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SOME OBSERVATIONS ON LOCAL UNIFORM BOUNDEDNESS PRINCIPLES II

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The field of automatic continuity has for more than a quarter of a century been a rich one, generating results that are not only interesting, but sometimes deep and surprising, as well as having practical applications. Because of the fundamental result that a linear map between normed linear spaces is continuous if and only if it is bounded, many results that are described as being results concerning automatic continuity are proved by arriving at contradictions involving boundedness.

Boundedness and continuity are not inextricably intertwined. Especially when the operators considered are nonlinear, there is no a priori reason to suppose that the phenomena are related. Additionally, when one is considering applications of the theory to signal processors, for example, there are important practical interpretations of both continuity and boundedness.

Since the amplitude of a signal (its norm in the appropriate space) can be intuitively described as soft (small norm) or loud (large norm), the following descriptions of continuity and boundedness properties are quite natural. Continuity at 0, for instance, can be described as follows: as the input signal becomes softer (its amplitude converges to 0), so does the output signal. Boundedness, however, can be described by saying that the output signals generated by soft input signals are soft.

Of the two properties, boundedness and continuity, it would appear that boundedness is the more important from an applications standpoint. One is often interested in the maximum stress to which a system can be subjected, and this maximum stress is described in terms of the loudness of the signals the system will generate. Interest in this dichotomy is heightened by the importance of nonlinear phenomena in many disparate fields of science and engineering.

Several of the results presented in this paper deal with automatic boundedness specifically, rather than automatic continuity, and involve non-linear operators. The paper is divided into several sections. The first deals with non-linear operators between linear spaces which preserve some vestigial remnants of linear properties. The remaining sections prove some local uniform boundedness theorems for complete metric spaces under some weaker-than-usual hypotheses which involve both the functions and the hypothesis of pointwise-boundedness.

1. Nonlinear maps between linear spaces

A proof originally given by Pták ([9]) enabled the following extension of the Banach-Steinhaus Theorem to be proved ([10]).

Theorem 1. *Let X be a Banach space, and Y a normed space. Let $\{T_a : a \in A\}$ be a pointwise-bounded collection of linear maps of X into Y, and assume that for each $a \in A$, there is a closed subspace S_a of X such that $T_a|S_a$ is continuous. Then there is a finite subset $a_1, \ldots, a_N \in A$ such that $\{T_a : a \in A\}$ is uniformly bounded on $\bigcap_{k=1}^{N} S_{a_k}$.*

It has been known for some time that this result can be obtained as a consequence of the Gliding Hump Theorem in automatic continuity ([7]). Recently, Máté has shown that the Gliding Hump Theorem can actually be obtained from a σ-convex version of the above result ([6]), and that Theorem 1 can also be used to obtain automatic continuity results concerning causal operators ([5], [7]).

An example given in [11] shows that it is not possible to obtain a general nonlinear local uniform boundedness extension of Theorem 1, even in the case when $X = Y = \mathbb{R}$, the real line, and the maps in question are 0 on their respective σ-convex subsets. However, one way to strengthen Theorem 1 is by enlarging the class of functions to which the theorem applies.

A sublinear map T between normed spaces satisfies the following inequality

$$\left\| T\left(\sum_{k=1}^{n} x_k \right) \right\| \leqslant \sum_{k=1}^{n} \|Tx_k\|$$

and a homogeneous map T between normed spaces satisfies the following equality (where c is a scalar)

$$\|T(cx)\| = |c|\,\|Tx\|.$$

We shall show that Theorem 1 can be proved when the maps T satisfy growth rate conditions generalizing sublinearity and homogeneity.

The following condition will replace sublinearity.

Definition 1. Let $\mathbb{R}^+$ denote the non-negative reals, and let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous monotonic function such that $\lim_{u\to 0} g(u) = 0$. This function g will be fixed throughout this section. An operator T between normed spaces is said to be additively bounded if

$$\left\| T\left(\sum_{k=1}^{n} x_k\right)\right\| \leqslant \sum_{k=1}^{n} g(\|Tx_k\|).$$

It is multiplicatively bounded if

$$\left\| T\left(\sum_{k=1}^{n} x_k\right)\right\| \leqslant \prod_{k=1}^{n} g(\|Tx_k\|).$$

While additive boundedness is basically a generalization of sublinearity, multiplicative boundedness is a charactreristic of the growth rate of either a real-valued polynomial of a real variable or an exponential function.

The following condition will replace homogeneity.

Definition 2. Let h and H be monotonic non-negative real-valued functions on the real numbers $\mathbb{R}$. T is said to be nearly homogeneous if

$$h(c)\|Tx\| \leqslant \|T(cx)\| \leqslant H(c)\|Tx\|.$$

The functions h and H will be fixed throughout this section.

Near homogeneity is also a growth rate condition, and it is most naturally exhibited by monomials of the form $f(x) = x^r$, where $r \geqslant 1$.

We now prove Theorem 1 for maps with the above properties. The proof is an adaptation of Pták's original idea ([9]).

Theorem 2. *Let X be a Banach space, $\{Y_a : a \in A\}$ be normed spaces. Let $\{S_a : a \in A\}$ be a collection of closed, bounded, and σ-convex subsets of X. Suppose that for each $a \in A$, $T_a : x \to Y_a$ is an additively bounded, nearly homogeneous function which is continuous on S_a. Suppose further that*

$$M[x] = \sup\{\|T_a x\| : a \in A\} < \infty \quad \text{for each } x \in X.$$

Then there exist $a_1, \ldots, a_N \in A$ such that

$$\sup\left\{\|T_a x\| : a \in A,\ x \in \bigcap_{k=1}^{N} S_{a_k}\right\} < \infty.$$

Proof. Suppose first that $\{T_a: a \in A\}$ is unbounded on the bounded σ-convex set S, and that d, $M > 0$. Then there exist $a \in A$, $x \in S$ such that $||x|| < d$, $||T_a x|| > M$. We can assume without loss of generality that $d < 1$ and that there exists $D > 1$ such that $u \in S \to ||u|| < D$. By assumption, choose $y \in S$ and $a \in A$ such that $||T_a y|| > M/h(d/2D)$. Let $x = (d/2D)y$. Since S is σ-convex, $x \in S$. Also, $||T_a((d/2D)y)|| > h(d/2D)||T_a y|| > M$. This is, of course, a well-known property of linear maps.

Since $\lim_{u\to 0} g(u) = 0$, for each n choose σ_n such that $u < \sigma_n \Rightarrow g(u) < 1/2^n$. Assume that the theorem is false. Then there is an $x_1 \in X$ and $a_1 \in A$ with $||T_{a_1} x_1|| > g(1)$.

Assume now that $a_1, \ldots, a_n \in A$ and $x_1, \ldots, x_n \in X$ have been chosen. Since the theorem has been assumed false, and $T_{a_j}|S_{a_j}$ is continuous, for $1 \leqslant j \leqslant n$ choose $d_{n+1,j}$ such that $x \in S_{a_j}$, $||x|| < d_{n+1,j} \Rightarrow ||T_{a_j} x|| < Q_{n+1}$, where $Q_{n+1} = \sigma_{n+1}/H(-1/2^{n+1})$. Now choose $x_{n+1} \in \bigcap_{j=1}^{n} S_{a_j}$ and $a_{n+1} \in A$ such that $||x_{n+1}|| < d_{n+1,j}$ for $1 \leqslant j \leqslant n$, and $||T_{a_{n+1}} x_{n+1}|| > W_{n+1}/h(1/2^{n+1})$, where $W_{n+1} = g(n+1) + \sum_{j=1}^{n} g(H(-2^{-j})M[x_j]) + g(1)$.

Let $x_0 = \sum_{n=1}^{\infty} 2^{-n} x_n$. For any n, $\sum_{j=n+1}^{\infty} 2^{-j} x_j \in S_n$, since $j > n \Rightarrow x_j \in S_n$, and S_n is σ-convex. Let $\varepsilon > 0$; since $T_{a_n}|S_{a_n}$ is continuous and g is continuous, choose N such that

$$g\left(\left\|T_{a_n}\left(\sum_{k=n+1}^{\infty} -2^{-k} x_k\right)\right\|\right) < g\left(\left\|T_a\left(\sum_{k=n+1}^{N} -2^{-k} x_k\right)\right\|\right) + \varepsilon.$$

We see that

$$\left\|T_{a_n}\left(\sum_{k=n+1}^{N} -2^{-k} x_k\right)\right\| \leqslant \sum_{k=n+1}^{N} g(||T_{a_n}(-2^{-k} x_k)||) \leqslant \sum_{k=n+1}^{\infty} g(H(-2^{-k})||T_{a_n} x_k||).$$

But $||x_k|| < d_{n,k} \Rightarrow H(-2^{-k})||T_{a_n} x_k|| < \sigma_k \Rightarrow g(H(-2^{-k})||T_{a_n} x_k||) < 1/2^k$. Thus, for any $\varepsilon > 0$, $\left\|T_{a_n}\left(\sum_{k=n+1}^{\infty} -2^{-k} x_k\right)\right\| < \sum_{k=n+1}^{N} 2^{-k} + \varepsilon$. Consequently, $\left\|T_{a_n}\left(\sum_{k=n+1}^{\infty} -2^{-k} x_k\right)\right\| \leqslant 1$.

Note that $2^{-n} x_n = x_0 + \sum_{j=1}^{n-1} -2^{-j} x_j + \sum_{j=n+1}^{\infty} -2^{-j} x_j$. If we regard the expression on the right as the sum of $n+1$ terms (the infinite sum being regarded as one term), we have

$$\begin{aligned}
h(2^{-n})\|T_{a_n}x_n\| &\leqslant \|T_{a_n}(2^{-n}x_n)\| \\
&\leqslant g(\|T_{a_n}x_0\|) + \sum_{j=1}^{n-1} g(\|T_{a_n}(-2^{-j}x_j)\|) \\
&\quad + g\left(\left\|T_{a_n}\left(\sum_{j=n+1}^{\infty} -2^{-j}x_j\right)\right\|\right) \\
&\leqslant g(\|T_{a_n}x_0\|) + \sum_{j=1}^{n-1} g\big(H(-2^{-j})M[x_j]\big) + g(1) \\
&= g(\|T_{a_n}x_0\|) + W_n - g(n).
\end{aligned}$$

We therefore conclude that

$$W_n < h(2^{-n})\|T_{a_n}x_n\| \leqslant g(\|T_{a_n}x_0\|) + W_n - g(n).$$

Therefore, $g(\|T_{a_n}x_0\|) > g(n)$ for all n. Since g is monotonic, this implies that $\|T_{a_n}x_0\| > n$ for all n, and this contradiction establishes the theorem. □

In Theorem 2, letting the range space Y_a vary with the index may seem to be a purely technical modification, but it is precisely this extension which enables Máté ([6]) to obtain the Gliding Hump Theorem as a consequence of the σ-convex linear version of Theorem 2.

Corollary 2.1. *Theorem 2 holds if additive boundedness is replaced by multiplicative boundedness.*

P r o o f. The proof is almost identical to the proof of Theorem 2. The only difference is that the quantity W_{n+1} is defined using multiplication instead of addition. Instead of W_{n+1} being a sum of $n+2$ expressions, it is a product of the same $n+2$ expressions. All the estimates are the same, and the final inequality is obtained by division rather than subtraction. □

Perhaps the most widely known result on local uniform boundedness is Osgood's Theorem, which states that a pointwise-bounded family of continuous functions on a complete metric space must be locally uniformly bounded (it is shown in [11] that the correct phrasing of Osgood's Theorem leads to an equivalence between it and the Baire Category Theorem).

One way of extending this result is to widen the class of functions to which it applies. Consequently, one would look for a class of functions satisfying a condition weaker than continuity. However, this condition cannot be too much weaker. A set E is said to be almost open ([4], p. 211) if there exist an open set U and sets H and K of the first category such that $E = (U \setminus H) \cup K$. A function is almost continuous if the inverse image of an open set is almost open.

The following example shows that Osgood's Theorem fails to hold under the assumption that the functions are almost continuous. Let $X = [0, 1]$, and let $\{r_n : n = 1, 2, \ldots\}$ be an ordering of the rationals in X. For each integer n, define a function f_n by $f_n(x) = 0$ if $x \neq r_n$ and $f_n(r_n) = n$. It is easy to verify that each f_n is almost continuous, and also that $\{f_n : n = 1, 2, \ldots\}$ is pointwise-bounded. Since $f_n(r_n) = n$, and each open subset of X contains infinitely many rationals, $\{f_n : n = 1, 2, \ldots\}$ cannot be locally uniformly bounded. This example consists of functions that are continuous at all points but one, and at first glance it appears that the simplicity of this example would make it difficult to extend the class of functions for which local uniform boundedness theorems hold. Nonetheless, one can introduce a class of functions for which Osgood's Theorem, and variations, can be proved. If E is a set in a topological space, let $\mathrm{Int}(E)$ denote its interior.

The following definitions are taken from [8], in which the reader can obtain further references to feeble properties.

Definition 3. A map $T\colon X \to Y$ from one topological space into another is called feebly continuous (FC) if for every open subset F of Y containing points of the range, $\mathrm{Int}(f^{-1}(F)) \neq 0$.

Feeble continuity is a rather pathological property. We present two examples indicative of this pathology.

Let $T\colon [0, 1) \to [0, 1)$ be a map of the half-open unit interval onto itself. If $a < b$ and $c < d$, define

$$T_{abcd}x = c + (d - c)T\big((x - a)/(b - a)\big)$$

T_{abcd} maps $[a, b)$ onto $[c, d)$ in a manner which preserves almost any interesting pathology T might exhibit.

Now let $I_n = [(n-1)/n, n/(n+1))$; cleary, $[0,1)$ is the disjoint union of these intervals. Let $J_1 = [0,1)$, $J_2 = [0,1/2)$, $J_3 = [1/2,1)$, $J_4 = [0,1/3)$, $J_5 = [1/3,2/3)$, $J_6 = [2/3,1)$, and so on. Use the maps T_{abcd} to map I_n onto J_n for each integer n. It is clear that the mapping thus constructed is a feebly continuous map of $[0,1)$ onto itself, as any open subset of $[0,1)$ must contain some interval of the form J_n, and the inverse image of J_n clearly contains the interior of the interval I_n. Obviously, whatever pathology the maps T_{abcd} exhibit is preserved in this construction. For instance, if T is not measurable, the inverse image under the constructed map of each open set is non-measurable.

Now consider the real-valued functions f and g defined by

$$\begin{aligned} f(x) &= \tan x & -\pi/2 < x < \pi/2 &\quad \text{or} \\ & & \pi/2 < x < 3\pi/2, &\quad x \text{ rational} \\ &= \tan x + 1 & \pi/2 < x < 3\pi/2, &\quad x \text{ irrational} \\ g(x) &= -\tan x & -\pi/2 < x < 3\pi/2, &\quad x \neq \pi/2 \end{aligned}$$

If U is any open interval, $f^{-1}(U) \cap (-\pi/2, \pi/2)$ is open, so f is feebly continuous. Clearly, g is continuous. However, $f+g$ is 1 at irrational points in $(\pi/2, 3\pi/2)$, and is 0 otherwise. It is easy to see that the inverse image under $f+g$ of $(1/2, 3/2)$ consists of irrationals only, and so has no interior. This example demonstrates that the sum of a feebly continuous function and a continuous function need not be a feebly continuous.

It is not possible to prove a general local uniform boundedness principle for feebly continuous functions. Let $X = [0,1]$, and let $\{r_n : n = 1, 2, \ldots\}$ be an ordering of the rationals in X. Define

$$\begin{aligned} f_n(x) &= n^2 x & 0 \leqslant x \leqslant 1/n \\ &= 0 & x > 1/n,\ x \text{ irrational, or } x = r_k \text{ for } k < n \\ &= n & x = r_k \text{ for } k \geqslant n \end{aligned}$$

Each f_n is feebly continuous, since if $0 \leqslant a < b \leqslant n$, then $(a/n^2, b/n^2) \subset f_n^{-1}((a,b))$. $\{f_n : n = 1, 2, \ldots\}$ is pointwise-bounded, since for each x, $f_n(x) = 0$ for all but finitely many n. However, these functions are not uniformly bounded on any open U. Choose N such that $U \backslash [0, 1/N]$ is non-empty; then it contains infinitely many rationals, which we shall denote $\{r_{n_k} : k = 1, 2, \ldots\}$. If $n_p > N$, $f_{n_p}(r_{n_p}) = n_p$.

It is possible to prove a local uniform boundedness principle for real-valued feebly continuous functions under some highly restrictive conditions. The following definition is motivated by a basic definition, which can be found in [1], from the theory of function algebras.

Definition 4. Let A be a set of real or complex-valued functions on a topological space X, and let $K > 0$. A point $p \in X$ is an independent point of height K for A

if every closed subset F of X not containing p and every $\varepsilon > 0$, there exists $f \in A$ such that $|f(p)| > K$ and $|f(x)| < \varepsilon$ for $x \in F$.

We first present a combinatorial-type lemma which will be needed in the next theorem. X denotes a topological space.

Lemma 1. *Let $\{E_n : n = 1, 2, \ldots\}$ be a sequence of subsets of X with the following property: if $n > k$, then either $\overline{E}_n$ is a subset of E_k or is disjoint from $\overline{E}_k$. Then there exists either a subsequence $\{E_{n_j} : j = 1, 2, \ldots\}$ whose closures are pairwise disjoint, or a subsequence $\{E_{n_j} : j = 1, 2, \ldots\}$ such that $\overline{E}_{n_{j+1}} \subset E_{n_j}$ for all j.*

Proof. Let $S = \{n :$ there are infinitely many $\overline{E}_k$ which are subsets of $E_n\}$. Suppose first that there is an integer N such that, if $n \geqslant N$, then $n \in S$. Let $n_1 = N$, and choose $n_2 > n_1$ such that $\overline{E}_{n_2} \subset E_{n_1}$. Having chosen $n_1 < \ldots < n_p$, observe that $n_p \in S$, and choose $n_{p+1} > n_p$ such that $\overline{E}_{n_{p+1}} \subset E_{n_p}$. The subsequence $\{E_{n_p} : p = 1, 2, \ldots\}$ satisfies $\overline{E}_{n_{p+1}} \subset E_{n_p}$.

The other possibility is that there exists a subsequence of integers $N_1 < N_2 < \ldots$ which do not belong to S. Since each of the integers N_p does not belong to S, for each integer p there is an integer k_p such that, if $k > k_p$, then $\overline{E}_k$ is disjoint from $\overline{E}_{N_p}$. Let $j_1 = 1$, $q_1 = N_{j_1}$. Having chosen $j_1 < \ldots < j_p$, choose $j_{p+1} > j_p$ such that $N_{j_{p+1}} > \max(k_{j_1}, \ldots, k_{j_p})$. Let $q_p = N_{j_p}$; $\{\overline{E}_{q_p} : p = 1, 2, \ldots\}$ is pairwise-disjoint. □

The following local uniform boundedness principle is reminiscent of theorems in function algebras.

Theorem 3. *Let X be either a complete metric space or a locally compact Hausdorff space, and let A be a collection of feebly continuous bounded complex-valued functions on X. Suppose that A is pointwise-bounded, σ-convex, and closed under multiplication. Assume further that there exists a $K > 0$ such that the set of independent points for A of height K is dense. Then A is locally uniformly bounded.*

Proof. We prove it when X is a complete metric space, the locally compact Hausdorff case, as usual, being similar. Let $\|f\|$ denote the uniform norm of f.

Assume the theorem is false. Then we can choose $g_1 \in A$ and $x_1 \in X$ such that $|g_1(x_1)| > 2$. Since g_1 is FC, the set $g_1^{-1}((2, \infty))$ has interior, which we can assume contains an open set E_1 of diameter < 1. Since the set of independent points of height K is dense, choose one such point $q_1 \in E_1$, and choose $f_1 \in A$ such that $|f_1(q_1)| > K$ and $|f_1(x)| < 1/\|g_1\|$ for $x \in X \setminus E_1$.

Suppose that functions $g_1, \ldots, g_n$ and $f_1, \ldots, f_n \in A$ have been chosen, as well as open sets $E_1, \ldots, E_n$ such that $\operatorname{diam}(E_k) < 1/k$, and $j < k \leqslant n \Rightarrow$ either $\overline{E}_k \subset E_j$, or $\overline{E}_k$ and $\overline{E}_j$ are disjoint. Since the theorem has been assumed false, we can find $x_{n+1} \in X$ and $g_{n+1} \in A$ such that $|g_{n+1}(x_{n+1})| > M_{n+1} = (2^{n+1}/K)(n + 2 + \sum_{j=1}^{n} 2^{-j}||f_j||\,||g_j||)$. Since g_{n+1} is FC, the set $g_{n+1}^{-1}((M_{n+1}, \infty)) = U_{n+1}$ has interior. If $\mathrm{Int}(U_{n+1}) \cap (\bigcup_{j=1}^{n} E_j) = \emptyset$, let E_{n+1} be an open subset of U_{n+1} of diameter less than $1/(n+1)$ and disjoint from $\bigcup_{j=1}^{n} \overline{E}_j$. If that intersection is non-empty, let p be the largest index such that $\mathrm{Int}(U_{n+1}) \cap E_p$ is non-empty, and let E_{n+1} be an open set of diameter less than $1/(n+1)$ such that $\overline{E}_{n+1} \subset \mathrm{Int}(U_{n+1}) \cap E_p$. Let $J_{n+1} = \{j : j < n+1, \overline{E}_j \cap \overline{E}_{n+1} = \emptyset\}$. Let $q_{n+1} \in E_{n+1}$ be an independent point of height K, and choose $f_{n+1} \in A$ such that $|f_{n+1}(q_{n+1})| > K$, and $x \in \bigcup_{j \in J_{n+1}} \overline{E}_j \Rightarrow |f_{n+1}(x)| < 1/||g_{n+1}||$. This completes the induction.

By Lemma 1, we can find a subsequence $\{n_j : j = 1, 2, \ldots\}$ such that either $\overline{E}_{n_{j+1}} \subset E_{n_j}$, or $\{\overline{E}_{n_j} : j = 1, 2, \ldots\}$ are pairwise disjoint. We can obtain a contradiction to the pointwise-boundedness of A in the former case in the usual manner, by examining the values $\{g_{n_j}(x_0) : j = 1, 2, \ldots\}$ at a point $x_0 \in \bigcap_{j+1}^{\infty} \overline{E}_{n_j}$.

In the latter case, since A is closed under multiplication and σ-convex, let $f = \sum_{j=1}^{\infty} 2^{-n_j} f_{n_j} g_{n_j}$. Then, for any p,

$$
\begin{aligned}
|f(q_{n_p})| &> 2^{-n_p}|f_{n_p}(q_{n_p})|\,|g_{n_p}(q_{n_p})| - \sum_{j=1}^{p-1} 2^{-n_j}|f_{n_j}(q_{n_p})|\,|g_{n_j}(q_{n_p})| \\
&\quad - \sum_{j=p+1}^{\infty} 2^{-n_j}|f_{n_j}(q_{n_p})|\,|g_{n_j}(q_{n_p})| \\
&> 2^{-n_p}|f_{n_p}(q_{n_p})|\,|g_{n_p}(q_{n_p})| - \sum_{j=1}^{p-1} 2^{-n_j}||f_{n_j}||\,||g_{n_j}|| \\
&\quad - \sum_{j=p+1}^{\infty} 2^{-n_j}|f_{n_j}(q_{n_p})|\,||g_{n_j}|| \\
&> 2^{-n_p}|f_{n_p}(q_{n_p})|\,|g_{n_p}(q_{n_p})| - \sum_{j=1}^{p-1} 2^{-n_j}||f_{n_j}||\,||g_{n_j}|| - 1 > n_p
\end{aligned}
$$

Therefore, f is not bounded, and this contradiction completes the proof. □

Althout the class FC does not yield general local uniform boundedness theorems, there is a subclass which will. Examples have already been given of feebly continuous functions whose restrictions to open sets are not feebly continuous. The following definition eliminates this pathology.

Definition 5. $f\colon X \to Y$ is hereditarily feebly continuous (HFC) if for each open subset E of X, $f|E\colon E \to Y$ is feebly continuous.

The following lemma provides more useful characterizations of hereditarily feebly continuous functions.

Lemma 2. *Let $T\colon X \to Y$. The following are equivalent.*

(1) *T is HFC.*

(2) *If U is an open subset of X, V is an open subset of Y, and $x \in U$ with $Tx \in V$, then there is an open subset W of U such that $T(W) \subset V$.*

(3) *For every open subset V of Y, there is an open subset U of X such that*

$$U \subset T^{-1}(V) \subset \bar{U}.$$

Proof. (1) $\Rightarrow$ (2): Assume that U and V are open, $x \in U$, and $Tx \in V$. Since T is HFC, $T|U$ is FC, and since $T^{-1}(V) \cap U$ is non-empty, it has non-empty interior W. Then $W \subset U$, and $T(W) \subset V$.

(2) $\Rightarrow$ (3): Let V be a non-empty open subset of Y which contains points of the range. Then $E = \operatorname{Int}(T^{-1}(V)) \neq \emptyset$. If $x \in T^{-1}(V)$, then $Tx \in V$, so if W is a neighborhood of x, there is a non-empty open subset Q of W such that $T(Q) \subset V$. Therefore $Q \subset E$, and so $Q \subset E \cap W$. Therefore, $x \in \bar{E}$.

(3) $\Rightarrow$ (1): Let U be an open subset of X. Assume that V is an open subset of Y containing points of $T(U)$. Choose an open set E such that $E \subset T^{-1}(V) \subset \bar{E}$. If $x \in U \cap T^{-1}(V)$, then $x \in \bar{E}$. Therefore, every neighborhood of x has a non-empty intersection with E, and so $W = U \cap E$ is non-empty. Therefore, $W \subset \operatorname{Int}((T|U)^{-1}(V))$, and so T is HFC. □

Although HFC may not seem to be a significantly weaker condition than continuity, there are some interesting discontinuous functions which are HFC. If $\mathbb{R}$ denotes the real numbers, piecewise-continuous functions from $\mathbb{R}$ into $\mathbb{R}$ are HFC. Define $f(x) = \sin 1/x$ and $g(x) = (1/x)\sin 1/x$ for non-zero x. To define $f, g\colon \mathbb{R} \to \mathbb{R}$, we must define $f(0)$ and $g(0)$. If $f(0)$ lies between -1 and 1, and $g(0)$ has any real value, both f and g are HFC (to see this, use (3) of Lemma 2).

Most uniform boundedness theorems that are proved for complete metric spaces make use of the fact that a decreasing sequence of closed sets whose diameters converge to 0 has a non-empty intersection. As we shall see, hereditary feeble continuity is sufficiently robust to enable similar arguments to be constructed.

3. Weakening the hypothesis of pointwise-boundedness

One hypothesis of uniform boundedness theorems which has not been subjected to close scrutiny is the hypothesis of pointwise-boundedness. Suppose that the maps under consideration are regarded as a collection of signal processors. The assumption that the maps are pointwise-bounded has a natural interpretation: the responses induced from the collection to any given input signal form a bounded set. One way to weaken this hypothesis is to assume that the collection is a union of sub-collections. One can then examine the responses of any sub-collection to a particular input signal. For a given input signal, the responses of some (but not all) of the sub-collections may form a bounded set. This situation can be termed partial pointwise-boundedness.

The results of this section concern circumstances for which partial pointwise-boundedness implies some sort of local uniform boundedness, and the exact form of boundedness that is exhibited. In the following theorem, X will be a complete metric space, Y a topological space, and $\{Y_n : n = 1, 2, \ldots\}$ an increasing closed cover of Y. $\{T_a : a \in A\}$ will denote a collection of hereditarily feebly continuous maps of X into Y. S denotes a set such that $s \in S \Rightarrow A_s \subset A$. If $x \in X$ and $a \in A$, define $m[x, s]$ to be the least integer n for which $T_a x \in Y_n$ for all $a \in A_s$ ($m[x, s] = +\infty$ if such an integer does not exist).

Just as the hypotheses of Theorem 1 give rise to a "good finite intersection", the same can be said of an appropriate assumption of partial pointwise-boundedness.

Theorem 4. *Suppose that, for each $x \in X$, $m[x, s]$ is finite for all but finitely many $s \in S$. Then there exist $s_1, \ldots, s_p$ in S, a non-empty open subset U of X, and an integer N such that $T_a x \in Y_N$ for $x \in U$ and $a \in \bigcap_{k=1}^{p} A_{s_k}$.*

Proof. The theorem is only of interest in the case where S is infinite, so assume that $\{s_n : n = 1, 2, \ldots\}$ are distinct elements of S. Assume that the theorem is false. Then there exist $x_1 \in X$ and $a_1 \in A_{s_1}$ such that $T_{a_1} x_1 \notin Y_1$. Since Y_1 is closed, choose an open set W_1 such that $T_{a_1} x_1 \in W_1$ and W_1 is disjoint from Y_1. By Lemma 2, there exists an open set $U_1 \subset X$ such that $T_{a_1}(U_1) \subset W_1$; we can assume that $\operatorname{diam}(U_1) < 1$. Now choose an open set V_1 whose closure is contained in U_1. At the nth step of the induction, asusme we have found open sets $V_n \subset \overline{V}_n \subset U_n \subset V_{n-1} \subset \ldots \subset V_1 \subset \overline{V}_1 \subset U_1$. Since the theorem has been assumed false, we can find $x_{n+1} \in V_n$ and $a_{n+1} \in \bigcap_{k=1}^{n+1} A_{s_k}$ such that $T_{a_{n+1}} x_{n+1} \notin Y_{n+1}$. Since Y_{n+1} is closed, there is an open set W_{n+1} such that $T_{a_{n+1}} x_{n+1} \in W_{n+1}$ and W_{n+1} is disjoint from Y_{n+1}. By Lemma 2, there is an open subset U_{n+1} of V_n, which we can assume has diameter less than $1/(n+1)$ such that $T_{a_{n+1}}(U_{n+1}) \subset W_{n+1}$. To complete the induction, choose an open V_{n+1} whose closure is contained in U_{n+1}.

Having constructed a decreasing sequence of closed sets whose diameters converge to 0, let $x_0 \in \bigcap_{n=1}^{\infty} \overline{V}_n$. For any integer n, if $k > n$, then $a_k \in A_{s_n}$. Note that $T_{a_k} x_0 \in T_{a_k}(U_k) \subset W_k$. Since W_k is disjoint from Y_k, $m[x_0, s_n] > k$ for $k > n$. Therefore $m[x_0, s_n]$ is infinite for all n, and this contradiction yields the desired result. □

The situation is slightly simpler if we start with a contable number of sub-collections.

Corollary 4.1. *Suppose that S is countable. If we assume that, for each $x \in X$, there is an $s \in S$ such that $m[s, x]$ is finite, then the conclusion of Theorem 4 holds.*

Proof. Let $S = \{s_n : n = 1, 2, \ldots\}$, and let $B_n = \bigcap_{k=1}^{n} A_{s_k}$. The hypotheses of Theorem 4 hold for the sub-collections $\{B_n : n = 1, 2, \ldots\}$, and the result follows since $\bigcap_{k=1}^{n} B_k = \bigcap_{k=1}^{n} A_{s_k}$. □

It might be wondered if it is possible to strengthen the conclusion of Theorem 4. In [11], it is shown that different classes of uniform boundedness theorems can be proved when the range space is a metric space. We shall now assume that Y is a metric space, and that the maps $\{T_a : a \in A\}$ are continuous. Fix $y_0 \in Y$, and let $m[s, x] = \sup\{d(T_a x, y_0) : a \in A_s\}$.

Theorem 5. *Suppose that, in addition to the above hypotheses, for each $x \in X$, there is a finite subset $S(x)$ of S such that $m[s, x]$ is finite for $s \in S \setminus S(x)$. Then there exists a finite subset S^* of S such that, for each $s \in S \setminus S^*$, there exists a non-empty open subset U_s such that*

$$\sup\{d(T_a x, y_0) : x \in U_s,\ a \in A_s\} < \infty.$$

Proof. We shall let d denote the metric in both X and Y. Assume that the theorem is false. Then there exist infinitely many distinct $\{s_n : n = 1, 2, \ldots\}$ such that for each n, the family $\{T_a : a \in A_{s_n}\}$ is unbounded on each non-empty open set. Let $\{t_n : n = 1, 2, \ldots\} = \{s_1 ; s_1, s_2 ; s_1, s_2, s_3 ; \ldots\}$. The key feature of $\{t_n : n = 1, 2, \ldots\}$ is that each s_k appears infinitely often therein.

Choose $x_1 \in X$, $a_1 \in A_{t_1}$ such that $d(T_{a_1} x_1, y_0) > 2$. Since T_{a_1} is continuous at x_1, choose neighborhoods U_1 and V_1 of x_1 such that $x_1 \in V_1 \subset \overline{V}_1 \subset U_1$ and $\operatorname{diam}(U_1) < 1$.

Suppose $x_1, \ldots, x_p \in X$, $a_1 \in A_{t_1}, \ldots, a_p \in A_{t_p}$ and open sets $V_n \subset \overline{V}_n \subset U_n \subset V_{n-1} \subset \ldots \subset V_1 \subset \overline{V}_1 \subset U_1$ have been chosen with $x_1 \in V_1, \ldots, x_p \in V_p$.

Because the maps $T_{a_1}, \ldots, T_{a_p}$ are continuous at x_p, choose a neighborhood U_{p+1} of x_p of diameter $< 1/(p+1)$, $U_{p+1} \subset V_p$, and $x \in U_{p+1} \Rightarrow d(T_{a_j}x, T_{a_j}x_p) < 1/2^{p+1}$ for $1 \leqslant j \leqslant p$. Now choose a point $x_{p+1} \in U_{p+1}$ and $a_{p+1} \in A_{t_{p+1}}$ such that $d(T_{a_{p+1}}x_{p+1}, y_0) > p+2$. Note that $d(T_{a_j}x_{p+1}, T_{a_j}x_p) < 1/2^{p+1}$ for $1 \leqslant j \leqslant p$. Let V_{p+1} be an open neighborhood of x_{p+1} with $V_{p+1} \subset \overline{V}_{p+1} \subset U_{p+1}$.

As in Theorem 4, choose $x_0 \in \bigcap_{k=1}^{\infty} \overline{V}_k$. If $n > p$, we have $d(T_{a_p}x_p, y_0) \leqslant d(T_{a_p}x_{p+1}, y_0) + d(T_{a_p}x_{p+1}, T_{a_p}x_p) \leqslant \ldots \leqslant d(T_{a_p}x_n, y_0) + \sum_{k=p}^{n-1} d(T_{a_p}x_{k+1}, T_{a_p}x_k) < d(T_{a_p}x_n, y_0) + \sum_{k=p}^{n-1} 1/2^{k+1} < d(T_{a_p}x_n, y_0) + 1$.

Therefore $n > p \Rightarrow d(T_{a_p}x_n, y_0) > d(T_{a_p}x_p, y_0) - 1 > p$. Since T_a is continuous and $x_n \to x_0$, for each p we have $d(T_{a_p}x_0, y_0) \geqslant p$. But $a_p \in A_{t_p}$, and so infinitely many $a_p \in A_{s_n}$ for each n. Therefore, $m[s_n, x_0]$ is infinite for all n, a contradiction which proves the theorem. □

This result is best possible, in that the dependence of the open set $U(s)$ on the choice of $s \in S \setminus S^*$ cannot be eliminated. Let X denote the interval [0,1], and let $\{r_n : n = 1, 2, \ldots\}$ be an ordering of the rationals in (0,1). Let A be the set of pairs of positive integers. If n, k are positive integers, let f_{nk} be a continuous function which is 0 on $[0, r_n] \cup [r_n + (1-r_n)/(k+1), 1]$ and peaks up to k on $\big(r_n, r_n + (1-r_n)/(k+1)\big)$. For each integer n, let A_n index the functions $\{f_{nk} : k = 1, 2, \ldots\}$. The family of functions $\{f_a : a \in A_n\}$ is pointwise-bounded for each x in the interval [0,1], but fails to be uniformly bounded in any neighborhood of r_n. Since each open subset of [0,1] contains infinitely many rationals, for any open subset U of [0,1] the family $\{f_a : a \in A_n\}$ will fail to be uniformly bounded on U for infinitely many n.

In the statement of Theorem 5, each of the maps T_a takes values in Y. An examination of the proof shows that it is possible for each map T_a to take values in a separate metric space Y_a. As we noted previously, this modification may be of more than technical interest.

4. Doubly-indexed boundedness and category theorems

The following definition describes a type of cover which plays an interesting role in the construction of Baire Category Theorems and local uniform boundedness principles.

Definition 6. Let X be a topological space. An infinite collection of subsets of X is called a thorough cover of X if each point of X belongs to all but finitely many members of the cover.

Fogelgren and McCoy ([3]) initially showed that complete metric space possess the following property: given any thorough closed cover of the space, there exists a non-empty open set contained in all but finitely many members of the cover. In an attractive note, Fletcher and Lindgren ([2]) have shown that this property holds in any second category space, and is equivalent to one of the well-known formulations of the Baire Category Theorem, that the space cannot be a countable union of closed, nowhere dense sets.

In this section, we prove a local uniform boundedness theorem using a doubly-indexed collection of sets which has as a corollary both the standard and Fogelgren-McCoy formulations of the Baire Category Theorem.

Theorem 6. *Let X be a complete metric space, Y a topological space, and let A and S be sets. For each $a \in A$, let T_a be a continuous map of X into Y. For each $s \in S$ and each positive integer n, let $Y_{s,n}$ be a closed subset of Y. Suppose that, for each $x \in X$, there exists a positive integer $n = n(x)$ and a finite subset $S(x)$ of S such that $T_a x \in Y_{s,n}$ for $s \in S \setminus S(x)$ and $a \in A$. Then there exist an integer N, a non-empty open subset U of X, and a finite subset S^* of S such that, if $x \in U$, $a \in A$, and $s \in S \setminus S^*$, then $T_a x \in \bigcup_{n=1}^{N} Y_{s,n}$.*

P r o o f. Suppose the theorem is false. Then there exist elements $a_1 \in A$ and $s_1 \in S$, and a point $x_1 \in X$, such that $T_{a_1} x_1 \notin Y_{1,s_1}$. Since T_{a_1} is continuous, choose an open set U_1 of diameter < 1 such that $x \in U_1 \Rightarrow T_{a_1} x \notin Y_{1,s_1}$. Now choose an open set V_1 such that $\overline{V}_1 \subset U_1$.

Assume that $s_1, \ldots, s_n \in S$, elements $a_1, \ldots, a_n \in A$, and open $V_n \subset \overline{V}_n \subset U_n \subset V_{n-1} \subset \ldots \subset V_1 \subset \overline{V}_1 \subset U_1$ have been chosen. Since the theorem has been assumed false, there exist $s_{n+1} \in S \setminus \{s_1, \ldots, s_n\}$, $a_{n+1} \in A$, and a point $x_{n+1} \in V_n$ such that $T_{a_{n+1}} x_{n+1} \notin \bigcup_{k=1}^{n+1} Y_{k,s_{n+1}}$. Since this set is closed and $T_{a_{n+1}}$ is continuous, choose an open set U_{n+1} with diameter $< 1/(n+1)$ such that $x \in U_{n+1} \Rightarrow T_{a_{n+1}} x \notin \bigcup_{k=1}^{n+1} Y_{k,s_{n+1}}$. We can assume that $x_{n+1} \in U_{n+1} \subset V_n$. Now choose an open set V_{n+1} such that $\overline{V}_{n+1} \subset U_{n+1}$.

Let $x_0 \in \bigcap_{n=1}^{\infty} \overline{V}_n$. Note that, if $n \geqslant k$, $T_{a_n} x_0 \notin Y_{k,s_n}$. Fix an integer p, and let S^* be a finite subset of S. Choose an integer n such that $n \geqslant p$ and $s_n \notin S^*$. Then $T_{a_n} x_0 \notin Y_{p,s_n}$. Consequently, there exists no integer p for which there exists a finite subset S^* of S such that $T_a x_0 \in Y_{p,s}$ for all $a \in A$ and $s \in S \setminus S^*$, a contradiction which establishes the theorem. □

A consequence of this theorem is the following corollary, which gives both the standard and Fogelgren-McCoy formulation of the Baire Category Theorem for complete metric spaces.

Corollary 6.1. *Let X be a complete metric space, and S a set. For each positive integer n and each $s \in S$, let X_{ns} be a closed subset of S. Suppose that for each $x \in X$, there exists an integer $n = n(x)$ and a finite subset $S(x)$ of S such that $x \in X_{ns}$ for each $s \in S \setminus S(x)$. Then there exists an integer N, a finite subset S^* of S, and a non-empty open subset U of X such that, if $s \in S \setminus S^*$, then $U \subset \bigcup_{n=1}^{N} X_{ns}$.*

Proof. Let $Y = X$, $Y_{ns} = X_{ns}$, and let $\{T_a : a \in A\}$ consist of the identity map $i: X \to Y$, and apply Theorem 6. □

Corollary 6.1 implies both versions of the Baire Category Theorem. If a complete metric space X is the union of a countable collection $\{F_n : n = 1, 2, \ldots\}$ of closed sets, for each pair of positive integers n and k, let $X_{nk} = F_n$. If $\{F_s : s \in S\}$ is a collection of closed sets such that each point of x belongs to all but finitely many members of the collection, for each positive integer n and each $s \in S$, let $X_{ns} = F_s$. In each instance, application of Corollary 6.1 yields the desired result.

5. Collective properties and local uniform boundedness

Consider the following statement of Osgood's Theorem: a pointwise-bounded family of continuous functions on a complete metric space is locally uniformly bounded. The hypotheses placed on the functions themselves are of two types. The requirement of continuity constrains each function separately; pointwise-boundedness, however, is a restriction on the entire collection, and local uniform boundedness is a collective property.

It is possible to define collective extensions of the properties of continuity and feeble continuity in such a way that Osgood's Theorem holds for collections of functions satisfying these properties.

Definition 7. Let X and Y be topological spaces, $\{T_a : a \in A\}$ a family of maps from X to Y. The family is said to be collectively continuous if, given open subsets U of X and V of Y, a point $x \in U$, and an index $a \in A$ such that $T_a x \in V$, there exists an index $b \in A$ and an open set $W \subset U$ such that $x \in W$ and $T_b(W) \subset V$. The family is collectively hereditarily feebly continuous if, with the above hypotheses, there exists an index $b \in A$ and an open set $W \subset U$ such that $T_b(W) \subset V$.

Note that a family of (hereditarily feebly) continuous functions is collectively (hereditarily feebly) continuous; in this case, the index b is the same as the given index a.

We present a simple version of Osgood's Theorem for collectively hereditarily feebly continuous families. It is not clear whether this theorem is anything more than a curiosity.

Theorem 7. *Let X be a complete metric space, Y a topological space, $\{Y_n : n = 1, 2, \ldots\}$ an increasing closed cover of Y. Let $\{T_a : a \in A\}$ be a collectively hereditarily feebly continuous family of maps from X to Y. Assume that, for each $x \in X$, there is an integer $n = n(x)$ such that $a \in A \Rightarrow T_a x \in Y_n$. Then $\{T_a : a \in A\}$ is locally unifomly bounded.*

Proof. The proof is familiar and straightforward. Suppose the theorem is false. Choose $x_1 \in X$ and $a_1 \in A$ such that $T_{a_1} x_1 \notin Y_1$. Let W_1 be an open set disjoint from Y_1 containing $T_{a_1} x_1$. Choose an open set U_1 of diameter less than 1 and an index $b_1 \in A$ such that $T_{b_1}(U_1) \subset W_1$. Now choose an open set V_1 such that $V_1 \subset \overline{V}_1 \subset U_1$.

Assume inductively that $b_1, \ldots, b_n \in A$, and open sets $V_n \subset \overline{V}_n \subset U_n \subset V_{n-1} \subset \ldots \subset V_1 \subset \overline{V}_1 \subset U_1$ have been chosen. Since the theorem has been assumed false, we can find an $x_{n+1} \in V_n$ and an index $a_{n+1} \in A$ such that $T_{a_{n+1}} x_{n+1} \notin Y_{n+1}$. Let W_{n+1} be an open set disjoint from Y_{n+1} containing $T_{a_{n+1}} x_{n+1}$. Choose an index $b_{n+1} \in A$ and an open subset U_{n+1} of V_n which has diameter less than $1/(n+1)$, such that $T_{b_{n+1}}(U_{n+1}) \subset W_{n+1}$. Now choose an open set V_{n+1} such that $V_{n+1} \subset \overline{V}_{n+1} \subset U_{n+1}$.

To conclude the proof, let $x_0 \in \bigcap_{n=1}^{\infty} \overline{V}_n$. Then, for each n, $T_{b_n} x_0 \in T_{b_n}(\overline{V}_n) \subset W_n$. Since W_n is disjoint from Y_n and the $\{Y_n : n = 1, 2, \ldots\}$ cover Y, this is a contradiction. □

References

[1] *P. G. Dixon*: Approximate indentities in Banach Algebras, Math. Proc. of the Cambridge Phil. Soc. *107* (1990), 557–571.

[2] *P. Fletcher and W. F. Lindgren*: A note on spaces of second category, Arch. Math. *24* (1973), 186–187.

[3] *J. R. Folgelgren and R. A. McCoy*: Some topological properties defined by homeomorphism groups, Arch. Math. *22* (1971), 528–533.

[4] *J. Kelley*: General topology, Van Nostrand, Princeton, 1955.

[5] *L. Máté*: On the continuity of Causal Operators and the Pták-Stein Theorem, Periodica Math. Hung. *20* (1989), 219–230.

[6] *L. Máté*: Remarks on the Pták-Stein Theorem, Periodica Math. Hung. *23* (1991), 59–63.

[7] *M. Neumann and V. Pták*: Automatic continuity, local type and causality, Studia Math. *82* (1985), 61–90.

[8] *D. Noll*: The preservation of Baire Category under preimages, Proc. Amer. Math. Soc. *107* (1989), 847–854.
[9] *V. Pták*: A uniform boundedness theorem and mappings into spaces of linear operators, Studia Math. *31* (1968), 425–431.
[10] *J. Stein*: Several theorems on boundedness and equicontinuity, Proc. Amer. Math. Soc. *26* (1971), 415–419.
[11] *J. Stein*: Some observations on local uniform boundedness principles, Czech. Math. J. *41* (1991), 64–74.

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