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THE PERRON PRODUCT INTEGRAL IN LIE GROUPS

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1. INTRODUCTION

The product integral for matrix-valued functions, defined on a compact interval of the real line, was introduced by Volterra ([21] and [22]) and completed by Schlesinger ([18] and [19]) and Rasch [17]. The main motivation for this construction was the study of linear differential equations with variable coefficients and the discussion of such systems in the complex plane.

The possibility of an extension for more general setting was perceived by Birkhoff [1], and it was accomplished by Hamilton [6] for functions taking its values in the Lie algebra of a Lie group of finite dimension. In [16] it was presented a self-contained survey of this Riemann-type product integral, using all the power of the theory of finite-dimensional Lie groups.

Recently, Jarník and Kurzweil [10] in an elementary way constructed a Perron-type product integral for matrix-valued functions and they applied it to study systems of linear differential equations. The purpose of this paper is to extend the construction of the Perron product integral to functions taking its values in a Lie algebra associated with a finite-dimensional Lie group. Its main results are the following: 1) a multiplicative property (Theorem 3.6); 2) the relation with the Perron summation integral (Theorem 3.8); 3) an existence theorem (Corollary 3.10); and 4) a continuity property (Theorem 3.12). These properties extend some of the results of [10].

This paper is organized as follows: In the Section 2 we present some basic notions of manifolds, Lie algebras and Lie groups and some key results with precise references for the proofs. In the Section 3 we construct the Perron product integral and we deduce some of its fundamental properties.

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2. PRELIMINARIES

Write $\mathbf{N} = \{0, 1, 2, \ldots\}$ and let $\mathbf{R}$ denote the real line.

Let M be a Hausdorff topological space. A *chart* on M is a triplet $c = (U, \varphi, n)$ where U is an open subset of M, $n \in \mathbf{N} \setminus \{0\}$ and φ is a homeomorphism of U onto the open subset $\varphi(U)$ of $\mathbf{R}^n$. The natural number n is called *dimension* of c and the open set U is called the *domain* of c, and we write $U = \mathrm{Dom}(c)$. Let $\mathscr{C}$ be a set of charts on M of the same dimension n. We say that $M = (M, \mathscr{C})$ is a C^∞ *manifold of dimension* n if the following conditions are satisfied:

(i) $M = \cup\{\mathrm{Dom}(c) \colon c \in \mathscr{C}\}$

(ii) If $c = (U, \varphi, n)$ and $c' = (V, \chi, n)$ are two elements of $\mathscr{C}$ such that $U \cap V \neq \emptyset$, then the function $\chi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \mathbf{R}^n$ is C^∞.

(iii) If $\mathscr{C}'$ is a set of charts on M of the same dimension n such that $\mathscr{C} \subseteq \mathscr{C}'$, then $\mathscr{C}' = \mathscr{C}$.

According to [2, Theorem 1.3, p. 54], a C^∞ manifold of dimension n is completely determined for any set $\mathscr{C}$ of charts on M of the same dimension n satisfying the conditions (i) and (ii).

Let M be a C^∞ manifold of dimension n and let $c = (U, \varphi, n)$ be a chart on M. For every $i \in \{1, 2, 3, \ldots, n\}$, put $x^i = \mathrm{pr}_i \circ \varphi \colon U \to \mathbf{R}$. Then the n-tuple of real-valued functions $(x^i)_{1 \leqslant i \leqslant n}$ is called a *local coordinate system* on c. For each point $y \in U$, the n-tuple of real numbers $(x^i(y))_{1 \leqslant i \leqslant n}$ is called the *local coordinates of* y with respect to c.

We present three examples of C^∞ manifolds:

1. Let V be a vector space over $\mathbf{R}$ of dimension n. Then V is a metrizable topological space. Let φ be a linear isomorphism from V onto $\mathbf{R}^n$. Then $\mathscr{C} = \{(V, \varphi, n)\}$ satisfies the conditions (i) and (ii), and it determines a structure of C^∞ manifold of dimension n on V.

2. Let M be a C^∞ manifold of dimension n and let N be a non-empty open subset of M. Consider a set $\mathscr{C}$ of charts on M of the same dimension n satisfying the conditions (i) and (ii). Then $\mathscr{C}' = \{(U \cap N, \varphi|U \cap N, n) \colon (U, \varphi, n) \in \mathscr{C}\}$ determines a structure of C^∞ manifold of dimension n on N, which is said then to be an *open submanifold* of M.

3. Let M and M' be C^∞ manifolds of dimension m and n, respectively. Then, with the product topology, $M \times M'$ becomes a Hausdorff topological space. Consider a set $\mathscr{C}$ of charts on M of the same dimension m and a set $\mathscr{C}'$ of charts on M' of the same dimension n, both satisfying the conditions (i) and (ii). Then $\mathscr{C}'' = \{(U \times V, \varphi \times \chi, m+n) \colon (U, \varphi, m) \in \mathscr{C} \text{ and } (V, \chi, n) \in \mathscr{C}'\}$ determines a structure of C^∞ manifold of dimension $m+n$ on $M \times M'$, and it is called the *product manifold* of M and M'.

Let M and M' be C^∞ manifolds of dimension m and n, respectively, and let $f\colon M \to M'$. We say that

a) f is a C^∞ *function* if, for every $x \in M$, there exist a chart (U, φ, m) on M and a chart (V, χ, n) on M' such that $x \in U$, $f(U) \subseteq V$ and the function $\chi \circ f \circ \varphi^{-1}\colon \varphi(U) \to \mathbf{R}^n$ is C^∞.

b) f is a *diffeomorphism* if f is bijective and if f and f^{-1} are C^∞ functions. It is clear that the composition of two C^∞ functions is again C^∞.

Let M be a C^∞ manifold of dimension n and let U be a non-empty open subset of M. We denote by $C^\infty(U)$ the set of all C^∞ functions $f\colon U \to \mathbf{R}$. It is clear that $C^\infty(U)$ is an associative algebra over $\mathbf{R}$ with unity. Let $x \in M$. We consider the set $\mathscr{F}(x)$ of all real-valued C^∞ functions, each defined on some open neighbourhood of x. If $f_1, f_2 \in \mathscr{F}(x)$ we write $f_1 \sim f_2$ if they agree on some open subset of M containing x. Then $\sim$ is an equivalence relation on $\mathscr{F}(x)$ and each element of the quotient set $C^\infty(x) = \mathscr{F}(x)/_\sim$ is called a *germ of C^∞ functions at x*. If $f \in \mathscr{F}(x)$, its corresponding germ will be denoted by f_x. It is easy to verify that $C^\infty(x)$ is an associative algebra over $\mathbf{R}$ with unity. We define the *tangent space $T_x(M)$ to M at x* to be the set of all linear forms v_x on $C^\infty(x)$ satisfying the Leibniz rule:

$$v_x(f_x \cdot g_x) = v_x(f_x)g(x) + f(x)v_x(g_x) \quad \text{for all} \quad f, g \in \mathscr{F}(x).$$

Every element of $T_x(M)$ is called a *tangent vector to M at x*. It is easy to see that $T_x(M)$ is a vector space over $\mathbf{R}$.

2.1 Lemma. *If M is a C^∞ manifold of dimension n and $x \in M$, then the tangent space $T_x(M)$ is also of dimension n.*

For a proof see [14, Theorem, pp. 41–42].

Let M be a C^∞ manifold of dimension n. A C^∞ *vector field on M* is a linear function $X : C^\infty(M) \to C^\infty(M)$ such that $X(f \cdot g) = (Xf) \cdot g + f \cdot (Xg)$. We denote by $\chi(M)$ the set of all C^∞ vector fields on M. It is clear that $\chi(M)$ is a vector space over $\mathbf{R}$ such that $X \circ Y - Y \circ X \in \chi(M)$ for all $X, Y \in \chi(M)$.

Let K be a field of characteristic 0. A *Lie algebra over K* is a vector space $\mathscr{A}$ over K endowed with a bilinear function from $\mathscr{A} \times \mathscr{A}$ to $\mathscr{A}$, usually denoted by $(X, Y) \to [X, Y]$, which satisfies the following two identities:

(i) $[X, X] = 0$;

(ii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for any elements X, Y, Z in $\mathscr{A}$.

We note that if $\mathscr{A}$ is an associative algebra over K, then the vector space $\mathscr{A}$ over K endowed with the bilinear function $(X, Y) \to XY - YX$ is a Lie algebra over K. For example, if $n \in \mathbf{N} \setminus \{0\}$, the set $M_n(\mathbf{R})$ of all $n \times n$ real matrices is a Lie algebra over $\mathbf{R}$.

2.2 Lemma. *If M is a C^∞ manifold of dimension n, then $\chi(M)$ endowed with the Lie product $[X,Y] = X \circ Y - Y \circ X$ is a Lie algebra over* $\mathbf{R}$.

For a proof see [2, Theorem 7.4, p. 153].

A *Lie group of dimension* n is a C^∞ manifold G of dimension n which is also endowed with a group structure such that the function $(g_1, g_2) \to g_1 g_2^{-1}$ from the product manifold $G \times G$ to G is C^∞. For example, if $n \in \mathbf{N} \setminus \{0\}$, the set $GL(n, \mathbf{R})$ of all invertible elements of $M_n(\mathbf{R})$ is a Lie group of dimension n^2 under matrix multiplication (see [2, Example 1.6, pp. 56–57]).

2.3 Lemma. *Let G be a Lie group of dimension n. Then G is a complete metrizable topological group with a left invariant metric ϱ.*

The proof follows from [4, Proposition 1, p. 97] and [15, Theorem, p. 34] taking into account Property 2.3.3 of [3, p. 18] in the proof of [4, Lemme 1, p. 96].

Let G be a Lie group of dimension n and let $g \in G$. Define the function $L_g : G \to G$ by the formula: $L_g(x) = gx$. It is easy to see that L_g is a diffeomorphism and the proof of the following lemma is straightforward:

2.4 Lemma. *Let G be a Lie group of dimension n, let $g \in G$ and let $X \in \chi(G)$. Define $((L_g)_* X)(f)(x) = X(f \circ L_{g^{-1}})(gx)$ for all $f \in C^\infty(G)$ and all $x \in G$. Then $(L_g)_* X \in \chi(G)$.*

Let G be a Lie group of dimension n and let $X \in \chi(G)$. We say that X is a *left invariant C^∞ vector field* on G if $(L_g)_* X = X$ for all $g \in G$. We denote by $L(G)$ the set of all left invariant C^∞ vector fields on G. It is clear that $X \in L(G)$ if and only if $Xf \circ L_g = X(f \circ L_g)$ for all $f \in C^\infty(G)$ and all $g \in G$. From this observation we can deduce the following

2.5 Lemma. *Let G be a Lie group of dimension n. Then $L(G)$ is a Lie subalgebra of $\chi(G)$.*

The Lie algebra $L(G)$ is called the *Lie algebra of the Lie group G*. For example, $L(GL(n, \mathbf{R})) = M_n(\mathbf{R})$ (see [7, Lemma 15, p. 59] or [23, Example 3.10(b), pp. 86–87]).

2.6 Lemma. *Let G be a Lie group of dimension n with neutral element e. Then there exists a linear isomorphism from $L(G)$ onto $T_e(G)$, and therefore*

$$\dim(L(G)) = n.$$

For a proof see [14, Theorem 1, pp. 190–191] or [23, Proposition 3.7 (a), p. 85].

2.7 Lemma. *Let G be a Lie group of dimension n with neutral element e. Then there exists a C^∞ function $\exp : L(G) \to G$ with the following properties:*

a) $\exp(s+t)X = \exp sX \cdot \exp tX$ *for all $s,t \in \mathbf{R}$ and all $X \in L(G)$.*

b) *There are open neighbourhoods U of e in G and V of 0 in $L(G)$ such that* exp *is a diffeomorphism from V onto U.*

For a proof see [20, pp. 84–88] or [23, pp. 102–103].

In the case where $G = GL(n,\mathbf{R})$, it can be shown that $\exp X = I + \frac{X}{1!} + \frac{X^2}{2!} + \ldots$ where I is the identity matrix in G (see [23, Example 3.35, pp. 105–107]).

3. The Perron product integral

A closed interval of the real line is said to be ***non-degenerate*** if it contains more than one point. Let $\mathscr{I}(\mathbf{R})$ denote the set of all non-degenerate closed intervals of the real line. For $K \in \mathscr{I}(\mathbf{R})$ we denote by $\mathscr{I}(K)$ the set of all elements of $\mathscr{I}(\mathbf{R})$ contained in K.

Let $K \in \mathscr{I}(\mathbf{R})$. A ***subdivision*** of K is a non-empty finite subset Δ of $K \times \mathscr{I}(K)$ such that

(i) If $(t,J) \in \Delta$, then $t \in J$.

(ii) If (t,J) and (t',J') are two distinct members of Δ, then $\mathring{J} \cap \mathring{J}' = \emptyset$.

(iii) $\cup\{J : (t,J) \in \Delta\} = K$.

If $K \in \mathscr{I}(\mathbf{R})$ we denote by $\sigma(K)$ the set of all subdivisions of K. For $[a,b] \in \mathscr{I}(\mathbf{R})$ it is clear that every element of $\sigma([a,b])$ can be written in the form

$$\Delta = \{(t_i,[x_{i-1},x_i]) : i \in \{1,2,\ldots,n\}\}$$

where $n \in \mathbf{N} \setminus \{0\}$, $t_i \in [x_{i-1},x_i]$ and

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

Every point x_i for $1 \leqslant i \leqslant n-1$ is called a ***tag*** of Δ.

Let $[a,b] \in \mathscr{I}(\mathbf{R})$, let c be a real number such that $a < c < b$, let

$$\Delta_1 = \{(t_i,[x_{i-1},x_i] : i \in \{1,2,\ldots,n\}\}$$

be an element of $\sigma([a,c])$ and let

$$\Delta_2 = \{(s_j,[y_{j-1},y_j]) : j \in \{1,2,\ldots,m\}\}$$

be an element of $\sigma([c,b])$. Put $s_j = t_{n+j}$ and $y_j = x_{n+j}$ for all $j \in \{1,2,\dots,m\}$. Since $x_n = c = y_0$, the set

$$\Delta = \{(t_i,[x_{i-1},x_i])\colon i \in \{1,2,\dots,m+n\}\}$$

is a subdivision of $[a,b]$ and this fact is denoted by $\Delta = \Delta_1 \circ \Delta_2$.

Let $K \in \mathscr{I}(\mathbf{R})$. Then

a) Every function from K to $]0,+\infty[$ is called a *gauge* on K.

b) If δ is a gauge on K, a subdivision Δ of K is said to be *δ-fine* if $J \subseteq]t-\delta(t), t+\delta(t)[$ for every $(t,J) \in \Delta$.

Write $\sigma(K,\delta) = \{\Delta \in \sigma(K)\colon \Delta \ \text{ is } \ \delta\text{-fine}\}$. It is well-known that $\sigma(K,\delta) \neq \emptyset$ for every $K \in \mathscr{I}(\mathbf{R})$ and every gauge δ on K (see [12, Lemma, p. 22] and [13, Compatibility Theorem, p. 38]).

If $(x_i)_{i\in\mathbf{N}}$ is a sequence of elements of a Lie group, then the symbol $\prod_{i=0}^{n} x_i$ is defined by the inductive formulas:

$$\prod_{i=0}^{0} x_i = x_0 \quad \text{and} \quad \prod_{i=0}^{n+1} x_i = x_{n+1}\cdot\prod_{i=0}^{n} x_i.$$

Now let G be a Lie group with neutral element e, let $L(G)$ be the Lie algebra of G, let $K \in \mathscr{I}(\mathbf{R})$, let $u\colon K \to L(G)$ and let $[a,b] \in \mathscr{I}(K)$. For each element $\Delta = \{(t_i,[x_{i-1},x_i])\colon i \in \{1,2,\dots,n\}\}$ of $\sigma([a,b])$, write

$$S(u,\Delta) = \prod_{i=1}^{n} \exp((x_i - x_{i-1})u(t_i)).$$

We say that u is *Perron product integrable* on $[a,b]$ if, for every $\varepsilon > 0$, there exists a gauge δ on $[a,b]$ such that $\varrho(S(u,\Delta_1),S(u,\Delta_2)) < \varepsilon$ whenever $\Delta_1,\Delta_2 \in \sigma([a,b],\delta)$, where ϱ is the left invariant metric on G given by Lemma 2.3.

Henceforth we fix an element $K \in \mathscr{I}(\mathbf{R})$.

3.1 Lemma. *Let $u\colon K \to L(G)$ and let $[a,b] \in \mathscr{I}(K)$. Then u is Perron product integrable on $[a,b]$ if and only if there exists an element $g \in G$ with the following property:*

(∗) For every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that $\varrho(g,S(u,\Delta)) < \varepsilon$ whenever $\Delta \in \sigma([a,b],\delta)$.

P r o o f. Since the sufficiency is trivial, it remains to show the necessity.

For every $n \in \mathbb{N} \setminus \{0\}$, let $W_n = \{S(u,\Delta)\colon \Delta \in \sigma([a,b],\delta)$ for some gauge δ on $[a,b]$ and $\varrho(S(u,\Delta),S(u,\Delta')) < \frac{1}{n}$ for all $\Delta' \in \sigma([a,b],\delta)\}$. Since u is Perron product integrable on $[a,b]$, every $W_n \neq \emptyset$, $n = 1,2,3,\ldots$. We shall show that

$$\operatorname{diam}(W_n) \leqslant \frac{2}{n} \qquad \text{for every } n \in \mathbb{N} \setminus \{0\}.$$

Let $S(u,\Delta_1)$, $S(u,\Delta_2)$ be two elements of W_n. Then $\Delta_i \in \sigma([a,b],\delta_i)$ for some gauge δ_i on $[a,b]$ and $\varrho(S(u,\Delta_i),S(u,\Delta_i')) < \frac{1}{n}$ for all $\Delta_i' \in \sigma([a,b],\delta_i)$ $(i = 1,2)$. Let $\delta = \min\{\delta_1,\delta_2\}$ and let $\Delta \in \sigma([a,b],\delta)$. Then

$$\Delta \in \sigma([a,b],\delta_1) \cap \sigma([a,b],\delta_2)$$

and therefore

$$\varrho\Big(S(u,\Delta_1),S(u,\Delta_2)) \leqslant \varrho(S(u,\Delta_1),S(u,\Delta)) + \varrho(S(u,\Delta),S(u,\Delta_2))\Big) < \frac{2}{n}.$$

Hence $\operatorname{diam}(W_n) \leqslant \frac{2}{n}$. We shall show that

$$W_{n+1} \subseteq W_n \qquad \text{for every } n \in \mathbb{N} \setminus \{0\}.$$

In fact, let $S(u,\Delta) \in W_{n+1}$. Then $\Delta \in \sigma([a,b]),\delta)$ for some gauge δ on $[a,b]$ and

$$\varrho(S(u,\Delta),(S(u,\Delta')) < \frac{1}{n+1} \quad \text{for all} \quad \Delta' \in \sigma([a,b],\delta).$$

Since $\frac{1}{n+1} < \frac{1}{n}$, it follows that $S(u,\Delta) \in W_n$.

Let $F_n = \overline{W_n}$ for all $n \in \mathbb{N} \setminus \{0\}$. Since $\operatorname{diam}(F_n) = \operatorname{diam}(W_n)$ it follows that $\lim\limits_{n\to\infty} \operatorname{diam}(F_n) = 0$. But (G,ϱ) is a complete metric space. Then, by Cantor Theorem [11, p. 413], there exists $g \in G$ such that $\bigcap\limits_{n=1}^{\infty} F_n = \{g\}$. To prove the condition $(*)$, let $\varepsilon > 0$. Choose $n_0 \in \mathbb{N} \setminus \{0\}$ such that $\frac{2}{n_0} < \varepsilon$. Since $g \in F_{n_0} = \overline{W_{n_0}}$, there exists $S(u,\Delta_0) \in W_{n_0}$ such that $\varrho(g,S(u,\Delta_0)) < \frac{\varepsilon}{2}$. Then $\Delta_0 \in \sigma([a,b],\delta)$ for some gauge δ on $[a,b]$ and $\varrho(S(u,\Delta_0),\, S(u,\Delta')) < \frac{1}{n_0}$ for all $\Delta' \in \sigma([a,b],\delta)$. Let $\Delta \in \sigma([a,b],\delta)$. Then $\varrho(g,S(u,\Delta)) \leqslant \varrho(g,S(u,\Delta_0)) + \varrho(S(u,\Delta_0),S(u,\Delta)) < \varepsilon$. □

It is clear that, if u is Perron product integrable on $[a,b]$, then there exists a unique element $g \in G$ satisfying the condition $(*)$ of Lemma 3.1. This element is called the *Perron product integral of u over* $[a,b]$ and it is denoted by $(P)\prod\limits_a^b \exp(u(t)\mathrm{d}t)$.

Now consider the set $D([a,b])$ of all pairs (δ,Δ), where δ is a gauge on $[a,b]$ and $\Delta \in \sigma([a,b],\delta)$. It is clear that $D([a,b])$ is non-empty. If (δ_1,Δ_1) and (δ_2,Δ_2) are two elements of $D([a,b])$, we say that (δ_1,Δ_1) is *finer than* (δ_2,Δ_2) and we write $(\delta_1,\Delta_1)\geqslant(\delta_2,\Delta_2)$ if $\delta_1\leqslant\delta_2$. For example, if (δ_1,Δ_1) and (δ_2,Δ_2) are two elements of $D([a,b])$, $\delta=\min\{\delta_1,\delta_2\}$ and $\Delta\in\sigma([a,b],\delta)$, then (δ,Δ) is finer than (δ_1,Δ_1) and (δ_2,Δ_2). Since $(D[a,b],\geqslant)$ is a partially ordered set, the preceeding example shows that $(D([a,b],\geqslant)$ is a directed set. Let $u\colon K\to L(G)$ and let $[a,b]\in\mathcal{I}(K)$. For each $(\delta,\Delta)\in D([a,b])$ define $h(\delta,\Delta)=S(u,\Delta)$. Then h is a net in G. Since G is a Hausdorff topological space, the net h converges to at most one point.

3.2 Lemma. *Let $u:K\to L(G)$ and let $[a,b]\in\mathcal{I}(K)$. Then u is Perron product integrable on $[a,b]$ if and only if the net h converges. Moreover,*

$$\lim_{(\delta,\Delta)}h(\delta,\Delta)=(P)\prod_a^b\exp(u(t)\mathrm{d}t).$$

Proof. Suppose that u is Perron product integrable on $[a,b]$. Let $\varepsilon>0$. Then by Lemma 3.1, there exists a gauge δ_ε on $[a,b]$ such that

$$\varrho\Big((P)\prod_a^b\exp(u(t)\mathrm{d}t),S(u,\Delta)\Big)<\varepsilon\quad\text{whenever}\quad\Delta\in\sigma([a,b],\delta_\varepsilon).$$

Choose a δ_ε-fine subdivision Δ_ε of $[a,b]$. Then $(\delta_\varepsilon,\Delta_\varepsilon)\in D([a,b])$. Let $(\delta,\Delta)\in D([a,b])$ be such that $(\delta,\Delta)\geqslant(\delta_\varepsilon,\Delta_\varepsilon)$. Then Δ is δ_ε-fine, and therefore

$$\varrho\Big((P)\prod_a^b\exp(u(t)\mathrm{d}t),h(\delta,\Delta)\Big)<\varepsilon.$$

Hence the net h converges to $(P)\prod_a^b\exp(u(t)\mathrm{d}t)$.

Suppose that the net h converges and let $g=\lim_{(\delta,\Delta)}h(\delta,\Delta)$. Let $\varepsilon>0$. Then there exists $(\delta_\varepsilon,\Delta_\varepsilon)\in D([a,b])$ such that $(\delta,\Delta)\in D([a,b])$ and $(\delta,\Delta)\geqslant(\delta_\varepsilon,\Delta_\varepsilon)$ imply $\varrho(g,h(\delta,\Delta))<\varepsilon$. Let $\Delta\in\sigma([a,b]),\delta_\varepsilon)$. Then $(\delta_\varepsilon,\Delta)\in D([a,b])$ and $(\delta_\varepsilon,\Delta)\geqslant(\delta_\varepsilon,\Delta_\varepsilon)$. Therefore $\varrho(g,S(u,\Delta))<\varepsilon$. By Lemma 3.1, u is Perron product integrable on $[a,b]$. □

3.3 Lemma. *Let $u:K\to L(G)$ and let $[a,b]\in\mathcal{I}(K)$. Then u is Perron product integrable on $[a,b]$ if and only if, for every $\varepsilon>0$, there exists a gauge δ on $[a,b]$ such that $\varrho(S(u,\Delta_1)^{-1},S(u,\Delta_2)^{-1})<\varepsilon$ whenever $\Delta_1,\Delta_2\in\sigma([a,b],\delta)$.*

Proof. Suppose that u is Perron product integrable on $[a,b]$. By Lemma 3.2, the net h converges, and therefore h is Cauchy in $(G,\mathcal{U}_R)$, where $\mathcal{U}_R$ is the right uniformity on G. Let $\varepsilon > 0$. Then $V = \{g \in G : \varrho(g, e) < \varepsilon\}$ is a neighbourhood of e in G, and therefore $U = \{(g_1, g_2) \in G \times G : g_2 \cdot g_1^{-1} \in V\}$ is an element of $\mathcal{U}_R$. So there exists $(\delta, \Delta) \in D([a, b])$ such that $(\delta', \Delta'), (\delta'', \Delta'') \in D([a, b])$ and (δ', Δ'), $(\delta'', \Delta'') \geqslant (\delta, \Delta)$ imply $h(\delta'', \Delta'') \cdot h(\delta', \Delta')^{-1} \in V$. Let Δ_1, $\Delta_2 \in \sigma([a, b], \delta)$. Since

$$(\delta, \Delta_1), (\delta, \Delta_2) \in D([a, b]) \quad \text{and} \quad (\delta, \Delta_1), (\delta, \Delta_2) \geqslant (\delta, \Delta),$$

we get $S(u, \Delta_2) \cdot S(u, \Delta_1)^{-1} \in V$. Hence

$$\varrho\left(S(u, \Delta_1)^{-1}, S(u, \Delta_2)^{-1}\right) = \varrho\left(S(u, \Delta_2) \cdot S(u, \Delta_1)^{-1}, e\right) < \varepsilon.$$

To prove the sufficiency, let U be an element of the right uniformity $\mathcal{U}_R$ on G. Then there exists a neighbourhood V of e in G such that

$$U = \{(g_1, g_2) \in G \times G : g_2 g_1^{-1} \in V\}.$$

Let $\varepsilon > 0$ be such that $\{g \in G : \varrho(g, e) < \varepsilon\} \subseteq V$. By hypothesis, there exists a gauge δ on $[a, b]$ such that

$$\varrho\left(S(u, \Delta_1)^{-1}, S(u, \Delta_2)^{-1}\right) < \varepsilon$$

whenever Δ_1, $\Delta_2 \in \varrho([a, b], \delta)$. Let $\Delta \in \sigma([a, b], \delta)$. For $i = 1, 2$, let $(\delta_i, \Delta_i) \in D([a, b])$ be such that $(\delta_i, \Delta_i) \geqslant (\delta, \Delta)$. Then Δ_1, $\Delta_2 \in \sigma([a, b], \delta)$, and therefore

$$\varrho\Big(S(u, \Delta_2) S(u, \Delta_1)^{-1}, e\Big) = \varrho\Big(S(u, \Delta_1)^{-1}, S(u, \Delta_2)^{-1}\Big) < \varepsilon.$$

Hence $h(\delta_2, \Delta_2) \cdot h(\delta_1, \delta_1)^{-1} \in V$. Consequently, h is a Cauchy net in $(G, \mathcal{U}_R)$. Since $(G, \mathcal{U}_R)$ is a complete uniform space, it follows that the net h converges. By Lemma 3.2, u is Perron product integrable on $[a, b]$. □

3.4 Theorem. *Let $K = [a, b]$, let $u : K \to L(G)$ and let $[c, d] \in \mathcal{I}(K)$. If u is Perron product integrable on $[a, b]$, then u is Perron product integrable on $[c, d]$.*

Proof. We consider three cases:

1. $c = a$ and $d < b$.

Let $\varepsilon > 0$. Since u is Perron product integrable on $[a, b]$, there exists a gauge δ on $[a, b]$ such that $\varrho(S(u, \Delta'), S(u, \Delta'')) < \varepsilon$ whenever $\Delta', \Delta'' \in \sigma([a, b], \delta)$. Let $\Delta \in \sigma([d, b], \delta\big|[d, b])$. For Δ_1, $\Delta_2 \in \sigma([c, d], \delta\big|[c, d])$, write $\Delta_3 = \Delta_1 \circ \Delta$ and $\Delta_4 = \Delta_2 \circ \Delta$. It is easy to verify that Δ_3 and Δ_4 are two δ-fine subdivisions of $[a, b]$, $S(u, \Delta_3) = S(u, \Delta) \cdot S(u, \Delta_1)$ and $S(u, \Delta_4) = S(u, \Delta) \cdot S(u, \Delta_2)$. Since ϱ is a left invariant metric, we have $\varrho(S(u,\Delta_1),S(u,\Delta_2)) = \varrho(S(u,\Delta)\cdot S(u,\Delta_1),\ S(u,\Delta)\cdot S(u,\Delta_2)) < \varepsilon$, and therefore u is Perron product integrable on $[c,d]$.

2. $c > a$ and $d = b$.

Let $\varepsilon > 0$. Since u is Perron product integrable on $[a,b]$, by Lemma 3.3, there exists a gauge δ on $[a,b]$ such that $\varrho(S(u,\Delta')^{-1}, S(u,\Delta'')^{-1}) < \varepsilon$ whenever Δ', $\Delta'' \in \sigma([a,b],\delta)$. Let $\Delta \in \sigma([a,c],\delta|[a,c])$. For Δ_1, $\Delta_2 \in \sigma([c,d],\delta|[c,d])$, write $\Delta_3 = \Delta\circ\Delta_1$ and $\Delta_4 = \Delta\circ\Delta_2$. It is easy to verify that Δ_3 and Δ_4 are two δ-fine subdivisions of $[a,b]$, $S(u,\Delta_3) = S(u,\Delta_1)\cdot S(u,\Delta)$ and $S(u,\Delta_4) = S(u,\Delta_2)\cdot S(u,\Delta)$. Then $S(u,\Delta_3)^{-1} = S(u,\Delta)^{-1}\cdot S(u,\Delta_1)^{-1}$ and $S(u,\Delta_4)^{-1} = S(u,\Delta)^{-1}\cdot S(u,\Delta_2)^{-1}$, and therefore

$$\varrho\Big(S(u,\Delta_1)^{-1}, S(u,\Delta_2)^{-1}\Big) = \varrho\Big(S(u,\Delta)^{-1}\cdot S(u,\Delta_1)^{-1}, S(u,\Delta)^{-1}\cdot S(u,\Delta_2)^{-1}\Big) < \varepsilon.$$

By Lemma 3.3, u is Perron product integrable on $[c,d]$.

3. $c > a$ and $d < b$.

Since u is Perron product integrable on $[a,b]$, the first case implies that u is Perron product integrable on $[a,d]$. Let $v = u|[a,d]$. Since v is product integrable on $[a,d]$, the second case implies that v is Perron product integrable on $[c,d]$. So u is Perron product integrable on $[c,d]$. □

3.5 Lemma. *Let $[a,b]\in\mathscr{I}(K)$, let c be a real number such that $a<c<b$ and let $D_c([a,b]) = \{(\delta,\Delta)\in D([a,b])\colon c$ is a tag of $\Delta\}$. Then $D_c([a,b])$ is a cofinal subset of $(D([a,b]),\geqslant)$.*

Proof. Let $(\delta,\Delta)\in D([a,b])$. Define a gauge δ' on $[a,b]$ by

$$\delta'(t) \leqslant \min(|t-c|,\delta(t)) \quad \text{for} \quad t\neq c \quad \text{and} \quad \delta'(c)\leqslant\delta(c).$$

Let $\Delta_0 = \{(t_i,[x_{i-1},x_i])\colon i\in\{1,2,\dots,n\}\}$ be a δ'-fine subdivision of $[a,b]$. We may suppose that c is not a tag of Δ_0. Then there exists $j\in\{1,2,\dots,n\}$ such that $x_{j-1}<c<x_j$. Because Δ_0 is δ'-fine, the above conditions imply that $t_j = c$. Put

$\Delta_1 = \{(t_1,[x_0,x_1]),(t_2,[x_1,x_2]),\dots,(t_j,[x_{j-1},t_j])\}$,

$\Delta_2 = \{(t_j,[t_j,x_j]),(t_{j+1},[x_j,x_{j+1}]),\dots,(t_n,[x_{n-1},x_n])\}$,

$\Delta' = \Delta_1\circ\Delta_2$.

Since Δ_1 is $\delta'|[a,c]$-fine and Δ_2 is $\delta'|[c,b]$-fine, Δ' is δ'-fine. So $(\delta',\Delta')\in D_c([a,b])$ and $(\delta',\Delta')\geqslant(\delta,\Delta)$. □

3.6 Theorem. *Let $K=[a,b]$, let c be a real number such that $a<c<b$ and let $u\colon K\to L(G)$. If u is Perron product integrable on $[a,c]$ and $[c,b]$, then u is Perron*

product integrable on $[a,b]$ *and*

$$(P)\prod_a^b \exp(u(t)\mathrm{d}t) = (P)\prod_c^b \exp(u(t)\mathrm{d}t)\cdot(P)\prod_a^c \exp(u(t)\mathrm{d}t).$$

Proof. By Lemma 3.5 $D_c([a,b])$ is a cofinal subset of $(D([a,b],\geqslant)$. Let $k(\delta,\Delta) = S(u,\Delta)$ for all $(\delta,\Delta)\in D_c([a,b])$. Then k is a subnet of the net h. We shall show that

$$\lim_{(\delta,\Delta)} k(\delta,\Delta) = (P)\prod_c^b \exp(u(t)\mathrm{d}t)\cdot(P)\prod_a^c \exp(u(t)\mathrm{d}t).$$

Let $(\delta,\Delta)\in D_c([a,b])$ be such that

$$\Delta = \{(t_i,[x_{i-1},x_i])\colon i\in\{1,2,\dots,n\}\}$$

where $x_j = c$ with $1\leqslant j\leqslant n-1$. Put

$$\delta_1(\delta) = \delta\big|[a,c], \qquad \delta_2(\delta) = \delta\big|[c,b],$$

$$\Delta_1(\Delta) = \{(t_i,[x_{i-1},x_i])\colon i\in\{1,2,\dots,j\}\}$$

and

$$\Delta_2(\Delta) = \{(t_i,[x_{i-1},x_i])\colon i\in\{j+1,j+2,\dots,n\}\}.$$

Then

$$(\delta_1(\delta),\Delta_1(\Delta))\in D([a,c]) \quad\text{and}\quad (\delta_2(\delta),\Delta_2(\Delta))\in D([c,b]).$$

Let

$$D_1([a,c]) = \{(\delta_1(\delta),\Delta_1(\Delta))\colon (\delta,\Delta)\in D_c([a,b])\}$$

and

$$D_2([c,b]) = \{(\delta_2(\delta),\Delta_2(\Delta))\colon (\delta,\Delta)\in D_c([a,b])\}.$$

Let $(\delta_0,\Delta_0)\in D([a,c])$ and let $(\delta_{00},\Delta_{00})\in D([c,b])$. Define: 1. $\Delta = \Delta_0\circ\Delta_{00}$;
2. $\delta'(t) = \delta_0(t)$ if $a\leqslant t<c$, $\delta''(t) = \delta_{00}(t)$ if $c<t\leqslant b$ and $\delta'(c) = \delta''(c) = \min\{\delta_0(c),\delta_{00}(c)\}$;
3. $\delta(t) = \delta'(t)$ if $t\in[a,c]$ and $\delta(t) = \delta''(t)$ if $t\in[c,b]$. Then it is clear that

$$(\delta,\Delta)\in D_c([a,b]),\delta' = \delta_1(\delta),\delta'' = \delta_2(\delta),\Delta_0 = \Delta_1(\Delta) \quad\text{and}\quad \Delta_{00} = \Delta_2(\Delta).$$

This shows that $D_1([a,c])$ is a cofinal subset of $(D([a,c],\geqslant)$ and $D_2([c,b])$ is a cofinal subset of $(D([c,b],\geqslant)$. Let $v = u\big|[a,c]$ and $w = u\big|[c,b]$. Put $h_1(\delta_1,\Delta_1) = S(v,\Delta_1)$

for all $(\delta_1, \Delta_1) \in D_1([a,c])$ and $h_2(\delta_2, \Delta_2) = S(w, \Delta_2)$ for all $(\delta_2, \Delta_2) \in D_2([c,b])$. Since u is Perron product integrable on $[a,c]$ and $[c,b]$, we have

$$\lim_{(\delta_1,\Delta_1)\in D_1([a,c])} h_1(\delta_1, \Delta_1) = (P)\prod_a^c \exp(u(t)\mathrm{d}t)$$

and

$$\lim_{(\delta_2,\Delta_2)\in D_2([c,b])} h_2(\delta_2, \Delta_2) = (P)\prod_c^b \exp(u(t)\mathrm{d}t).$$

But $k(\delta, \Delta) = h_2\Big(\delta_2(\delta), \Delta_2(\Delta)\big) \cdot h_1(\delta_1(\delta), \Delta_1(\Delta)\Big)$ for all $(\delta, \Delta) \in D_c([a,b])$. Then

$$\lim_{(\delta,\Delta)} k(\delta, \Delta) = (P)\prod_c^b \exp(u(t)\mathrm{d}t) \cdot (P)\prod_a^c \exp(u(t)\mathrm{d}t).$$

Now let $\varepsilon > 0$. Then there exists $(\delta_0, \Delta_0) \in D_c([a,b])$ such that

$$\varrho\Big(k(\delta, \Delta), (P)\prod_c^b \exp(u(t)\mathrm{d}t) \cdot (P)\prod_a^c \exp(u(t)\mathrm{d}t)\Big) < \varepsilon$$

whenever $(\delta, \Delta) \in D_c([a,b])$ and $(\delta, \Delta) \geqslant (\delta_0, \Delta_0)$. Define a gauge δ' on $[a,b]$ satisfying the following conditions: 1. $\delta'(t) \leqslant \delta_0(t)$ for all $t \in [a,b]$; 2. $t + \delta'(t) < c$ if $t < c$; 3. $t - \delta(t') > c$ if $t > c$. Let Δ be any δ'-fine subdivision of $[a,b]$. We may suppose that c is not a tag of Δ. Then we can write $\Delta = \{(t_i, [x_{i-1}, x_i]) \colon i \in \{1, 2, \dots, n\}\}$ where $x_{j-1} < c < x_j$ for some $j \in \{1, 2, \dots, n\}$. Since Δ is δ'-fine, the conditions 1 and 2 imply that $c = t_j$. Define

$$\begin{aligned}\Delta' = \{&(t_1, [x_0, x_1]), \dots, (t_{j-1}, [x_{j-2}, x_{j-1}]), (t_j, [x_{j-1}, t_j]),\\ &(t_j, [t_j, x_j]), (t_{j+1}, [x_j, x_{j+1}]), \dots, (t_n, [x_{n-1}, x_n])\}.\end{aligned}$$

Then

$$(\delta', \Delta') \in D_c([a,b]) \quad \text{and} \quad (\delta', \Delta') \geqslant (\delta_0, \Delta_0).$$

So

$$\varrho\Big(k(\delta', \Delta'), (P)\prod_c^b \exp(u(t)\mathrm{d}t) \cdot (P)\prod_a^c \exp(u(t)\mathrm{d}t)\Big) < \varepsilon.$$

But

$$k(\delta', \Delta') = \prod_{i=j+1}^{n} \exp((x_i - x_{i-1})u(t_i)) \cdot \exp((x_j - t_j)u(t_j)) \cdot \exp((t_j - x_{j-1})u(t_j))$$
$$\times \prod_{i=1}^{j-1} \exp((x_i - x_{i-1}))u(t_i)$$
$$= \prod_{i=1}^{n} \exp((x_i - x_{i-1}))u(t_i) = S(u, \Delta).$$

Then

$$\varrho\Big(S(u,\Delta), (P)\prod_{c}^{b} \exp(u(t)\mathrm{d}t) \cdot (P)\prod_{a}^{c} \exp(u(t)\mathrm{d}t)\Big) < \varepsilon.$$

By Lemma 3.1, u is Perron product integrable on $[a, b]$ and

$$(P)\prod_{a}^{b} \exp(u(t)\mathrm{d}t) = (P)\prod_{c}^{b} \exp(u(t)\mathrm{d}t) \cdot (P)\prod_{a}^{c} \exp(u(t)\mathrm{d}t).$$

□

We denote by $\|\cdot\|$ any norm on $L(G)$. Let $K \in \mathscr{I}(\mathbf{R})$, let $u\colon K \to L(G)$ and let $[a, b] \in \mathscr{I}(K)$. For each element $\Delta = \{(t_i, [x_{i-1}, x_i])\colon i \in \{1, 2, \ldots, n\}\}$ of $\sigma([a, b])$ we write $s(u, \Delta) = \sum_{i=1}^{n}(x_i - x_{i-1})u(t_i)$. We say that u is *Perron summation integrable* on $[a, b]$ if there exists an element $X \in L(G)$ with the following property:

(**) For every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that $\|X - s(u, \Delta)\| < \varepsilon$ whenever $\Delta \in \sigma([a, b], \delta)$.

It is easy to see that, if u is Perron summation integrable on $[a, b]$, then there exists a unique element $X \in L(G)$ satisfying the condition (**). This element is called the *Perron summation integral* of u over $[a, b]$ and it is denoted by $(P)\int_a^b u(t)\mathrm{d}t$.

For each $(\delta, \Delta) \in D([a, b])$ define $j(\delta, \Delta) = s(u, \Delta)$. Then j is a net in $(L(G), \|\cdot\|)$. A trivial modification of the argument used in the proof of Lemma 3.2, yields the following Lemma:

3.7 Lemma. *Let $u\colon K \to L(G)$ and let $[a, b] \in \mathscr{I}(K)$. Then u is Perron summation integrable on $[a, b]$ if and only if the net j converges. Moreover,*

$$\lim_{(\delta,\Delta)} j(\delta, \Delta) = (P)\int_a^b u(t)\mathrm{d}t.$$

3.8 Theorem. *Let $[a,b] \in \mathcal{I}(K)$ and let $u\colon K \to L(G)$ be a function such that $[u(s), u(t)] = 0$ for all s, $t \in [a,b]$. If u is Perron summation integrable on $[a,b]$, then u is Perron product integrable on $[a,b]$ and*

$$(P)\prod_a^b \exp(u(t)\mathrm{d}t) = \exp\Big((P)\int_a^b u(t)\mathrm{d}t\Big).$$

Proof. Let $(\delta, \Delta) \in D([a,b])$ be such that

$$\Delta = \{(t_i, [x_{i-1}, x_i])\colon i \in \{1,2,\dots,n\}\}.$$

Then

$$h(\delta,\Delta) = S(u,\Delta) = \prod_{i=1}^n \exp((x_i - x_{i-1})u(t_i)) = \exp\Big(\sum_{i=1}^n (x_i - x_{i-1})u(t_i)\Big)$$

because $[u(t_i), u(t_j)] = 0$ for all $i, j \in \{1,2,\dots,n\}$. So $h(\delta,\Delta) = \exp(j(\delta,\Delta))$. The continuity of exp and Lemmas 3.2 and 3.7 imply the result. □

Let $[a,b] \in \mathcal{I}(K)$. A *Riemann partition* of $[a,b]$ is a finite family $\pi = \{I_j\}_{j=1}^n$ of elements of $\mathcal{I}(K)$ such that $\bigcup_{j=1}^n I_j = [a,b]$ and $\mathring{I}_j \cap \mathring{I}_k = \emptyset$ if $j \neq k$. If $I \in \mathcal{I}(K)$ we denote by $|I|$ the length of I.

Let π be a Riemann partition of $[a,b]$. Then

a) The positive real number $\|\pi\| = \max\{|I|\colon I \in \pi\}$ is called the *mesh* of π.

b) A *choice function* for π is any function $c\colon \pi \to K$ such that $c(I) \in I$ for all $I \in \pi$.

Consider the set $D_R([a,b])$ of all pairs (π, c), where π is a Riemann partition of $[a,b]$ and c is a choice function for π. If (π, c) and (π', c') are two elements of $D_R([a,b])$, we say that (π', c') is *finer than* (π, c), and we write $(\pi', c') \geqslant (\pi, c)$, if $\|\pi'\| \leqslant \|\pi\|$. It is clear that $D_R([a,b], \geqslant)$ is a directed set.

Let $u\colon K \to L(G)$. For each $(\pi, c) \in D_R([a,b])$ with

$$\pi = \{[x_{i-1}, x_i] : a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}, \qquad n \in \mathbb{N} \setminus \{0\},$$

we write

$$S_R(u, (\pi, c)) = \prod_{i=1}^n \exp((x_i - x_{i-1})u(c([x_{i-1}, x_i]))).$$

Then the function $(\pi, c) \to S_R(u, (\pi, c))$ is a net in G. If this net converges, we say that u is *Riemann product integrable* on $[a,b]$ and the limit $\lim\limits_{(\pi,c)} S_R(u, (\pi, c))$ is called the *Riemann product integral* over $[a,b]$ and we denote it by $\prod_a^b \exp(u(t)\mathrm{d}t)$.

3.9 Theorem. *Let $[a,b] \in \mathscr{I}(K)$ and let $u\colon K \to L(G)$. If u is Riemann product integrable on $[a,b]$, then u is Perron product integrable on $[a,b]$ and*

$$(P)\prod_a^b \exp(u(t)\mathrm{d}t) = \prod_a^b \exp(u(t)\mathrm{d}t).$$

Proof. Write $g = \prod_a^b \exp(u(t)\mathrm{d}t)$. Let $\varepsilon > 0$. Then we can choose $(\pi_0, c_0) \in D_R([a,b])$ such that $\varrho(S_R(u,(\pi,c)),g) < \varepsilon$ whenever $(\pi,c) \in D_R([a,b])$ and $(\pi,c) \geqslant (\pi_0,c_0)$.

Let $\eta = \|\pi_0\|$. Let δ be a gauge on $[a,b]$ such that $\delta(t) = \frac{1}{2}\eta$ for all $t \in [a,b]$. Let $\Delta \in \sigma([a,b],\delta)$ be such that $\Delta = \{(t_i,[x_{i-1},x_i])\colon i \in \{1,2,\dots,n\}\}$, where $n \in \mathbb{N}\setminus\{0\}$. Put $c([x_{i-1},x_i]) = t_i$ for all $i \in \{1,2,\dots,n\}$. Then c is a choice function for the Riemann partition $\pi = \{[x_{i-1},x_i]\colon i \in \{1,2,\dots,n\}\}$ of $[a,b]$. Because Δ is δ-fine, we have $(\pi,c) \geqslant (\pi_0,c_0)$. So $\varrho(S_R(u,(\pi,c)),g) < \varepsilon$. Since $S_R(u,(\pi,c)) = S(u,\Delta)$, we get $\varrho(S(u,\Delta),g) < \varepsilon$. By Lemma 3.1, u is Perron product integrable on $[a,b]$ and $(P)\prod_a^b \exp(u(t)\mathrm{d}t) = g$. □

3.10 Corollary. *Let $K = [a,b]$ and let $u\colon K \to L(G)$. If u is bounded and continuous a.e. on K, then u is Perron product integrable on $[a,b]$.*

Proof. It is an immediate consequence of Theorem 3.9 and the Existence Theorem of [16, p. 326]. □

Let $[a,b] \in \mathscr{I}(K)$ and let $u\colon K \to L(G)$ be a Perron product integrable function on $[a,b]$. If $t_1, t_2 \in [a,b]$, we define

$$(P)\prod_{t_1}^{t_2} \exp(u(t)\mathrm{d}t) = e \quad \text{if} \quad t_1 = t_2$$

and

$$(P)\prod_{t_1}^{t_2} \exp(u(t)\mathrm{d}t) = \Big((P)\prod_{t_2}^{t_1} \exp(u(t)\mathrm{d}t)\Big)^{-1} \quad \text{if} \quad t_1 > t_2.$$

Then it is easy to verify that

$$(P)\prod_{t_1}^{t_3} \exp(u(t)\mathrm{d}t) = (P)\prod_{t_2}^{t_3} \exp(u(t)\mathrm{d}t)\cdot(P)\prod_{t_1}^{t_2} \exp(u(t)\mathrm{d}t)$$

for all $t_1, t_2, t_3 \in [a,b]$.

3.11 Lemma. *Let $K = [a,b]$, let $[c,d] \in \mathcal{I}(K)$ and let $u\colon K \to L(G)$ be a Perron product integrable function on $[a,b]$. Then*

a) $\lim\limits_{\substack{s\to c\\ c<s<d}} (P)\prod\limits_s^d \exp(u(t)\mathrm{d}t) = (P)\prod\limits_c^d \exp(u(t)\mathrm{d}t)$,

b) $\lim\limits_{\substack{s\to d\\ c<s<d}} (P)\prod\limits_c^s \exp(u(t)\mathrm{d}t) = (P)\prod\limits_c^d \exp(u(t)\mathrm{d}t)$.

Proof. a) Let $\varepsilon > 0$. By Theorem 3.4 u is Perron product integrable on $[c,d]$. Then, by Lemma 3.1, there exists a gauge δ on $[c,d]$ such that

$$\varrho\Big((P)\prod_c^d \exp(u(t)\mathrm{d}t), S(u,\Delta)\Big) < \frac{\varepsilon}{3}$$

whenever $\Delta \in \sigma([c,d],\delta)$. Let s be any real number such that $c < s < d$. Since u is Perron product integrable on $[s,d]$, there exists a gauge δ_s on $[s,d]$ such that $\delta_s \leqslant \delta|[s,d]$ and

$$\varrho\Big((P)\prod_s^d \exp(u(t)\mathrm{d}t), S(u,\Delta_s)\Big) < \frac{\varepsilon}{3}$$

whenever $\Delta_s \in \sigma([s,d],\delta_s)$. Let $\varphi(t) = \exp((t-c)\cdot u(c))$ for all $t \geqslant c$. Since φ is continuous at c, there exists $\eta > 0$ such that $\eta < \min\{d-c, \delta(c)\}$ and $c < t < c+\eta$ implies $\varrho(\varphi(t), e) < \varepsilon/3$. Let s be a real number such that $c < s < c+\eta$ and let Δ_s be a δ_s-fine subdivision of $[s,d]$. Let $\Delta = \{(c,[c,s])\} \circ \Delta_s$. Then Δ is a δ-fine subdivision of $[c,d]$. Since $S(u,\Delta) = S(u,\Delta_s)\cdot\exp((s-c)u(c))$, we get

$$\begin{aligned}
&\varrho\Big((P)\prod_c^d \exp(u(t)\mathrm{d}t), (P)\prod_s^d \exp(u(t)\mathrm{d}t)\Big)\\
&\quad\leqslant \varrho\Big((P)\prod_c^d \exp(u(t)\mathrm{d}t), S(u,\Delta)) + \varrho(S(u,\Delta_s)\cdot\exp((s-c)u(c)), S(u,\Delta_s)\Big)\\
&\qquad + \varrho(S(u,\Delta_s), (P)\prod_s^d \exp(u(t)\mathrm{d}t))\\
&\quad< \frac{\varepsilon}{3} + \varrho(\varphi(s), e) + \frac{\varepsilon}{3} < \varepsilon.
\end{aligned}$$

Hence

$$\lim_{\substack{s\to c\\ c<s<d}} (P)\prod_s^d \exp(u(t)\mathrm{d}t) = (P)\prod_c^d \exp(u(t)\mathrm{d}t).$$

b) A trivial modification of the argument used in a). □

3.12 Theorem. *Let $K = [a,b]$, let $u\colon K \to L(G)$ be a Perron product integrable function on $[a,b]$ and let*

$$\varphi(s) = (P)\prod_a^s \exp(u(t)\mathrm{d}t) \quad \textit{for all} \quad s \in K.$$

Then $\varphi\colon K \to G$ is uniformly continuous.

Proof. It suffices to show that φ is right-continuous on $K \setminus \{b\}$ and left-continuous on $K \setminus \{a\}$. Lemma 3.11 b) implies immediately the left-continuity of u on $K \setminus \{a\}$. Let $c \in K \setminus \{b\}$. By Lemma 3.11 a) we have

$$\lim_{\substack{s \to c \\ c<s<b}} (P)\prod_s^b \exp(u(t)\mathrm{d}t) = (P)\prod_c^b \exp(u(t)\mathrm{d}t).$$

Since

$$(P)\prod_a^s \exp(u(t)\mathrm{d}t) = (P)\prod_b^s \exp(u(t)\mathrm{d}t)\cdot(P)\prod_a^b \exp(u(t)\mathrm{d}t)$$

for all $s \in K$ and the function $(g_1, g_2) \to g_1^{-1}g_2$ from $G \times G$ to G is continuous, it follows that

$$\begin{aligned}\lim_{\substack{s \to c \\ c<s<b}} (P)\prod_a^s \exp(u(t)\mathrm{d}t) &= (P)\prod_b^c \exp(u(t)\mathrm{d}t)\cdot(P)\prod_a^b \exp(u(t)\mathrm{d}t)\\ &= (P)\prod_a^c \exp(u(t)\mathrm{d}t)\end{aligned}$$

and therefore $\lim_{s\downarrow c}\varphi(s) = \varphi(c)$. □

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