Vlastimil Pták; Pavla Vrbová

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CONTRACTIVE COUPLINGS

VLASTIMIL PTÁK AND PAVLA VRBOVÁ, Praha

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It is the aim of the present note to introduce and investigate a generalized notion of coupling intended as an abstract framework for the study of functional models in a subsequent communication. In the first part a necessary and sufficient condition for the existence of couplings of Hilbert space operators with a prescribed angle between their domains of definition is given. The second part of the paper is devoted to the problem of coupling of contractions under an additional requirement concerning the minimal isometric dilations of these contractions; it turns out that this problem is closely related to the lifting problem of intertwining relations that characterize generalized Hankel operators [6, 7].

The problem to be treated may be formulated as follows. Given two bounded linear operators A_1 and A_2 acting on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and a contraction $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ what are the conditions for the existence of a Hilbert space $\mathscr{K}$ and an operator $U \in B(\mathscr{K})$ such that

$$\begin{aligned}
&(1) \qquad \mathscr{K} \text{ contains } \mathscr{H}_1 \text{ and } \mathscr{H}_2,\\
&(2) \qquad U|\mathscr{H}_1 = A_1, \quad U^*|\mathscr{H}_2 = A_2,\\
&(3) \qquad X = P(\mathscr{H}_2)|\mathscr{H}_1.
\end{aligned}$$

(By $P(\mathscr{H})$ we shall denote the orthogonal projection onto $\mathscr{H}$.) In section three a proof is given of the fact that a solution exists if and only if the intertwining relation $XA_1 = A_2^*X$ is satisfied. In that case a solution U exists whose norm does not exceed the maximum of the norms of A_1 and A_2. In other words, if A_1 and A_2 are contractions then the set of contractive solutions U is nonvoid. Moreover, a parametrization of all contractive solutions U on a certain model space can be given.

The construction of this model space—denoted by the authors by $\mathscr{P}(X)$—is described in section two.

It turns out that the space $\mathscr{P}(X)$ is one of the possible equivalent representations of a Hilbert space $\mathscr{K}$ with the following properties (we assume that $\mathscr{H}_1$, $\mathscr{H}_2$ and the contraction $X: \mathscr{H}_1 \to \mathscr{H}_2$ are given)

(i) $\mathscr{K}$ contains $\mathscr{H}_1$ and $\mathscr{H}_2$,

(ii) $\mathscr{K} = \mathscr{H}_1 \vee \mathscr{H}_2$,

(iii) $X = P(\mathscr{H}_2)|\mathscr{H}_1$.

It is not difficult to realize that representations of spaces $\mathscr{K}$ with properties (i)–(iii) as well as the problem of coupling as described above are closely related to the theory of functional models (see [4]). The authors intend to collect their results concerning applications of these ideas to this theory in another communication.

The note is organized as follows. As has been already said above section one collects the material from dilation theory needed in the sequel. Section three is devoted to the existence and description of the solution and to an analysis of the particular case when isometric solutions exist. In section four we impose further restrictions on the space $\mathscr{K}$. We investigate the particular case when $\mathscr{K}$ contains the minimal coisometric extensions of the contractions A_1 and A_2. It turns out that this problem is closely connected with the notion of a generalized Hankel operator introduced recently by the authors [6]. More precisely, in this case the necessary condition $XA_1 = A_2^*X$ and boundedness of X are not sufficient for the existence of a solution. Boundedness has to be replaced by a stronger condition: the authors introduced in [6] and [7] the condition of $\mathscr{R}$-boundedness; this stronger condition guarantees the possibility of lifting the intertwining relation $XA_1 = A_2^*X$ to a relation with required properties.

The idea of combining two spaces in this manner appears first—in the particular case of semiunitary operators—in a paper of Adamyan and Arov where the suggestive name "couplings" was introduced for this notion. Many questions investigated in the theory of unitary dilations may be formulated in terms of couplings of suitable mappings and spaces. The ideas of Adamyan and Arov were further developed by a number of authors. The connections between couplings, dilation theory and mappings of positive type were investigated by many authors, notably by R. Arocena and M. Cotlar. Some of their results are related to the questions treated in the first part of the present paper. In distinction to their work we aim at the most general framework for model theory in the first part of our paper, the second part being devoted to couplings on which more stringent conditions are imposed; such conditions appear first in the authors' investigation of Hankel-like operators.

The paper arose from discussions the authors held with V. M. Adamyan and D. Z. Arov at the 20th Seminar on functional analysis held in May 1989 in Liptovský Ján. The authors wish to acknowledge a debt of gratitude to them for their stimulating lectures and contributions to the programme of the meeting.

1. Notation and preliminaries

Given a contraction $T \in B(\mathscr{H})$ on a Hilbert space $\mathscr{H}$ we define the defect operator $D(T)$ by the formula $D(T) = (1-T^*T)^{1/2}$. Then $|h|^2 = |D(T)h|^2+|Th|^2$ for $h \in \mathscr{H}$. Also, we define the asymptotic modulus $A(T)$ by the formula $A(T) = (\lim T^{*n}T^n)^{1/2}$. The limit exists in the strong operator topology and $|A(T)h| = \lim |T^n h|$ for $h \in \mathscr{H}$. This notion was used first by R. G. Douglas [2]. In the theory of generalized Hankel operators, the authors defined the notion of $\mathscr{A}$-boundedness for operators $X : \mathscr{H}_1 \to \mathscr{H}_2$ as follows.

Given two contractions $T_1 \in B(\mathscr{H}_1)$, $T_2 \in B(\mathscr{H}_2)$, an operator $X : \mathscr{H}_1 \to \mathscr{H}_2$ is said to be $\mathscr{A}$-bounded (with respect to T_1 and T_2) if

$$|(Xh_1, h_2)| \leqslant \beta |A(T_1^*)h_1|\,|A(T_2^*)h_2|$$

for all $h_1 \in \mathscr{H}_1$, $h_2 \in \mathscr{H}_2$ and a suitable constant β.

Since $|A(T)x| \leqslant |x|$ for every contraction T and every x, an $\mathscr{A}$-bounded operator is always bounded; the condition is stronger than boundedness, however.

In the theory of generalized Hankel operators the construction of a symbol is based on a lifting theorem for $\mathscr{A}$-bounded operators [6], [7] and this theorem will also be used in an essential manner in the present paper. For our purposes it will be convenient, however, to restate it in a different form:

The classical commutant lifting theorem may be either stated in terms of isometric dilations [8] or in terms of coisometric extensions [4]. The same remark applies in the case of the lifting theorem for $\mathscr{A}$-bounded operators; we give the coisometric version below. Recall first the notions of the minimal isometric dilation and the minimal coisometric extension. If $T \in B(\mathscr{H})$ is a contraction then there exists an isometry U defined on a Hilbert space containing $\mathscr{H}$ such that $T^n = P(\mathscr{H})U^n|\mathscr{H}$ for $n \geqslant 0$. Moreover, the restriction $V = U|\bigvee_{n\geqslant 0} U^n\mathscr{H}$ is uniquely determined up to unitary equivalence and satisfies $V^*|\mathscr{H} = T^*$. The isometry V is known as the minimal isometric dilation of T. Considering the adjoints we may restate these facts in the following manner: given a contraction $T \in B(\mathscr{H})$ there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a coisometry $W \in B(\mathscr{K})$ such that $T = W|\mathscr{H}$ and $\mathscr{K} = \bigvee_{n\geqslant 0} W^{*n}\mathscr{H} = \operatorname{Inv}(\mathscr{H}, W^*)$. A coisometry W extending T and possessing this minimal property is uniquely determined up to unitary equivalence and is called the minimal coisometric extension of T.

(1, 1) *Let $A_1 \in B(\mathscr{H}_1)$ and $A_2 \in B(\mathscr{H}_2)$ be two contractions and let $U_1 \in B(\mathscr{K}_1)$ and $U_2 \in B(\mathscr{K}_2)$ be their minimal coisometric extensions. Suppose $X : \mathscr{H}_1 \to \mathscr{H}_2$*

satisfies

$$XA_1 = A_2^* X$$

and

$$|(Xh_1, h_2)| \leqslant |A(A_1)h_1|\,|A(A_2)h_2|$$

for all $h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2$.

Then there exists a $Y : \mathcal{K}_1 \to \mathcal{K}_2$ *such that*

$$YU_1 = U_2^* Y,$$
$$P(\mathcal{H}_2)Y\big|\mathcal{H}_1 = X,$$
$$|Y| \leqslant 1.$$

The following fact will be frequently used in the sequel: the minimal isometric dilation of a coisometry is a unitary operator. More precisely, if V is an isometry and W its minimal unitary extension then the minimal isometric dilation of V^* is the unitary operator W^*.

(1, 2) *The minimal isometric dilation of a coisometry is unitary.*

Proof. Let $V \in B(\mathcal{H})$ be a coisometry and let $U \in B(\mathcal{K})$ be its minimal isometric dilation. Thus $V^* = U^*\big|\mathcal{H}$, $\mathcal{K} = \bigvee_{n \geqslant 0} U^n \mathcal{H}$ and $U^*U = 1$; let us show that $UU^* = 1$. For $k \geqslant 1$ we have $(UU^* - 1)U^k h = 0$. It suffices to prove $UU^*h = h$ for all $h \in \mathcal{H}$. If $h \in \mathcal{H}$ we have $h = VV^*h = P(\mathcal{H})UU^*h$, in particular $|UU^*h| \leqslant |h| = |P(\mathcal{H})UU^*h|$. Hence $UU^*h \in \mathcal{H}$ and, consequently, $h = UU^*h$. □

(1, 3) *Suppose* U *is a contraction on a Hilbert space* $\mathcal{H}$. *Let* $\mathcal{H}_0$ *be a subspace of* $\mathcal{H}$ *invariant with respect to* U. *Denote by* $\mathcal{K}$ *the smallest* U^* *invariant subspace of* $\mathcal{H}$ *containing* $\mathcal{H}_0$, $\mathcal{K} = \mathrm{Inv}(U^*, \mathcal{H}_0)$.

If $U^*\big|\mathcal{K}$ *is isometric then* $\mathcal{K}$ *is reducing with respect to* U.

Proof. For $k \in \mathcal{K}$, we have

$$P(\mathcal{K})UU^*k = (U^*\big|\mathcal{K})^* U^*k = k.$$

It follows that

$$\begin{aligned}|P(\mathcal{K}^\perp)UU^*k|^2 &= |UU^*k|^2 - |P(\mathcal{K})UU^*k|^2 \\ &\leqslant |k|^2 - |P(\mathcal{K})UU^*k|^2 = |k|^2 - |k|^2 = 0.\end{aligned}$$

In particular, $UU^*k = k$. To prove that $U\mathscr{X} \subset \mathscr{X}$ it suffices to show that $UU^{*m}h_0 \in \mathscr{X}$ for every $h_0 \in \mathscr{H}_0$ and every $m \geqslant 0$. If $m = 0$, we have $Uh_0 \in \mathscr{H}_0 \subset \mathscr{X}$ by our assumption; for $m > 1$,

$$UU^{*m}h_0 = UU^*(U^{*m-1}h_0) = U^{*m-1}h_0 \in \mathscr{X}.$$

The proof is complete. □

2. The space $\mathscr{P}(X)$

This section is devoted to the construction of a representation of a Hilbert space which is spanned by two of its subspaces.

Given two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and a contraction $X : \mathscr{H}_1 \to \mathscr{H}_2$, we denote by $\mathscr{P}(X)$ the Hilbert space constructed as follows.

Consider the direct sum $\mathscr{H}_1 \oplus \mathscr{H}_2$ and the subspace $\mathscr{M} \subset \mathscr{H}_1 \oplus \mathscr{H}_2$

$$\mathscr{M} = \{(h_1, -Xh_1); h_1 \in \ker D(X)\}.$$

Define on $\mathscr{H}_1 \oplus \mathscr{H}_2$ a scalar product

$$\begin{aligned}(h_1 + h_2, h_1' + h_2')_{\mathscr{P}(X)} = (h_1, h_1')_{\mathscr{H}_1} + (h_2, h_2')_{\mathscr{H}_2} \\ + (h_2, Xh_1')_{\mathscr{H}_2} + (Xh_1, h_2')_{\mathscr{H}_2}.\end{aligned}$$

It is easy to verify that this scalar product is nonnegative definite and the corresponding quadratic form is

$$|h_1 + h_2|^2_{\mathscr{P}(X)} = |Xh_1 + h_2|^2_{\mathscr{H}_2} + |(1 - X^*X)^{1/2}h_1|^2_{\mathscr{H}_1}.$$

Its kernel turns out to be exactly $\mathscr{M}$; it follows that the scalar product is, in fact, a positive definite scalar product on $\mathscr{H}_1 \oplus \mathscr{H}_2/\mathscr{M}$. The space $\mathscr{P}(X)$ is defined as the completion of $\mathscr{H}_1 \oplus \mathscr{H}_2/\mathscr{M}$ in this scalar product.

Since

$$|h_1 \oplus h_2 + m|^2_{\mathscr{P}(X)} = |Xh_1 + h_2|^2_{\mathscr{H}_2} + |D(X)h_1|^2_{\mathscr{H}_1}$$

for every $m \in \mathscr{M}$ we have

$$\begin{aligned}|h_1 \oplus 0 + m|^2_{\mathscr{P}(X)} &= |Xh_1|^2_{\mathscr{H}_2} + |D(X)h_1|^2_{\mathscr{H}_1} = |h_1|^2_{H_1}, \\ |0 \oplus h_2 + m|^2_{\mathscr{P}(X)} &= |h_2|^2_{\mathscr{H}_2}\end{aligned}$$

In this manner the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are isometrically imbedded in $\mathcal{P}(X)$. More precisely, the mappings

$$h_1 \mapsto h_1 \oplus 0 + \mathcal{M}$$

and

$$h_2 \mapsto 0 \oplus h_2 + \mathcal{M}$$

realize isometric imbeddings of $\mathcal{H}_1$ and $\mathcal{H}_2$ into $\mathcal{P}(X)$.

Identifying the elements of $\mathcal{H}_1$ and $\mathcal{H}_2$ with the corresponding classes it is not difficult to see that the mappings P and Q defined by the relations

$$(1) \qquad \begin{aligned} P(h_1 + h_2) &= Xh_1 + h_2, \\ Q(h_1 + h_2) &= (1 - X)h_1 \end{aligned}$$

are indeed operators in $(\mathcal{H}_1 \oplus \mathcal{H}_2)/\mathcal{M}$. It turns out that the operator P is identical with the orthogonal projection onto $\mathcal{H}_2$ in $\mathcal{P}(X)$; and, $Q = P(\mathcal{H}_2^{\perp})$. To see that, it suffices to observe that, for each $h_1 \in \mathcal{H}_1$ and each $h_2 \in \mathcal{H}_2$ the sum $Xh_1 + h_2 \in \mathcal{H}_2$ and the product $((1 - X)h_1, h_2)_{\mathcal{P}(X)} = 0$. For the length of $(1 - X)h_1$ in $\mathcal{P}(X)$ we obtain

$$|(1 - X)h_1|^2_{\mathcal{P}(X)} = |D(X)h_1|^2_{\mathcal{H}_1}.$$

In particular, relations (1) yield $X = P(\mathcal{H}_2)|\mathcal{H}_1$. Summing up, we have obtained the following

(2, 4) *Let $X : \mathcal{H}_1 \to \mathcal{H}_2$ be a contraction. The space $\mathcal{P}(X)$ possesses the following properties*

1° *$\mathcal{P}(X)$ contains both $\mathcal{H}_1$ and $\mathcal{H}_2$ and is generated by their union*

2° *$P(\mathcal{H}_2)|\mathcal{H}_1 = X$.*

Remark. Having in mind the identifications of $\mathcal{H}_1$ and $\mathcal{H}_2$ made in the previous considerations it is easy to see that the space $\mathcal{P}(X)$ is the only Hilbert space satisfying 1° and 2°.

Indeed, let $\mathcal{X}$ be such that $\mathcal{X} = \mathcal{H}_1 \vee \mathcal{H}_2$ and such that $X = P(\mathcal{H}_2)|\mathcal{H}_1$. Then $\mathcal{H}_1 + \mathcal{H}_2$ is a dense set in both $\mathcal{X}$ and $\mathcal{P}(X)$. Moreover, for $h_1, h_2 \in \mathcal{H}$ we have

$$\begin{aligned} |h_1 + h_2|^2_{\mathcal{P}(X)} &= |h_1|^2_{\mathcal{H}_1} + 2\,\mathrm{Re}(Xh_1, h_2)_{\mathcal{H}_2} + |h_2|^2_{\mathcal{H}_2} \\ &= |h_1|^2_{\mathcal{X}} + 2\,\mathrm{Re}(Xh_1, h_2)_{\mathcal{X}} + |h_2|^2_{\mathcal{X}} \\ &= |h_1|^2_{\mathcal{X}} + 2\,\mathrm{Re}(P(\mathcal{H}_2)h_1, h_2)_{\mathcal{X}} + |h_2|^2_{\mathcal{X}} = |h_1 + h_2|^2_{\mathcal{X}}; \end{aligned}$$

thus the topology on $\mathcal{X}$ coincides with that on $\mathcal{P}(X)$.

This section is devoted to the solution of the following problem. We are given two operators $A_1 \in B(\mathscr{H}_1)$, $A_2 \in B(\mathscr{H}_2)$ and an operator $X: \mathscr{H}_1 \to \mathscr{H}_2$. What are the conditions for the existence of a Hilbert space $\mathscr{K}$ containing $\mathscr{H}_1$ and $\mathscr{H}_2$ and an operator $U \in B(\mathscr{K})$ such that $P(\mathscr{H}_2)|\mathscr{H}_1$ coincides with X and

$$U|\mathscr{H}_1 = A_1, \quad U^*|\mathscr{H}_2 = A_2.$$

The following proposition shows that there is a necessary condition connecting A_1, A_2 and X.

(3, 1) *Suppose $\mathscr{H}_1$, $\mathscr{H}_2$ are two Hilbert spaces and let operators $A_1 \in B(\mathscr{H}_1)$ and $A_2 \in B(\mathscr{H}_2)$ be given. Suppose there exists a Hilbert space $\mathscr{K}$ containing both $\mathscr{H}_1$ and $\mathscr{H}_2$ and an operator $U \in B(\mathscr{K})$ such that*

$$U|\mathscr{H}_1 = A_1,$$
$$U^*|\mathscr{H}_2 = A_2.$$

Then $X = P(\mathscr{H}_2)|\mathscr{H}_1$, satisfies

$$XA_1 = A_2^*X.$$

Proof. Observe that

$$A_2^* = P(\mathscr{H}_2)U|\mathscr{H}_2.$$

Since $\mathscr{H}_2$ is invariant with respect to U^* we have

$$P(\mathscr{H}_2)U = P(\mathscr{H}_2)UP(\mathscr{H}_2) = A_2^*P(\mathscr{H}_2).$$

It follows that

$$XA_1 = P(\mathscr{H}_2)U|\mathscr{H}_1 = A_2^*P(\mathscr{H}_2)|\mathscr{H}_1 = A_2^*X.$$

The proof is complete. □

It is to be expected that U or U^* will be uniquely determined, at least on $\mathscr{H}_1 + \mathscr{H}_2$, if more information is available. We have seen that alone the existence of a U implies an assertion about the position of $\mathscr{H}_1$ and $\mathscr{H}_2$ in $\mathscr{K}$. The following remark shows that, for a given $\mathscr{K}$, uniqueness of U or of U^* may be proved under additional conditions on the imbedding of $\mathscr{H}_1$ and $\mathscr{H}_2$ in $\mathscr{K}$.

(3,2) *Suppose* A_1, A_2, U *satisfy the assumptions of the preceding lemma. If, in addition,* $\mathcal{H}_2$ *is reducing with respect to* U *then* U *is uniquely determined on* $\mathcal{H}_1 \vee \mathcal{H}_2$,

$$U(h_1 + h_2) = A_1 h_1 + A_2^* h_2.$$

In the case that $\mathcal{H}_1$ *is reducing with respect to* U *the operator* U^* *is uniquely determined on* $\mathcal{H}_1 \vee \mathcal{H}_2$,

$$U^*(h_1 + h_2) = A_1^* h_1 + A_2 h_2.$$

The proof of this statement is based on the following two observations.

If $\mathcal{H}_2$ is reducing, we have $U\mathcal{H}_2 \subset \mathcal{H}_2$ so that

$$U|\mathcal{H}_2 = (U^*|\mathcal{H}_2)^* = A_2^*.$$

If $\mathcal{H}_1$ is reducing then $U^*\mathcal{H}_1 \subset \mathcal{H}_1$ whence

$$U^*|\mathcal{H}_1 = (U|\mathcal{H}_1)^* = A_1.$$

As the first step towards solving our problem let us impose the additional condition that $\mathcal{K}$ be as small as possible, i.e. $\mathcal{K} = \mathcal{H}_1 \vee \mathcal{H}_2$. We have seen in the preceding section $\mathcal{K}$ can then be identified with the space $\mathcal{P}(X)$. Also, if we are looking for U of norm not exceeding the maximum of norms of A_1 and A_2 we may restrict ourselves to the case of contractions and reformulate the problem as follows:

let $A_1 \in B(\mathcal{H}_1)$ and $A_2 \in B(\mathcal{H}_2)$ be contractions, let $X : \mathcal{H}_1 \to \mathcal{H}_2$ be a contraction satisfying $XA_1 = A_2^* X$.

Does there exist a contraction $U \in B(\mathcal{P}(X))$ such that $U|\mathcal{H}_1 = A_1$ and $U^*|\mathcal{H}_2 = A_2$? We intend to show that the answer is affirmative and we shall parametrize all contractive solutions.

(3,3) *Given contractions* $A_1 \in B(\mathcal{H}_1)$, $A_2 \in B(\mathcal{H}_2)$ *and a contraction* $X : \mathcal{H}_1 \to \mathcal{H}_2$ *such that*

$$XA_1 = A_2^* X$$

denote by $\mathscr{C}(A_1, A_2, X)$ *the set of all* $U \in B(\mathcal{P}(X))$ *such that* $U|\mathcal{H}_1 = A_1$ *and* $U^*|\mathcal{H}_2 = A_2$.

Then $\mathscr{C}(A_1, A_2, X)$ *contains at least one contraction and the set of all contractions in* $\mathscr{C}(A_1, A_2, X)$ *consists of all operators of the form*

$$A_2^* P(\mathcal{H}_2) + C(D(A_2^*)P(\mathcal{H}_2) + P(\mathcal{H}_2^{\perp})) \tag{2}$$

where $C\colon \mathscr{P}(X) \to \mathscr{H}_2^{\perp}$ is an arbitrary contraction such that $C(D(A_2^*)P(\mathscr{H}_2) + P(\mathscr{H}_2^{\perp}))|\mathscr{H}_1 = P(\mathscr{H}_2^{\perp})A_1$.

Proof. We observe first that each $U \in \mathscr{C}(A_1, A_2, X)$ satisfies

$$PU = A_2^*P$$

where P is the orthogonal projection of $\mathscr{P}(X)$ onto $\mathscr{H}_2$. Indeed, the relation $U^*|\mathscr{H}_2 = A_2$ implies $A_2^* = PU|\mathscr{H}_2$ and $U^*P = PU^*P$. It follows that

$$PU = PUP = A_2^*P.$$

Now suppose $U \in \mathscr{C}(A_1, A_2, X)$ is a contraction. Writing Q for $1 - P$, we have then, for each $x \in \mathscr{P}(X)$,

$$\begin{aligned} |A_2^*Px|^2 + |QUx|^2 &= |PUx|^2 + |QUx|^2 \\ &= |Ux|^2 \leqslant |Px|^2 + |Qx|^2. \end{aligned}$$

Thus

$$\begin{aligned} |QUx|^2 &\leqslant |D(A_2^*)Px|^2 + |Qx|^2 \\ &= |(D(A_2^*)P + Q)x|^2 \end{aligned}$$

for every $x \in \mathscr{P}(X)$. It follows that $QU = CK$ where $K = D(A_2^*)P + Q$ and C is a contraction mapping the closure of $\operatorname{Ran} K$ into $\mathscr{H}_2^{\perp}$. We have thus

$$\begin{aligned} U &= PU + QU = PU + CK \\ &= A_2^*P + CK. \end{aligned}$$

At the same time, the restriction $CK|\mathscr{H}_1$ satisfies the relation

$$CK|\mathscr{H}_1 = QA_1.$$

This is obvious since

$$QA_1 = QU|\mathscr{H}_1 = CK|\mathscr{H}_1.$$

Conversely, suppose we are given a contraction $C\colon \mathscr{P}(X) \longrightarrow \mathscr{H}_2^{\perp}$ such that $CK|\mathscr{H}_1 = QA_1$. Set $V = A_2^*P + CK$ and let us prove that V is a contraction and $V \in \mathscr{C}(A_1, A_2, X)$. To see that V is a contraction, we argue as follows. For every $x \in \mathscr{P}(X)$,

$$\begin{aligned} |Vx|^2 &= |A_2^*Px|^2 + |CKx|^2 \\ &\leqslant |A_2^*Px|^2 + |Kx|^2 = |A_2^*Px|^2 + |D(A_2^*)Px|^2 + |Qx|^2 \\ &= |Px|^2 + |Qx|^2 = |x|^2. \end{aligned}$$

For the restriction $V|\mathscr{H}_1$ we obtain

$$V|\mathscr{H}_1 = A_2^*P|\mathscr{H}_1 + CK|\mathscr{H}_1 = A_2^*X + QA_1$$
$$= XA_1 + QA_1 = PA_1 + QA_1 = A_1$$

To prove the identity $V^*|\mathscr{H}_2 = A_2$ we argue as follows:

Since $\operatorname{Ran} C \subset \mathscr{H}_2^\perp$ we have $PV = A_2^*P$. Taking adjoints we obtain $V^*P = PA_2$ and, consequently, $V^*|\mathscr{H}_2 = A_2$.

It remains to show that the set of contractions C from $\operatorname{Ran} K$ into $\mathscr{H}_2^\perp$ satisfying $CK|\mathscr{H}_1 = QA_1$ is non-empty. To this end it is sufficient to show that

$$|QA_1h_1| \leqslant |(D(A_2^*)P + Q)h_1|$$

for $h_1 \in \mathscr{H}_1$. Indeed,

$$|QA_1h_1|^2 = |A_1h_1 - XA_1h_1|^2 = |A_1h_1|^2 + |XA_1h_1|^2$$
$$- 2\operatorname{Re}(A_1h_1, XA_1h_1) = |A_1h_1|^2 + |XA_1h_1|^2 - 2|XA_1h_1|^2$$
$$= |A_1h_1|^2 - |XA_1h_1|^2 \leqslant |h_1|^2 - |A_2^*Xh_1|^2 = |Ph_1|^2 + |Qh_1|^2$$
$$- |A_2^*Ph_1|^2 = |D(A_2^*)Ph_1|^2 + |Qh_1|^2 = |(D(A_2^*)P + Q)h_1|^2.$$

The proof is complete. □

We have seen in (3, 3) that the set $\mathscr{C}(A_1, A_2, X)$ always contains contractions. It is not difficult to see that stronger postulates on U cannot be imposed unless additional assumptions are made concerning A_1, A_2 and X.

Clearly a $U \in \mathscr{C}$ cannot be isometric except in the case that A_1 is already isometric itself. Thus isometry of A_1 is a trivial necessary condition for the existence of isometric couplings.

The following proposition describes isometries in $\mathscr{C}(A_1, A_2, X)$.

(3, 4) *The mapping $U \in \mathscr{C}(A_1, A_2, X)$ given by formula (2) is an isometry if and only if the corresponding C is isometric on the range of $K = D(A_2^*)P(\mathscr{H}_2) + P(\mathscr{H}_2^\perp)$. In particular, a necessary condition for the existence of an isometry in $\mathscr{C}(A_1, A_2, X)$ is the inequality* $\dim \operatorname{Ran} K \leqslant \dim \mathscr{H}_2^\perp$.

If A_1 is an isometry then the operator $Kh_1 \longrightarrow QA_1h_1$ is an isometry mapping $K\mathscr{H}_1$ into $\mathscr{H}_2^\perp$. Then $\mathscr{C}$ contains an isometry if and only if

$$\dim(D(X)\mathscr{H}_1)^- \ominus D(X)A_1\mathscr{H}_1 \geqslant \dim(K\mathscr{P}(X))^- \ominus K\mathscr{H}_1$$

Proof. If $U = A_2^*P + CK$ is isometric then

$$\begin{aligned}0 = |x|^2 - |Ux|^2 &= |x|^2 - |A_2^*Px|^2 - |CKx|^2 \\ &= |Px|^2 + |Qx|^2 - |A_2^*Px|^2 - |CKx|^2 \\ &= |Qx|^2 + |D(A_2^*)Px|^2 - |CKx|^2 = |Kx|^2 - |CKx|^2\end{aligned}$$

for all $x \in \mathscr{K}$. This proves the first assertion.

For $h_1 \in \mathscr{H}_1$ we have

$$A_2^*Ph_1 = A_2^*Xh_1 = XA_1h_1 = PA_1h_1.$$

Consequently,

$$\begin{aligned}|Kh_1|^2 &= |D(A_2^*)Ph_1|^2 + |Qh_1|^2 \\ &= |Ph_1|^2 - |A_2^*Ph_1|^2 + |Qh_1|^2 \\ &= |h_1|^2 - |PA_1h_1|^2.\end{aligned}$$

If A_1 is an isometry it follows that

$$|Kh_1|^2 = |A_1h_1|^2 - |PA_1h_1|^2 = |QA_1h_1|^2;$$

consequently

$$Kh_1 \to QA_1h_1$$

defines an isometric mapping of $(K\mathscr{H}_1)^-$ into $\mathscr{H}_2^\perp$. This isometry can be extended to an isometry of $(K\mathscr{P}(X))^-$ into $\mathscr{H}_2^\perp$ if and only if $\dim \mathscr{H}_2^\perp \ominus QA_1\mathscr{H}_1 \geqslant \dim(K\mathscr{P}(X))^- \ominus K\mathscr{H}_1$. It remains to show that

$$\dim \mathscr{H}_2^\perp \ominus QA_1\mathscr{H}_1 = \dim(D(X)\mathscr{H}_1)^- \ominus (D(X)A_1\mathscr{H}_1)^-.$$

To this end it is sufficient to observe that

$$|P(\mathscr{H}_2^\perp)h_1|^2_{\mathscr{P}(X)} = |(1-X)h_1|^2_{\mathscr{P}(X)} = |h_1|^2_{\mathscr{H}_1} - |Xh_1|^2_{\mathscr{H}_1} = |D(X)h_1|^2_{\mathscr{H}_1}$$

for all $h_1 \in \mathscr{H}_1$. □

Using (3,4) it is easy to show that isometry of A_1 alone does not guarantee the existence of an isometry in $\mathscr{C}(A_1, A_2, X)$.

Example. Let $\mathscr{H}_1$, $\mathscr{H}_2$ be nonzero Hilbert spaces, let $A_1 \in B(\mathscr{H}_1)$ be unitary, $A_2 = 0$ and $X = 0$. Then $D(X) = 1_{\mathscr{H}_1}$, $D(A_2^*) = 1_{\mathscr{H}_2}$, $\mathscr{H}_1 \perp \mathscr{H}_2$ and $\mathscr{P}(X) = \mathscr{H}_1 \oplus \mathscr{H}_2$. Further, $\mathscr{K} = D(A_2^*)P(\mathscr{H}_2) + P(\mathscr{H}_2^\perp) = 1_{\mathscr{H}_1 \oplus \mathscr{H}_2}$ whence $(K\mathscr{P}(X))^- \ominus K\mathscr{H}_1 = \mathscr{H}_2$. On the other hand, $(D(X)\mathscr{H}_1)^- \ominus D(X)A_1\mathscr{H}_1 = \mathscr{H}_1 \ominus \mathscr{H}_1 = (0)$.

(4, 1) *Let U be a contraction on a Hilbert space $\mathscr{K}$. Suppose $\mathscr{H}_1$ is a subspace of $\mathscr{K}$ invariant with respect to U and $\mathscr{H}_2$ a subspace of $\mathscr{K}$ invariant with respect to U^*.*

Then the operator $X = P(\mathscr{H}_2)|\mathscr{H}_1$ satisfies

$$X(U|\mathscr{H}_1) = (U^*|\mathscr{H}_2)^* X.$$

Set

$$\mathscr{K}_1 = \operatorname{Inv}(\mathscr{H}_1, U^*) \text{ and } \mathscr{K}_2 = \operatorname{Inv}(\mathscr{H}_2, U)$$

and suppose that the restrictions $U^*|\mathscr{K}_1$ and $U|\mathscr{K}_2$ are isometric. Then

(i) *the spaces $\mathscr{K}_1$ and $\mathscr{K}_2$ are reducing with respect to U,*

(ii) *the operator X satisfies the estimate*

$$|(Xh_1, h_2)| \leqslant |A(U|\mathscr{H}_1)h_1|\,|A(U^*|\mathscr{H}_2)h_2|$$

for all $h_1 \in \mathscr{H}_1$, $h_2 \in \mathscr{H}_2$.

Moreover, if $U|\mathscr{H}_1$ is an isometry then $U|\mathscr{K}_1 \vee \mathscr{K}_2$ is an isometry. If $U^*|\mathscr{H}_2$ is an isometry then $U^*|\mathscr{K}_1 \vee \mathscr{K}_2$ is an isometry. If both $U|\mathscr{H}_1$ and $U^*|\mathscr{H}_2$ are isometric then U is unitary on $\mathscr{K}_1 \vee \mathscr{K}_2$.

Proof. The first assertion is an immediate consequence of lemma (1, 3). If $X = P(\mathscr{H}_2)|\mathscr{H}_1$, the relation

$$X(U|\mathscr{H}_1) = (U^*|\mathscr{H}_2)^* X$$

can be obtained applying lemma (3, 1).

Choose natural numbers n, m and set

$$P_1 = U^m U^{*m} \text{ and } P_2 = U^{*n} U^n.$$

Since $\mathscr{K}_1$ and $\mathscr{K}_2$ are U-reducing it follows that both P_1 and P_2 map $\mathscr{K}_1$ into $\mathscr{K}_2$. This implies that

$$P_1 P_2|\mathscr{K}_1 = P_2|\mathscr{K}_1.$$

Since $U^*|\mathscr{K}_1$ and $U|\mathscr{K}_2$ are isometric operators and $\mathscr{K}_1$, $\mathscr{K}_2$ are reducing it follows that

$$UU^*|\mathscr{K}_1 = 1_{\mathscr{K}_1} \text{ and } U^*U|\mathscr{K}_2 = 1_{\mathscr{K}_2}.$$

As a consequence of these facts, we get, for $h_1 \in \mathscr{H}_1$, $h_2 \in \mathscr{H}_2$,

$$\begin{aligned}(Xh_1, h_2) &= (h_1, h_2) = (h_1, P_2h_2) = (P_2h_1, h_2)\\ &= (P_1P_2h_1, h_2) = (P_2h_1, P_1h_2) = (U^{*n}U^nh_1, U^mU^{*m}h_2)\\ &= (U^{*n}(U|\mathscr{H}_1)^nh_1, U^m(U^*|\mathscr{H}_2)^mh_2).\end{aligned}$$

Thus

$$|(Xh_1, h_2)| \leqslant |(U|\mathscr{H}_1)^nh_1|\,|(U^*|\mathscr{H}_2)^mh_2|$$

so that, passing to the limit, we obtain the estimate

$$|(Xh_1, h_2)| \leqslant |A(U|\mathscr{H}_1)h_1|\,|A(U^*|\mathscr{H}_2)h_2|.$$

Further, for $k_1 \in \mathscr{K}_1$, $k_2 \in \mathscr{K}_2$, we have

$$\begin{aligned}|U(k_1 + k_2)|^2 &= |Uk_1|^2 + |Uk_2|^2 + 2\operatorname{Re}(Uk_1, Uk_2)\\ &= (U^*Uk_1, k_1) + (U^*Uk_2, k_2) + 2\operatorname{Re}(k_1, U^*Uk_2)\\ &= |k_1|^2 + |k_2|^2 - |(1 - U^*U)^{1/2}k_1|^2 + 2\operatorname{Re}(k_1, k_2)\\ &= |k_1 + k_2|^2 - |(1 - U^*U)^{1/2}k_1|^2.\end{aligned}$$

If $U|\mathscr{H}_1$ is an isometry its minimal coisometric extension $U|\operatorname{Inv}(\mathscr{H}_1, U^*)$ is a unitary operator according to (1,2). In other words, $U^*U|\mathscr{K}_1 = 1_{\mathscr{K}_1}$ and, consequently, $U|\mathscr{K}_1 \vee \mathscr{K}_2$ is an isometry. Similarly,

$$\begin{aligned}|U^*(k_1 + k_2)|^2 &= |U^*k_1|^2 + |U^*k_2|^2 + 2\operatorname{Re}(U^*k_1, U^*k_2)\\ &= (UU^*k_1, k_1) + (UU^*k_2, k_2) + 2\operatorname{Re}(UU^*k_1, k_2)\\ &= |k_1 + k_2|^2 - |(1 - UU^*)^{1/2}k_2|^2.\end{aligned}$$

□

If $U^*|\mathscr{H}_2$ is an isometry then $U^*|\operatorname{Inv}(\mathscr{H}_2, U)$ is again a unitary operator according to (1,2), in other words, $UU^*|\mathscr{K}_2 = 1_{\mathscr{K}_2}$. Consequently, $U^*|\mathscr{K}_1 \vee \mathscr{K}_2$ is an isometry.

(4,2) *Let $\mathscr{H}_1$, $\mathscr{H}_2$ be two Hilbert spaces and suppose $A_1 \in B(\mathscr{H}_1)$, $A_2 \in B(\mathscr{H}_2)$ are contractions. Let $X : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator. Then the following assertions are equivalent.*

1° *There exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}_1$ and $\mathscr{H}_2$ and a contraction $U \in B(\mathscr{K})$ with the following properties:*

$$\mathscr{H}_1 \text{ is } U \text{ invariant and } U|\mathscr{H}_1 = A_1,$$

$$\mathscr{H}_2 \text{ is } U^* \text{ invariant and } U^*|\mathscr{H}_2 = A_2,$$

$$X = P(\mathcal{H}_2)|\mathcal{H}_1.$$

The restrictions $U^*|\operatorname{Inv}(\mathcal{H}_1, U^*)$ *and* $U|\operatorname{Inv}(\mathcal{H}_2, U)$ *are isometric operators and* $\mathcal{K} = \operatorname{Inv}(\mathcal{H}_1, U^*) \vee \operatorname{Inv}(\mathcal{H}_2, U)$.

2° *The operator* X *satisfies the intertwining relation*

$$XA_1 = A_2^* X$$

and the estimate

$$|(Xh_1, h_2)| \leqslant |A(A_1)h_1|\,|A(A_2)h_2|$$

for all $h_1 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_2$.

Proof. The implication 1° → 2° is a part of the preceding proposition. The proof of the converse will proceed in several steps.

According to (1,1) there exist spaces $\mathcal{K}_1$, $\mathcal{K}_2$, coisometries $U_1 \in B(\mathcal{K}_1)$, $U_2 \in B(\mathcal{K}_2)$ such that $\mathcal{K}_1 = \operatorname{Inv}(\mathcal{H}_1, U_1^*)$, $\mathcal{K}_2 = \operatorname{Inv}(\mathcal{H}_2, U_2^*)$ and an operator $\widetilde{X} : \mathcal{K}_1 \to \mathcal{K}_2$ with the following properties:

$$\begin{aligned} \widetilde{X}U_1 &= U_2^* \widetilde{X}, \\ U_1|\mathcal{H}_1 &= A_1, \\ U_2|\mathcal{H}_2 &= A_2, \\ |\widetilde{X}| &\leqslant 1 \end{aligned}$$

and

$$X = P(\mathcal{H}_2)\widetilde{X}|\mathcal{H}_1.$$

Set $\mathcal{K} = \mathcal{P}(\widetilde{X})$ so that $\mathcal{K} = \mathcal{K}_1 \vee \mathcal{K}_2$ contains $\mathcal{K}_1$ and $\mathcal{K}_2$ and, consequently, contains $\mathcal{H}_1$ and $\mathcal{H}_2$. Also, $\widetilde{X} = P(\mathcal{K}_2)|\mathcal{K}_1$ in $\mathcal{K}$ whence $X = P(\mathcal{H}_2)\widetilde{X}|\mathcal{H}_1 = P(\mathcal{H}_2)|\mathcal{H}_1$. We have then, for $k_1 \in \mathcal{K}_1$, $k_2 \in \mathcal{K}_2$

$$\begin{aligned} |U_1k_1 + U_2^*k_2|^2 &= |U_1k_1|^2 + |U_2^*k_2|^2 + 2\operatorname{Re}(U_1k_1, U_2^*k_2) \\ &\leqslant |k_1|^2 + |k_2|^2 + 2\operatorname{Re}(\widetilde{X}U_1k_1, U_2^*k_2) \\ &= |k_1|^2 + |k_2|^2 + 2\operatorname{Re}(U_2^*\widetilde{X}k_1, U_2^*k_2) \\ &= |k_1|^2 + |k_2|^2 + 2\operatorname{Re}(\widetilde{X}k_1, k_2) = |k_1 + k_2|^2. \end{aligned}$$

It follows that the mapping

$$U_0 : k_1 + k_2 \to U_1k_1 + U_2^*k_2$$

is well-defined and contractive. Denote by U the extension by continuity of U_0 to the whole of $\mathcal{K}$.

We shall need an explicit expression for U^*. Consider fixed elements $k_1, k_1' \in \mathscr{K}_1$, $k_2, k_2' \in \mathscr{K}_2$. Using the definition of the scalar product in $\mathscr{P}(\tilde{X})$ and the relation $\tilde{X}U_1 = U_2^*\tilde{X}$ we obtain

$$\begin{aligned}(U(k_1+k_2),k_2') &= (U_1k_1 + U_2^*k_2, k_2')\\ &= (\tilde{X}U_1k_1, k_2') + (k_2, U_2k_2')\\ &= (U_2^*\tilde{X}k_1, k_2') + (k_2, U_2k_2')\\ &= (\tilde{X}k_1, U_2k_2') + (k_2, U_2k_2')\\ &= (k_1 + k_2, U_2k_2').\end{aligned}$$

Since U_1 and U_2 are coisometries we also have the relation $\tilde{X}U_1^* = U_2\tilde{X}$; we shall use it to compute U^*k_1'.

$$\begin{aligned}(U(k_1+k_2),k_1') &= (U_1k_1 + U_2^*k_2, k_1')\\ &= (k_1, U_1^*k_1') + (U_2^*k_2, \tilde{X}k_1')\\ &= (k_1, U_1^*k_1') + (k_2, U_2\tilde{X}k_1')\\ &= (k_1, U_1^*k_1') + (k_2, \tilde{X}U_1^*k_1')\\ &= (k_1 + k_2, U_1^*k_1').\end{aligned}$$

The expression for U^* is thus

$$U^*(k_1' + k_2') = U_1^*k_1' + U_2k_2'.$$

It follows that both $\mathscr{K}_1$ and $\mathscr{K}_2$ are reducing with respect to U and

$$U|\mathscr{K}_1 = U_1, \quad U^*|\mathscr{K}_2 = U_2.$$

Consequently,

$$\mathscr{K}_1 = \mathrm{Inv}(\mathscr{H}_1, U_1^*) = \mathrm{Inv}(\mathscr{H}_1, U^*), \qquad U|\mathscr{H}_1 = U_1|\mathscr{H}_1 = A_1$$

and

$$\mathscr{K}_2 = \mathrm{Inv}(\mathscr{H}_2, U_2^*) = \mathrm{Inv}(\mathscr{H}_2, U), \qquad U^*|\mathscr{H}_2 = U_2|\mathscr{H}_2 = A_2.$$

□

(4,3) **Corollary.** *Suppose $A_1 \in B(\mathscr{H}_1)$ is an isometry, $A_2 \in B(\mathscr{H}_2)$ a contraction. Suppose further that the bounded operator $X : \mathscr{H}_1 \to \mathscr{H}_2$ satisfies*

$$XA_1 = A_2^*X,$$
$$|X^*h_2| \leqslant |A(A_2)h_2| \quad \text{for } h_2 \in \mathscr{H}_2.$$

Then there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}_1$ and $\mathscr{H}_2$ and an isometry $U \in B(\mathscr{K})$ such that

$$U|\mathscr{H}_1 = A_1,$$
$$U^*|\mathscr{H}_2 = A_2,$$
$$X = P(\mathscr{H}_2)|\mathscr{H}_1.$$

Proof. Since $A(A_1) = 1$, we have

$$|(Xh_1, h_2)| \leqslant |A(A_1)h_1|\,|A(A_2)h_2|.$$

It follows from the preceding theorem that there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}_1$ and $\mathscr{H}_2$ and a contraction $U \in B(\mathscr{K})$ with the following properties (we write, for brevity, $\mathscr{K}_1 = \operatorname{Inv}(\mathscr{H}_1, U^*)$, $\mathscr{K}_2 = \operatorname{Inv}(\mathscr{H}_2, U)$):

$$U|\mathscr{H}_1 = A_1,$$
$$U^*|\mathscr{H}_2 = A_2,$$
$$X = P(\mathscr{H}_2)|\mathscr{H}_1,$$

the restrictions $U^*|\mathscr{K}_1$ and $U|\mathscr{K}_2$ are isometries and $\mathscr{K} = \mathscr{K}_1 \vee \mathscr{K}_2$. Since $A_1 = U|\mathscr{H}_1$ is an isometry it follows, according to (4, 1), that U is also an isometry. □

Similarly we argue to obtain the following.

(4, 4) **Corollary.** *Suppose $A_1 \in B(\mathscr{H}_1)$ is a contraction, $A_2 \in B(\mathscr{H}_2)$ an isometry. Suppose that $X : \mathscr{H}_1 \to \mathscr{H}_2$ satisfies*

$$XA_1 = A_2^* X,$$

$$|Xh_1| \leqslant |A(A_1)h_1| \quad \text{for } h_1 \in \mathscr{H}_1.$$

Then there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}_1$ and $\mathscr{H}_2$ and a coisometry $U \in B(\mathscr{K})$ such that

$$U|\mathscr{H}_1 = A_1,$$
$$U^*|\mathscr{H}_2 = A_2,$$
$$X = P(\mathscr{H}_2)|\mathscr{H}_1.$$

(4, 5) **Corollary.** *Suppose $A_1 \in B(\mathscr{H}_1)$, $A_2 \in B(\mathscr{H}_2)$ are isometries, $X : \mathscr{H}_1 \to \mathscr{H}_2$ satisfies*

$$XA_1 = A_2^* X.$$

Then there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}_1$ and $\mathscr{H}_2$ and a unitary operator $U \in B(\mathscr{K})$ such that

$$U|\mathscr{H}_1 = A_1,$$
$$U^*|\mathscr{H}_2 = A_2,$$
$$X = P(\mathscr{H}_2)|\mathscr{H}_1.$$

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Authors' address: Institute of Mathematics, Czechoslovak Academy of Sciences, Žitná 25, 115 67 Praha 1, Czechoslovakia.