, which contains K, and $i, s\colon L \to K$ two functions such that $i(x) = s(x) = x$ for $x \in K$ and $i(x) < s(x)$ for $x \in L \setminus K$. Then*

$$x \leqslant y \ (\textit{in } L) \ \textit{ iff } \ s(x) \leqslant i(y) \ (\textit{in } K) \ \textit{ or } \ x = y$$

defines a partial ordering on L. With respect to this partial ordering L becomes a lattice and K is a sublattice of L. Moreover, if x, y are incomparable elements of L, then

$$\sup_L\{x, y\} = \sup_K\{s(x), s(y)\} \ \textit{ and } \ \inf_L(x, y) = \inf_K\{i(x), i(y)\}.$$

P r o o f. Since $i(x) = s(x) = x$ for $x \in K$, the relation defined on L coincides on K with the given partial ordering on K. Obviously, $\leqslant$ is reflexive on L.

$\leqslant$ is antisymmetric: Suppose that x, $y \in L$ with $x \leqslant y$, $y \leqslant x$, $x \neq y$. Then $s(x) \leqslant i(y)$ and $s(y) \leqslant i(x)$. Since $i(z) \leqslant s(z)$ for all $z \in L$, one obtains $s(x) \leqslant i(y) \leqslant s(y) \leqslant i(x) \leqslant s(x)$, hence $s(x) = i(x) = s(y) = i(y)$. It follows that x, $y \in K$, since $s(z) \neq i(z)$ for $z \in L \setminus K$. Consequently, $x = s(x) = s(y) = y$.

$\leqslant$ is transitive: Suppose that x, y, $z \in L$ with $x \leqslant y$, $y \leqslant z$ and $x \neq y \neq z$. Then $s(x) \leqslant i(y) \leqslant s(y) \leqslant i(z)$, hence $s(x) \leqslant i(z)$ and $x \leqslant z$.

Let x, y be incomparable elements of L. $a := \sup_K\{s(x), s(y)\}$ is the supremum of $\{x, y\}$ in L: Since $a \in K$, we have $s(a) = i(a) = a$. Therefore $s(x) \leqslant a = i(a)$, hence $x \leqslant a$ and just so $y \leqslant a$. Let $z \in L$ be an upper bound of $\{x, y\}$. Since x, y are incomparable, it follows that $z \neq x$ and $z \neq y$ and therefore $s(x) \leqslant i(z)$ and $s(y) \leqslant i(z)$, hence $s(a) = a \leqslant i(z)$. Consequently $a \leqslant z$. Similarly one gets that $\inf_K\{i(x), i(y)\}$ is the infimum of $\{x, y\}$. In particular, L is a lattice and K a sublattice of L. □

In the example (2.3), (L, u) is a compact Hausdorff uniform lattice. It follows from some statements in [W2] that L is (as lattice) complete and that u is order continuous, exhaustive and satisfies (F) and (σ) (see [W2] for the definitions). Therefore (L, u) has strong topological properties. On the other hand, the lattice L is not modular. It would be of interest to decide, whether such an example exists also in the modular case.

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