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CONVERGENCES $\mathfrak{L}_S^H$ FOR THE GROUP OF REAL NUMBERS

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For each subgroup H of the group R of real numbers and each subset S of the quotient group R/H a convergence $\mathfrak{L}_S^H$ for the group R is constructed. The relation of the system of convergences $\mathfrak{L}_S^H$ to the Čech-Stone compactification of discrete spaces is clarified. Necessary and sufficient conditions are given for $(R, \mathfrak{L}_S^H, +)$ to be a complete group with respect to the convergence $\mathfrak{L}_S^H$. This gives some views on the structure of the groups R and R/H.

The point of our considerations is the group $(R, +)$ of real numbers. We use the fact that R is a linearly ordered point set for which a convergence $\mathfrak{L}$ is defined by means of open intervals $(a, b) \subset R$ such that $\lim x_n = x$, $\lim y_n = y$ implies that $\lim(x_n - y_n) = x - y$. In this sense R is a convergence commutative group ([1]). It will be denoted $(R, \mathfrak{L}, +)$.

Recall that a convergence $\mathfrak{M}$ for a set M is a collection of pairs $(\langle x_n \rangle, x)$ where $\langle x_n \rangle$ is a sequence of points $x_n \in M$ and $x \in M$. We assume that the convergence $\mathfrak{M}$ satisfies the well known Fréchet axioms of convergence inclusive the axiom of the maximal convergence ($\mathfrak{M} = \mathfrak{M}^*$). A commutative group $(M, +)$ with a convergence $\mathfrak{M}$ will be denoted $(M, \mathfrak{M}, +)$. If $(\langle x_n \rangle, x) \in \mathfrak{M}$, $(\langle y_n \rangle, y) \in \mathfrak{M}$ implies that $(\langle x_n - y_n \rangle, x - y) \in \mathfrak{M}$ we have a convergence commutative group $(M, \mathfrak{M}, +)$ (abbr. *cc*-group). In such a group Cauchy sequences are defined to be sequences $\langle x_n \rangle$, $x_n \in M$, such that $(\langle x_n - x_{i_n} \rangle, 0) \in \mathfrak{M}$ whenever $\langle x_{i_n} \rangle \subset \langle x_n \rangle$. A *cc*-group $(M, \mathfrak{M}, +)$ is complete if each Cauchy sequence $\mathfrak{M}$-converges in M, more precisely, if $\langle x_n \rangle$, $x_n \in M$, is a Cauchy sequence then there is a point $x \in M$ such that $(\langle x_n \rangle, x) \in \mathfrak{M}$.

I.

Notation. We denote N the set of natural numbers, N^{-1} the set of numbers n^{-1}, $n \in N$, Q the group of rational and R the group of real numbers, H a subgroup of the group R and S a subset of the quotient group R/H. A subgroup of R is either discrete or dense. Points x_1 and x_2 of R are non-equivalent (with respect to H) if $(x_1 - x_2) \notin H$. In the section I we consider R/H as a set of points sometimes called indexes. They will be denoted by Greek letters ξ, η, ζ.

Let H be a subgroup of the group R and R/H the corresponding quotient group. Elements $\xi \in R/H$ are classes $T_\xi = a_\xi + H$ where a_ξ is a representative of the class T_ξ. We identify elements ξ with ordinals $\xi < \omega_H$ where ω_H is the least ordinal of the power $|R/H|$. We put $a_0 = 0$. Then $T_0 = H$. Notice that $R = UT_\xi$, $\xi \in R/H$, $|R/Q| = \exp(\omega)$, $|R/R| = 1$.

Definition D1. Let H be a subgroup of the group R. Functions $f\colon R/H \to N^{-1}$ are called generating functions. Adjoin to $S \subset R/H$ a class F_S^H (or simply F_S) of generating functions f such that the partial function f/S is a constant function. $f/\emptyset$ is considered as a constant partial function. If S contains only one index ξ we write F_ξ instead of $F_{\{\xi\}}$.

Lemma 1. *Let $S_1 \subset S_2 \subset R/H$. Then $F_{S_1} \supset F_{S_2}$.*

Proof. If $f \in F_{S_2}$ then f/S_2 is constant and f/S_1 as well. Hence $f \in F_{S_1}$. □

Definition D2. Let H be a subgroup of the group R. Let (a,b) be an open interval of real numbers. Denote $(a,b)_\xi = (a,b) \cap T_\xi$, $\xi \in R/H$. Let z be a point of R and S a subset of R/H. A set $W(z)$ is called a closure neighborhood or, simply, a neighborhood of the point z if there is a generating function $f \in F_S$ such that $W_f(z) \subset W(z)$ where

$$W_f(z) = U\big(z - f(\xi), z + f(\xi)\big)_\xi, \quad \xi \in R/H.$$

Remark. Let $(a,b) \subset R$, $z \in (a,b)$. Choose $m_0 \in N$ such that $m_0^{-1} < \min\{z - a, b - z\}$ and put $f(\xi) = m_0^{-1}$, $\xi \in R/H$. Then $f \in F_S^H$. Hence $W_f(z) \subset (a,b)$. Consequently, the open interval (a,b) in R is a closure neighborhood of each point $z \in (a,b)$.

The following are the main properties of closure neighborhoods $W_f(z)$, $f \in F_S^H$.
(i) $z \in W_f(z)$. (If $z \in T_{\xi_0}$ then $z \in \big(z - f(\xi_0), z + f(\xi_0)\big)_{\xi_0} \subset W_f(z)$, by D2).

(ii) If $W_{f_1}(z)$, $f_i \in F_S$, $i = 1, 2$, are neighborhoods of a point z, then $W_{f_1}(z) \cap W_{f_2}(z)$ is a neighborhood of the point z. ($W_{f_1}(z) \cap W_{f_2}(z) = W_{f_3}(z)$, where $f_3(\xi) = \min\{f_1(\xi), f_2(\xi)\}$, $\xi \in R/H$).

(iii) If $z_1 \neq z_2$ there are $f_i \in F_S$, $i = 1, 2$, such that $W_{f_1}(z_1) \cap W_{f_2}(z_2) = \emptyset$ (see Remark above).

From (i), (ii), (iii) we deduce that the system of closure neighborhoods $W_f(z)$, $f \in F_S^H$, of points $z \in R$ satisfies the axioms of Hausdorff topological spaces except the axiom of open neighborhoods which need not be fulfilled. This is shown in the following

Lemma 2. *Let H be a dense subgroup of the group R and $S \subset R/H$. Let $z \in R$. Then there is a complete system of open closure neighborhoods at the point z if and only if there is a finite $K \subset R/H$ such that $S = R/H - K$.*

P r o o f. Let $W_f(z)$, $f \in F_S^H$, be a neighborhood of the point z. Since the partial function $f/(R/H - K)$ is constant and K is finite there is a natural number p such that $p^{-1} < f(\xi)$, $\xi \in R/H$. Hence $(z - p^{-1}, z + p^{-1}) \subset W_f(z)$. It follows that the system of intervals $(z - n^{-1}, z + n^{-1})$, $n \in N$, is a complete system of open neighborhoods at the point z.

Now, assume that $R/H - S$ is infinite. Choose distinct $\xi_n \in (R/H - S)$. Define $f(\xi) = 1$, $\xi \neq \xi_n$, $f(\xi_n) = n^{-1}$, $n \in N$. Then $f \in F_S$ and we have a neighborhood $W_f(z)$. Let $W_g(z) \subset W_f(z)$, $g \in F_S$. Suppose that (on the contrary) $W_g(z)$ is open. The neighborhood $W_g(z)$ is infinite because H is dense. Choose a point $t \in W_g(z)$, $t \neq z$. Then there is, by the assumption, a neighborhood $W_h(t) \subset W_g(z)$, $h \in F_S^H$. Notice that $h(\xi) \leqslant g(\xi) \leqslant f(\xi)$, $\xi \in R/H$. There are $\varepsilon_n > 0$ such that

$$\begin{aligned}(t - \varepsilon_n, t + \varepsilon_n)_{\xi_n} &\subset \big(t - h(\xi_n), t + h(\xi_n)\big)_{\xi_n} \subset \big(z - g(\xi_n), z + g(\xi_n)\big)_{\xi_n} \\ &\subset \big(z - f(\xi_n), z + f(\xi_n)\big)_{\xi_n} \subset (z - n^{-1}, z + n^{-1}), \quad n \in N.\end{aligned}$$

Hence $t \in (z - n^{-1}, z + n^{-1})$ and so $t = z$. This is a contradiction. Thus $W_g(z)$ is not open.

We have seen above that the class F_S^H generates a complete system of closure neighborhoods $W_f(z)$ at the point z. By neighborhoods $W_f(z)$, $f \in F_S^H$, a convergence for the group R is defined in a well known way. □

Definition D3. Let H be a subgroup of the group R, $S \subset R/H$. Denote $\mathfrak{L}_S^H$ a collection of pairs $(\langle x_n \rangle, x)$, $x_n \in R$, $x \in R$, such that if $W_f(x)$, $f \in F_S^H$, is a neighborhood of the point x then $x_n \in W_f(x)$, $n \geqslant n_0$. If $(\langle x_n \rangle, x) \in \mathfrak{L}_S^H$ we say that the sequence $\langle x_n \rangle$ $\mathfrak{L}_S^H$-converges to the point x and write $\mathfrak{L}_S^H - \lim x_n = x$. The collection $\mathfrak{L}_S^H$ is called a convergence for R. (It will be sometimes denoted $\mathfrak{L}_S$.)

Fréchet axioms of convergence are clearly satisfied. From (iii) it follows that $\mathfrak{L}_S^H - \lim x_n = x$, $\mathfrak{L}_S^H - \lim y_n = y$ implies $x = y$. In view of (i) we have $\mathfrak{L}_S^H - \lim x = x$. If $(\langle x_n \rangle, x) \in \mathfrak{L}_S^H$, $\langle x_i \rangle \subset \langle x_n \rangle$, then $(\langle x_{i_n} \rangle, x) \in \mathfrak{L}_S^H$, by D3. From D3 it instantly follows that $\mathfrak{L}_S^H$ is a maximal convergence, i.e. $\mathfrak{L}_S^H = \mathfrak{L}_S^{H*}$.

Denote $\mathfrak{L}$ the usual metric convergence for R. We write simply $\lim x_n = x$ instead of $\mathfrak{L} - \lim x_n = x$. Note that $\mathfrak{L} = \mathfrak{L}_{R/H}^H$.

Lemma 3. *Let $S_1 \subset S_2 \subset R/H$. Then $\mathfrak{L}_{S_1}^H \subset \mathfrak{L}_{S_2}^H$.*

P r o o f. Let $(\langle x_n \rangle, z) \in \mathfrak{L}_{S_1}$. Let $W_f(z)$, $f \in F_{S_2}^H$, be a neighborhood of the point z. Define a generating function $g(\xi) = f(\xi)$, $\xi = S_1$, $g(\xi) \leqslant f(\xi)$, $\xi \in R/H - S_1$. The partial function g/S_1 is constant, by Lemma 1, and so $g \in F_{S_1}^H$. Since $x_n \in W_g(z)$, $n \geqslant n_0$, and $W_g(z) \subset W_f(z)$ we have $x_n \in W_f(z)$, $n \geqslant n_0$. Hence $(\langle x_n \rangle, z) \in \mathfrak{L}_{S_2}$.

The assertion $\mathfrak{L}_{S_1} \subset \mathfrak{L}_{S_2}$ implies $S_1 \subset S_2$ is not correct. Let H be a subgroup of R, $R \neq H$. Choose indexes $\xi_1 \neq \xi_2$ and put $S_1 = \{\xi_1\}$, $S_2 = \{\xi_2\}$. Then $\mathfrak{L}_{S_1} = \mathfrak{L}_{S_2}$, but $S_1 \not\subset S_2$. This example shows that the map $\varphi(S) = \mathfrak{L}_S^H$, $S \subset R/H$, $H \neq R$, is not one-to-one even when it preserves the order relation $\subset$, by Lemma 3. Next we investigate the structure of the system of classes $\varphi^{-1}(\mathfrak{L}_S^H)$, $S \subset R/H$.

Let H be a subgroup of the group R, $S \subset R/H$. Denote $R_S = UT_\xi$, $\xi \in S$. Notice that $R_S \subset R$, $R_\emptyset = \emptyset$, $R_{\{0\}} = H$, $R_{R/R} = R$. □

Lemma 4. *Let H be a subgroup of the group R, $S \subset R/H$. Then $\mathfrak{L}_S^H - \lim z_n = z$ if and only if $\lim z_n = z$ and there is a finite $K \subset R/H$ such that $z_n \in R_{S \cup K}$, $n \in N$.*

P r o o f. Let $\mathfrak{L}_S^H - \lim z_n = z$. Then $\lim z_n = z$ because $\mathfrak{L}_S^H \subset \mathfrak{L}$, by Lemma 3. Suppose that (on the contrary) there is a subsequence $\langle z_{i_n} \rangle \subset \langle z_n \rangle$, $z_{i_n} \neq z$, and distinct indexes $\eta_n \in (R/H - S)$ such that $z_{i_n} \in T_{\eta_n}$. Put $f(\xi) = 1$, $\xi \neq \eta_n$, and choose $f(\eta_n) \in N^{-1}$ such that $z_{i_n} \notin (z - f(\eta_n), f(\eta_n))$, $n \in N$. This is possible because $z_{i_n} \neq z$. Then $f \in F_S$ and we have a neighborhood $W_f(z)$ of z which contains no point z_i. Hence $\langle z_{i_n} \rangle$ does not $\mathfrak{L}_S^H$-converge to z. This is in contradiction with the assumption $(\langle z_n \rangle, z) \in \mathfrak{L}_S^H$.

Now, let $\lim z_n = z$, $z_n \in R_{S \cup K}$. We use the property $\mathfrak{L}_S = \mathfrak{L}_S^*$ to prove that $\mathfrak{L}_S - \lim z_n = z$. Let $\langle z_{i_n} \rangle$ be a subsequence of $\langle z_n \rangle$. Either there is a subsequence $\langle t_n \rangle \subset \langle z_{i_n} \rangle$ of non-equivalent points $t_n \in R_S$ and then $\mathfrak{L}_S - \lim t_n = z$ or it is not so, and there is an index $\xi_0 \in (S \cup K)$ and a subsequence $\langle u_n \rangle \subset \langle z_{i_n} \rangle$, $u_n \in T_{\xi_0}$. Hence $\mathfrak{L}_S - \lim u_n = z$. It follows that $(\langle z_n \rangle, z) \in \mathfrak{L}_S$. □

Lemma 5. *Let H be a subgroup of the group R. Let $S_i \subset R/H$, $i = 1, 2$. Let $S_1 \div S_2$ be a finite set. Then $\mathfrak{L}_{S_1}^H = \mathfrak{L}_{S_2}^H$.*

P r o o f. Let $(\langle z_n\rangle, z) \in \mathfrak{L}_{S_1}$ and $W_f(z)$, $f \in F_{S_2}$, be a neighborhood of the point $z \in R$. We are to prove that $z_n \in W_f(z)$, $n \geqslant n_0$. Notice that $S_1 \cup S_2 = (S_1 \div S_2) \cup (S_1 \cap S_2)$. The partial function $f/S_1 \cap S_2$ is constant, by Lemma 1, and $S_1 \div S_2$ is a finite set. Therefore the number $d = \min\{f(\xi)\}$, $\xi \in S_1 \cup S_2$, belongs to the set N^{-1}. Put $g(\xi) = d$, $\xi \in S_1 \cup S_2$, and $g(\xi) \leqslant f(\xi)$, $g(\xi) \in N^{-1}$, $\xi \in (R/H - (S_1 \cup S_2))$. Then $g \in F_{S_1}$ and so $z_n \in W_g(z)$, $n \geqslant n_0$. Hence $z_n \in W_f(z)$, $n \geqslant n_0$ and therefore $\mathfrak{L}_{S_1} \subset \mathfrak{L}_{S_2}$.

Analogously we prove that $\mathfrak{L}_{S_2} \subset \mathfrak{L}_{S_1}$. □

Lemma 6. *Let H be a subgroup of the group R. Let $S_i \subset R/H$, $i = 1, 2$. Let $\mathfrak{L}^H_{S_1} \subset \mathfrak{L}^H_{S_2}$. Then $S_1 - S_2$ is a finite set.*

P r o o f. First prove the following statement: If S_0 is an infinite subset of R/H then there is a sequence of non-equivalent points $x_n \in T_{\xi_n}$, $\xi_n \in S_0$, and a point $z \in R$ such that $\mathfrak{L}^H_{S_0} - \lim x_n = z$. Distinguish two cases. 1) H is dense. Let $\langle \xi_n\rangle$ be one-to-one sequence of indexes $\xi_n \in S_0$. Choose a point $z \in R$. Since H is dense there is a sequence $\langle x_n\rangle$ of non-equivalent points $x_n \in T_{\xi_n}$ with $\lim x_n = z$. Hence $\mathfrak{L}^H_{S_0} - \lim x_n = z$, by Lemma 4. 2) H is discrete. Denote d the least positive number of H. Choose numbers $b_\xi \in T_\xi$ such that $0 \leqslant b_\xi < d$, $\xi \in R/H$. Since S_0 is infinite there is a one-to-one sequence $\langle \xi_n\rangle$, $\xi_n \in S_0$, and a point $z \in R$, $0 \leqslant z \leqslant d$, such that $\lim b_{\xi_n} = z$. Denote $b_{\xi_n} = x_n$. Then $\langle x_n\rangle$ is a sequence of non-equivalent points x_n with $\mathfrak{L}^H_{S_0} - \lim x_n = z$, by Lemma 4.

Suppose that $S_1 - S_2$ is infinite and denote $S_0 = S_1 - S_2$. Then $S_0 \subset S_1$ and $(\langle x_n\rangle, z) \in \mathfrak{L}^H_{S_1}$, by Lemma 3 where $\langle x_n\rangle$ is the sequence constructed above. On the other hand, $(\langle x_n\rangle, z) \notin \mathfrak{L}^H_{S_2}$, by Lemma 4. This is a contradiction. □

Proposition 1. *Let H be a subgroup of the group R, $S_i \subset R/H$, $i = 1, 2$. Then $\mathfrak{L}^H_{S_1} = \mathfrak{L}^H_{S_2}$ if and only if $S_1 \div S_2$ is a finite set.*

Proof follows instantly from Lemmas 5 and 6.

From Proposition 1 it follows that there is a connection between convergences $\mathfrak{L}^H_S$ and some subsets of the Čech-Stone compactification of a discrete topological space. Consider R/H as a discrete topological space of isolated points ξ and denote $\beta^* S = \beta S - R/H$, where β is a topological operator in the Čech-Stone compactification $\beta(R/H)$. It is well known that $\beta^* S_1 = \beta^* S_2$ if and only if $S_1 \div S_2$ is finite. Hence $\mathfrak{L}^H_{S_1} = \mathfrak{L}^H_{S_2}$ if and only if $\beta^* S_1 = \beta^* S_2$, by Proposition 1.

Let H be a subgroup of the group R. We denote, as above, functions $\varphi(S) = \mathfrak{L}^H_S$, $S \subset R/H$. We have shown that φ is not one-to-one except in the case when $H = R$. From Proposition 1 it follows that S_1 and S_2 are equivalent (i.e. $S_2 \in \varphi^{-1}(\mathfrak{L}_{S_1})$) iff $S_1 \div S_2$ is finite. Now, define a quasi-order $\prec$ as follows: $S_1 \prec S_2$ if there is a finite $K \subset R/H$ such that $S_1 \subset S_2 \cup K$.

Lemma 7. *Let H be a subgroup of the group R. Then $S_1 \prec S_2$ if and only if $\mathfrak{L}^H_{S_1} \subset \mathfrak{L}^H_{S_2}$.*

P r o o f. Let $S_1 \prec S_2$. Then $S_1 \subset S_2 \cup K$. It follows $\mathfrak{L}_{S_1} \subset \mathfrak{L}_{S_2}$, by Lemma 3 and Proposition 1. Now, let $\mathfrak{L}_{S_1} \subset \mathfrak{L}_{S_2}$. According to Lemma 6 the set $S_1 - S_2$ is finite. Since $S_1 \subset S_2 \cup (S_1 - S_2)$ we have $S_1 \prec S_2$. □

Proposition 2. *Let H be a subgroup of the group R. There is a similar map (with respect to the inclusion $\subset$), on the system $\mathbb{L}^H$ of convergences $\mathfrak{L}^H_S$, $S \subset R/H$, onto the system of clopen sets $\beta^*(S)$ of the space $\beta^*(R/H)$.*

P r o o f. Denote $\psi(\mathfrak{L}^H_S) = \beta^* S$, $S \subset R/H$. Let S_1, S_2 be subsets of R/H, $\mathfrak{L}_{S_1} \subset \mathfrak{L}_{S_2}$. Then $S_1 \prec S_2$, by Lemma 7. Therefore, according to the definition of the quasi-order $\prec$ it follows that $\beta^* S_1 \subset \beta^* S_2$. It remains to prove the following implication: If A is a clopen subset of $\beta^*(R/H)$ then there is $S \subset R/H$ such that $\beta^* S = A$. This is true because there is a clopen set B in $\beta(R/H)$ such that $A = B \cap \beta^*(R/H)$ and so there is $S \subset R/H$ such that $A = \beta^*(S)$. □

R e m a r k. Notice that $\aleph_{\alpha_1} \leqslant \aleph_{\alpha_2}$ implies $\aleph_{\alpha_1} \cdot \aleph_{\alpha_2} = \aleph_{\alpha_2}$. Let $S \subset R/Q$. Denote $F = \{K;\ K \subset R/Q, K \text{ finite}\}$, $X_S = \{S \div K : K \in F\}$, $Y = \{S;\ S \subset R/Q\}$, $Z = Y/F$, $\aleph_{\alpha_1} = |X_S|$, $\aleph_{\alpha_2} = |Z|$. Clearly $\aleph_{\alpha_1} = \exp(\omega)$, $\aleph_{\alpha_2} = \exp(\exp(\omega))$. Then $|\mathbb{L}^Q| = |X_S|\ |Z| = \exp(\omega).\ \exp(\exp(\omega)) = \exp(\exp(\omega))$. Thus the number of convergences $\mathfrak{L}^Q_S$, $S \subset R/Q$, is $\exp(\exp(\omega))$.

Let H be a subgroup of the group R, $S \subset R/H$. We have seen that a closure topology for R is defined by means of the class F^H_S of generating functions. The corresponding closure operator will be denoted w^H_S (or simply w_S). Hence $w_S A = \{x \in R\colon A \cap W_f(x) \neq \emptyset, f \in F^H_S\}$. Another closure topology for R is defined by means of the convergence $\mathfrak{L}^H_S$. Denote λ^H_S (or λ_S) the corresponding closure operator: $\lambda^H_S A = \{x \in R;\ x = \mathfrak{L}^H_S - \lim x_n, x_n \in A, n \in N\}$. Hence we have closure spaces (R, w^H_S) and (R, λ^H_S).

Now, we are interested in the question what is the relation between closures λ^H_S and w^H_S. It is well known that there are closure spaces (P, u) and adjoint convergence spaces (P, λ_u) such that $u \neq \lambda_u$. It is not the case if $P = R$, $u = w^H_S$. We show that $w^H_S = \lambda^H_S$. It is evident that $\lambda_S A \subset w_S A$, $A \subset R$. Suppose that there is $z \in R$ and $A \subset R$ such that $z \in (w_S A - \lambda_S A)$. Then $z \notin A$ and there is no sequence of points $x_n \in A$ such that $\mathfrak{L}^H_S - \lim x_n = z$. In view of Lemma 4 there is a generating function $f \in F_S$ such that $A \cap \big(z - f(\xi), z + f(\xi)\big)_\xi = \emptyset$, $\xi \in R/H$. Hence $A \cap W_f(z) = \emptyset$. This is a contradiction. Consequently, $w_S = \lambda_S$.

Notice that ω_1-iterated closure $\lambda^{\omega_1}_S = w^{\omega_1}_S$ are topologies for R.

II.

In this section we investigate some convergence and group properties of the structures $(R, \mathfrak{L}_S^H, +)$. For this purpose we consider indexes ξ of the set R/H as elements of the group $(R/H, +)$. If ξ_1, ξ_2 are elements of R/H then $\xi_1 + \xi_2 = \xi_3$ where ξ_3 is uniquely determined by the addition $T_{\xi_1} + T_{\xi_2} = T_{\xi_3}$ in the group $(R/H, +)$. The inverse element to the element $\xi \in R/H$ is the element $\eta \in R/H$ such that $T_\eta = -T_\xi$. It will be denoted $-\xi$.

Now, we are going to examine conditions under which $(R, \mathfrak{L}_S^H, +)$ is a *cc*-group. First we give an example to show that $(R, \mathfrak{L}_S^H, +)$ need not be a *cc*-group even when R_S is a subgroup of the group R.

Example. Let $H = Q$ and let R_S be the group of algebraic numbers. Put $x_n = n^{-1}\sqrt{2}$, $y_n = \pi + n^{-1}$. Then $\lim x_n = 0$, $\lim y_n = \pi$ and $\mathfrak{L}_S^H - \lim x_n = 0$, $\mathfrak{L}_S^H - \lim y_n = \pi$, by Lemma 4. On the other hand, $\langle x_n + y_n \rangle$ is a sequence of non-equivalent transcendent numbers which, by the same lemma, does not $\mathfrak{L}_S^H$-converge to the point π.

Definition D4. Let $(M, \mathfrak{M}, +)$ be a commutative group with a convergence $\mathfrak{M}$ for M. We say that $(M, \mathfrak{M}, +)$ satisfies condition $(-)$ provided that the following implication holds

$$(-) \qquad \text{If } (\langle x_n \rangle, x) \in \mathfrak{M} \text{ then } (\langle -x_n \rangle, -x) \in \mathfrak{M}.$$

$(M, \mathfrak{M}, +)$ satisfies condition $(+)$ provided that

$$(+) \qquad \text{If } (\langle x_n \rangle, x) \in \mathfrak{M},\ (\langle y_n \rangle, y) \in \mathfrak{M} \text{ then } (\langle x_n + y_n \rangle, x + y) \in \mathfrak{M}.$$

It is clear that $(M, \mathfrak{M}, +)$ is a *cc*-group if and only if both the conditions $(-)$ and $(+)$ are satisfied.

Definition D5. Let H be a subgroup of the group R, $S \subset R/H$. We denote S^- the set of elements $\eta \in R/H$ such that $T_\eta = -T_\xi$, $\xi \in S$.

Lemma 8. *$|S| = |S^-|$, $|S - S^-| = |S^- - S|$, $(S_1 \cup S_2)^- = S_1^- \cup S_2^-$, $(S_1 \cap S_2)^- = S_1^- \cap S_2^-$, $x \in R_S$ if and only if $-x \in R_{S^-}$.*

Proof follows instantly from D5 and from the equivalence $\xi \in (S - S^-)$ if and only if $-\xi \in (S^- - S)$.

The properties $(-)$ and $(+)$ can be formulated by means of Čech-Stone operator β^*. In the proofs we use the equivalence

(i) $\beta^* S_1 \subset \beta^* S_2$ if and only if $S_1 \subset S_2 \cup K$ where K is finite. From (i) it follows
(ii) $\beta^* S_1 = \beta^* S_2$ if and only if $S_1 \div S_2$ is finite.

Lemma 9. *Let H be a subgroup of the group R, $S \subset R/H$. Then $(R, \mathfrak{L}_S^H, +)$ satisfies $(-)$ if and only if $\beta^* S = \beta^*(S^-)$.*

Proof. Let $\beta^* S = \beta^*(S^-)$. The set $S \div S^-$ is finite, by (ii). Let $\mathfrak{L}_S^H - \lim x_n = x$. In view of Lemma 4, there is a finite $K \subset R/H$ such that $x_n \in R_{S \cup K}$ and $\lim x_n = x$. Hence $\lim(-x_n) = -x$ and $-x_n \in R_{S^- \cup K^-}$, by Lemma 8. Notice that $S^- \cup K^- = (S^- \cap S) \cup (S^- - S) \cup K^- \subset S \cup K_1$ where $K_1 = (S^- - S) \cup K^-$. It follows that K_1 is a finite subset of R/H and $R_{S^- \cup K^-} \subset R_{S \cup K_1}$. Thus $\mathfrak{L}_S^H - \lim(-x_n) = -x$, by Lemma 4.

Let $\beta^* S \neq \beta^*(S^-)$. Then $S \div S^-$ is infinite and both the sets $S - S^-$ and $S^- - S$ are infinite, by Lemma 8. In view of statement (see the proof of Lemma 6) there is a sequence of non-equivalent points $x_n \in R_{S - S^-}$ and a point $z \in R$ such that $\mathfrak{L}_{S-S^-}^H - \lim x_n = z$. Notice that $\langle -x_n \rangle$ is a sequence of non-equivalent points $-x_n \in R_{S^- - S}$. Consequently, $-x_n \notin R_S$. From Lemma 4 it follows that the sequence $\langle -x_n \rangle$ does not $\mathfrak{L}_S^H$-converge to $-z$. □

Lemma 10. *Let H be a subgroup of the group R, $S \subset R/H$. $(R, \mathfrak{L}_S^H, +)$ satisfies $(+)$ if and only if $\beta^*\big((S \cup L_1) + (S \cup L_2)\big) \subset \beta^* S$ whenever L_1, L_2 are finite subsets of R/H.*

Proof. Let $\beta^*\big((S \cup L_1) + (S \cup L_2)\big) \subset \beta^* S$. Let $\mathfrak{L}_S^H - \lim x_n = x$, $\mathfrak{L}_S^H - \lim y_n = y$. There are finite subsets K_1, K_2 of R/H such that $x_n \in R_{S \cup K_1}$, $\lim x_n = x$, and $y_n \in R_{S \cup K_2}$, $\lim y_n = y$. Hence $\lim(x_n + y_n) = x + y$. Since $\beta^*\big((S \cup K_1) + (S \cup K_2)\big) \subset \beta^* S$ there is, according to (i) above, a finite $K \subset R/H$ such that $\big((S \cup K_1) + (S \cup K_2)\big) \subset S \cup K$. Consequently, $R_{((S \cup K_1) + (S \cup K_2))} \subset R_{S \cup K}$ and so $(x_n + y_n) \in R_{S \cup K}$, $\lim(x_n + y_n) = x + y$. We have $\mathfrak{L}_S^H - \lim(x_n + y_n) = x + y$, by Lemma 4.

Suppose that there are finite subsets K_1, K_2 of R/H such that $\beta^*\big((S \cup K_1) + (S \cup K_2)\big) \not\subset \beta^* S$. According to (i) we deduce $\big((S \cup K_1) + (S \cup K_2)\big) \not\subset S \cup K$ for every finite subset K of R/H. It follows that there is an infinite set of elements $\zeta_n' = \xi_n' + \eta_n'$, $\xi_n' \in (S \cup K_1)$, $\eta_n' \in (S \cup K_2)$ such that if K is finite then there is $n_K \in N$ such that $\zeta_n' \notin (S \cup K)$, $n \geqslant n_K$. Since the sequence $\langle \zeta_n' \rangle$ is one-to-one there is a sequence $\langle \zeta_n \rangle \subset \langle \zeta_n' \rangle$, $\zeta_n = \xi_n + \eta_n$ such that either $\langle \xi_n \rangle$, $\langle \eta_n \rangle$ are one-to-one or one of them, say $\langle \xi_n \rangle$, is one-to-one whereas the other is a constant one, i.e. $\eta_n = \eta$, $n \in N$. In the first case there is (in view of the statement in the proof of Lemma 6) a subsequence $\langle \xi_{i_n} \rangle \subset \langle \xi_n \rangle$, points $x \in R$ and $y \in R$, sequences $\langle x_n \rangle$, $x_n \in T_{\xi_{i_n}}$, and $\langle y_n \rangle$, $y_n \in T_{\eta_{i_n}}$, such that $\mathfrak{L}_S^H - \lim x_n = x$ and $\mathfrak{L}_S^H - \lim y_n = y$. In the second case we choose $y \in T_\eta$ and put $y_n = y$, $n \in N$. Then $\mathfrak{L}_S^H - \lim x_n = x$ and $\mathfrak{L}_S^H - \lim y_n = y$. In both cases we have a sequence $\langle x_n + y_n \rangle$ of non-equivalent points $(x_n + y_n) \in T_{\zeta_n}$ which does not $\mathfrak{L}_S^H$-converge to the point $x + y$ because there is no finite $K \subset R/H$ such that $\zeta_n \in (S \cup K)$, $n \geqslant n_0$. □

Lemma 11. *Let H be a subgroup of the group R and S a finite subset of R/H. Then $(R, \mathfrak{L}_S^H, +)$ is a cc-group.*

Proof. S and S^- are finite sets. Hence $\beta^*(S) = \emptyset$, $\beta^*(S^-) = \emptyset$. The condition $(-)$ is satisfied, by Lemma 9. Now, let L_1, L_2 be finite. Then $((S \cup L_1) + (S \cup L_2))$ is a finite subset of R/H and so $\beta^*\big((S \cup L_1) + (S \cup L_2)\big) = \emptyset$, $\beta^*(S) = \emptyset$. Hence $(+)$ is satisfied, by Lemma 10. □

Next we use lemmas 9 and 10 to answer the question: Given a subgroup $H \subset R$ does there exist more than two cc-groups $(R, \mathfrak{L}_S^H, +)$?

Lemma 12. *Let S be an infinite and K a finite subset of R/H. Let $\langle \xi_n \rangle$ be a one-to-one sequence of elements $\xi_n \in S \cup K$. Then there is n_0 such that $\xi_n \in S$, $n \geqslant n_0$.*

Proof. Since $\langle \xi_n \rangle$ is one-to-one the finite set K contains at most a finite number of elements ξ_n. □

Lemma 13. *Let H be a subgroup of the group R. Let S be an infinite subset of R/H. Let $(R, \mathfrak{L}_S^H, +)$ be a cc-group. Let $\langle \xi_n \rangle$ be a one-to-one sequence of elements $\xi_n \in S$. Let $\eta \in R/H$. Then there is n_0 such that $(\xi_n + \eta) \in S$, $n \geqslant n_0$.*

Proof. Put $L_1 = \emptyset$, $L_2 = \{\eta\}$. Then $\xi_n \in (S \cup L_1)$, $\eta \in (S \cup L_2)$. Since $(R, \mathfrak{L}_S^H, +)$ is a cc-group the condition $(+)$ is satisfied. We can apply Lemma 10. There is a finite $K \subset R/H$ such that $(\xi_n + \eta) \in S \cup K$, $n \in N$. Therefore $(\xi_n + \eta) \in S$, $n \geqslant n_0$, by Lemma 12. □

Lemma 14. *Let H be a subgroup of the group R. Let S and $R/H - S$ be infinite subsets of R/H. Then $(R, \mathfrak{L}_S^H, +)$ fails to be a cc-group.*

Proof. Suppose that, on the contrary, $(R, \mathfrak{L}_S^H, +)$ is a cc-group. Denote $S' = R/H - S$. Let $\langle \xi_n \rangle$, $\xi_n \in S$, $\langle \eta_n \rangle$, $\eta_n \in S'$, be one-to-one sequences. According to Lemma 12 there is n_1 such that $(\xi_n + \eta_1) \in S$, $n \geqslant n_1$. Put $m_1 = n_1$ and $\zeta_1 = \xi_{m_1} + \eta_1$. Suppose that we have chosen natural numbers $m_1 < m_2 < \ldots < m_p$ and non-equivalent elements $\zeta_i \in S$, $i \leqslant p$, where $\zeta_i = \xi_{m_i} + \eta_i$, $i \leqslant p$. Notice, that $\langle \xi_n + \eta_{p+1} \rangle$ is a one-to-one sequence such that $(\xi_n + \eta_{p+1}) \in S$, $n \geqslant n_0$, by Lemma 13. It follows that there is a natural number $m_{p+1} > m_p + n_0$ such that

$(\xi_{m_{p+1}} + \eta_{p+1}) \neq \zeta_i$, $i \leqslant p$. Put $\zeta_{p+1} = \xi_{m_{p+1}} + \eta_{p+1}$. Hence we have an increasing sequence $m_1 < m_2 < \ldots < m_{p+1}$ and a one-to-one sequence $\zeta_1, \zeta_2, \ldots, \zeta_{p+1}$ of elements of S. We have constructed, by means of mathematical induction, a one-to-one sequence of elements $\zeta_i \in S$, $\zeta_i = \xi_{m_i} + \eta_i$, $i \in N$. Elements ξ_{m_i} belong to the set S and elements $-\xi_{m_i}$ to the set $S^- = S \cap S^- \cup (S^- - S)$. $(R, \mathfrak{L}_S^H, +)$ satisfies $(-)$ and so $S^- - S$ is a finite set, by Lemma 9. Put $L_1 = \emptyset$, $L_2 = S^- - S$. Then $\zeta_i \in S \cup L_1$ and $-\xi_{m_i} \in S \cup L_2$. According to Lemma 10 there is a finite $K \subset R/H$ such that $(\zeta_i - \xi_{m_i}) \in S \cup K$, i.e. $\eta_i \in S \cup K$. The sequence $\langle \eta_i \rangle$ is one-to-one. According to Lemma 12 there is i_0 such that $\eta_i \in S$, $i \geqslant i_0$. On the other hand, $\eta_i \in S'$, $i \in N$. Thus we got a contradictory result.

There is a close connection between cc-groups $(R, \mathfrak{L}_S^H, +)$ and complete groups with respect to the convergence $\mathfrak{L}_S^H$. This is shown in the following lemma. □

Lemma 15. *$(R, \mathfrak{L}_S^H, +)$ is a cc-group if and only if it is a complete group.*

P r o o f. Let $(R, \mathfrak{L}_S^H, +)$ be a cc-group. By Lemma 14 there is a finite $K \subset R/H$ such that $S = K$ or $S = R/H - K$. If $S = R/H - K$ then $(R, \mathfrak{L}_S^H, +)$ is a complete because $\mathfrak{L}_S^H = \mathfrak{L}$. Now, suppose that S is a finite set. Let $\langle c_n \rangle$, $c_n \in R$, be a Cauchy sequence of points c_n in $(R, \mathfrak{L}_S^H, +)$. Distinguish two cases

1) There is a finite subset K_0 such that $c_n \in R_{K_0}$. The sequence $\langle c_n \rangle$ is a Cauchy sequence with respect to $\mathfrak{L}$, because $\mathfrak{L}_S^H \subset \mathfrak{L}$, by Lemma 3. Hence there is a point $x \in R$ such that $\lim c_n = x$. We have $\mathfrak{L}_S^H - \lim c_n = x$, according to Lemma 4.

2) There is a subsequence $\langle b_n \rangle \subset \langle c_n \rangle$ of non-equivalent points $b_n \in R$. We construct, analogously as in [1], a subsequence $\langle b_{i_n} \rangle \subset \langle b_n \rangle$ such that $\langle b_n - b_{i_n} \rangle$ does not $\mathfrak{L}_S^H$-converge to 0. Put $i_1 = 1$. Suppose that we have chosen points $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$, $i_1 < i_2 < \ldots < i_k$, such that no two numbers $t_m = b_m - b_{i_m}$, $m \leqslant k$, are equivalent. We prove that there is a point $b_{i_{k+1}}$, $i_{k+1} > i_k$, in the sequence $\langle b_n \rangle$ such that no two numbers t_m, $1 \leqslant m \leqslant k+1$ are equivalent. Let $q > k$. Suppose (indirect proof) that there is no point b_s, $i_k < s \leqslant i_k + q$ in the sequence $\langle b_n \rangle$ such that any two numbers $b_{k+1} - b_s$ and t_m, $m \leqslant k$, are non-equivalent. Denote $u_s = b_{k+1} - b_s$. Let $f\colon \{i_k < s \leqslant i_k + q\} \to \{1, 2, \ldots, k\}$ be a (one-valued) function such that u_s and $t_{f(s)}$ are equivalent numbers. Since $q > k$ there are $s_1 > i_k$ and $s_2 \leqslant i_k + q$, $s_1 < s_2$, such that $f(s_1) = f(s_2)$. Consequently, the numbers $u_{s_1}, t_{f(s_1)}$ are equivalent and also numbers $u_{s_2}, t_{f(s_2)}$ are equivalent. It follows that $(b_{f(s_1)} - b_{s_1}) \in H$, $(b_{f(s_2)} - b_{s_2}) \in H$. Hence $(b_{s_1} - b_{s_2}) \in H$ and so b_{s_1}, b_{s_2} are equivalent points. This is a contradiction because b_n are non-equivalent points. We conclude that there is $s_0 \in \{i_k + 1, i_k + 2, \ldots, i_k + q\}$ such that points b_{s_0}, b_{i_m}, $m \leqslant k$, are non-equivalent. Hence, it suffices to put $i_{k+1} = s_0$.

In such a way we have constructed a sequence $\langle b_n - b_{i_n} \rangle$ of non-equivalent points. Since S is finite it follows from Lemma 4 that the sequence $\langle b_n - b_{i_n} \rangle$ does not $\mathfrak{L}_S^H$-

converge to 0. Therefore $\langle c_n \rangle$ is not a Cauchy sequence with respect to $\mathfrak{L}_S^H$. The case 2) cannot occur.

Let $(R, \mathfrak{L}_S^H, +)$ be a complete group with respect to the convergence $\mathfrak{L}_S^H$. Then it is a *cc*-group, by the definition on p. 25.

Lemmas 11 and 14 give us a complete information about structures $(R, \mathfrak{L}_S^H, +)$ which are *cc*-groups. If $H = R$ then $(R, \mathfrak{L}, +)$ is the unique *cc*-group. If $H \neq R$ then there are exactly two different *cc*-groups, i.e. $(R, \mathfrak{L}_0^H, +)$ and $(R, \mathfrak{L}, +)$. $\square$

Closing remarks. If Q is a subgroup of H and H a subgroup of R, $H \neq R$, then there are two different completions $(R, \mathfrak{L}_S^H, +)$ of Q, namely, $(R, \mathfrak{L}_0^H, +)$ and $(R, \mathfrak{L}, +)$. It follows that there is more than one completions of Q. There would be interesting to know what is the number of completion of the group of rational numbers Q.

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Addendum after the proofs. P. Simon and R. Frič proved, independently from each other, that the number of completions of the group Q is $\exp(\exp(\omega))$ [2].

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