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James G. Raftery; Teo Sturm

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TOLERANCE NUMBERS, CONGRUENCE n-PERMUTABILITY AND BCK-ALGEBRAS

J. G. RAFTERY, Pietermaritzburg, and T. STURM, Durban

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INTRODUCTION

Tolerance relations (i.e., reflexive, symmetric and compatible binary relations) on algebras have attracted some attention in the literature of the last decade. (For example, see [3], [4], [5], [6] and other papers of Chajda; for results on weaker compatible relations, see [8], [14] and [15]; for the case of lattices, see [1] and [14].)

In this paper, we define the tolerance number $\mathrm{tn}(\mathbf{A})$ of an algebra $\mathbf{A}$ as the least positive integer n such that for any tolerance relation τ on $\mathbf{A}$, the n-th relational power τ^n of τ is transitive (i.e., τ^n is congruence on $\mathbf{A}$), provided such an n exists; otherwise we define $\mathrm{tn}(\mathbf{A})$ to be ω. We also define the tolerance number of a class of algebras of the same type as the supremum of the tolerance numbers of the algebras in the class. We prove that if $\mathrm{tn}(\mathbf{A}) = n$ then $\mathbf{A}$ is congruence $(n+1)$-permutable. (The converse fails.) The proof yields a Mal'cev-type characterization of the local condition "$\mathrm{tn}(\mathbf{A}) \leqslant n$" and the result implies that the varieties with tolerance number at most n are just the congruence $(n+1)$-permutable varieties. These facts generalize earlier descriptions of "tolerance trivial" algebras and varieties in [4], [6], [12] and [15].

The quasi-variety of all BCK-algebras, which is not a variety and which has no nontrivial congruence permutable subvariety, is a good case study. It turns out that every nontrivial variety of BCK-algebras has tolerance number 2, yet every nonzero countable cardinal is the tolerance number of some BCK-algebra. The theory of BCK-tolerances is applied to obtain characterizations of varieties of BCK-algebras.

We denote by ω the set of all non-negative integers, and by $\mathscr{F} = (F, ar)$ an arbitrary but fixed type of (universal) algebras with a set F of operation symbols and an arity function $\mathrm{ar}\colon F \to \omega$. All algebras considered in this section are assumed to be of type $\mathscr{F}$. We denote by $\mathbf{A} = (A; F)$ and by K a given algebra and a given class of algebras respectively. For binary relations τ, $\eta \subseteq A^2$ we write $\tau\eta$ for the relational product of τ and η and we define

$$\tau^0 = \mathrm{id}_A := \{(a,a)\colon a \in A\}; \ \tau^{n+1} = \tau^n\tau \quad (n \in \omega).$$

A *tolerance relation* (briefly a *tolerance*) on $\mathbf{A}$ is a binary reflexive and symmetric relation on A which is compatible with every operation in F. (A congruence on $\mathbf{A}$ is therefore just a transitive tolerance on $\mathbf{A}$.) We write $\mathrm{Tol}\,\mathbf{A}$ (resp. $\mathrm{Con}\,\mathbf{A}$) for the set of all tolerances (resp. congruences) on $\mathbf{A}$. Both of these are algebraic closure systems on the lattice of subsets of A^2 and hence algebraic lattices when ordered by set inclusion. The corresponding algebraic closure operators are denoted by T (resp. Θ). We write $T((a_1,b_1),\ldots,(a_n,b_n))$ for $T(\{(a_1,b_1),\ldots,(a_n,b_n)\})$. It is well known that

$$\Theta(\eta) = \bigcup_{n\in\omega}(T(\eta))^n \qquad (\eta \subseteq A^2).$$

We define the *tolerance number* $\mathrm{tn}(\mathbf{A})$ of $\mathbf{A}$ and the *tolerance number* $\mathrm{tn}(K)$ of K by:

$$\mathrm{tn}(\mathbf{A}) = \begin{cases} \min\{n\colon 0 < n \in \omega \text{ and } \tau^n \in \mathrm{Con}\,\mathbf{A} \text{ for every } \tau \in \mathrm{Tol}\,\mathbf{A}\} & \text{if this exists;} \\ \omega \text{ otherwise;} \end{cases}$$

$$\mathrm{tn}(K) = \sup\{\mathrm{tn}(\mathbf{B})\colon \mathbf{B} \in K\}$$

(where the supremum is taken in the well-ordered class of all ordinals). We therefore have $\mathrm{tn}(\emptyset) = 0$ and

$$1 \leqslant \mathrm{tn}(\mathbf{A}),\ \mathrm{tn}(K) \leqslant \omega \qquad (\text{for } K \neq \emptyset).$$

If $\mathrm{tn}(\mathbf{A}) = 1$, i.e. $\mathrm{Tol}\,\mathbf{A} = \mathrm{Con}\,\mathbf{A}$, we say that $\mathbf{A}$ is *tolerance trivial.* In general we say that K possesses a property of algebras if every element of K possesses this property.

The symbols n, m shall denote elements of ω throughout. By an *n-ary algebraic function* ($n > 0$) on $\mathbf{A}$ we shall mean an n-ary operation $G\colon A^n \to A$ such that for some $r \in \omega$, some $(n+r)$-ary $\mathscr{F}$-term t and some elements $b_1, \ldots, b_r \in A$, we have

$$G(a_1,\ldots,a_n) = t(a_1,\ldots,a_n,b_1,\ldots,b_r) \qquad (a_1,\ldots a_n \in A).$$

Our main result in this section (Theorem 1.2) extends results proved in [4], [6], [12], [15].

1.1. Lemma. (Chajda [3, Lemma 2]). *Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$. For $c, d \in A$, we have $(c,d) \in T((a_1,b_1), \ldots, (a_n,b_n))$ if and only if for some 2n-ary algebraic function G on* **A**, we have

$$G(a_1, \ldots, a_n, b_1, \ldots, b_n) = c$$
$$G(b_1, \ldots, b_n, a_1, \ldots, a_n) = d.$$

1.2. Theorem. *If* $\mathrm{tn}(\mathbf{A}) = n$ *then* **A** *is congruence* $(n+1)$*-permutable.*

Proof. For convenience, we assume that n is odd. The even case requires minor notational modification only. Suppose that $\mathrm{tn}(\mathbf{A}) = n$ and that $\theta, \varphi \in \mathrm{Con}\,\mathbf{A}$ with

$$(a,b) \in \underbrace{\theta\varphi\theta\varphi\ldots\theta\varphi}_{n+1 \text{ terms}}.$$

Then for some $c_0, c_1, \ldots, c_n, c_{n+1} \in A$, we have

$$a = c_0\theta c_1\varphi c_2 \ldots c_{n-1}\theta c_n\varphi c_{n+1} = b. \tag{1}$$

Thus, if $\tau = T((c_0,c_1),(c_1,c_2),\ldots,(c_{n-1},c_n),(c_n,c_{n+1}))$, we have $(a,b) \in \tau^{n+1}$. But $\tau^{n+1} = \tau^n$ by assumption, so there exist $d_0, d_1, \ldots, d_n \in A$ such that

$$a = d_0\tau d_1\tau d_2 \ldots d_{n-1}\tau d_n = b.$$

By the previous lemma, there exist $(2n+2)$-ary algebraic functions $G_1, \ldots, G_n$ on **A** such that

$$d_{i-1} = G_i(c_0, c_1, c_2, \ldots, c_{n-1}, c_n\,;\, c_1, c_2, c_3, \ldots, c_n, c_{n+1}),$$
$$d_i = G_i(c_1, c_2, c_3, \ldots, c_n, c_{n+1}\,;\, c_0, c_1, c_2, \ldots, c_{n-1}, c_n)$$

for $i = 1, \ldots, n$. Now since congruences on **A** are compatible with all of the G_i, it follows from (1) that

$$d_{i-1}\varphi G_i(c_0, c_2, c_2, c_4, c_4, c_6, \ldots, c_{n-1}, c_{n-1}, c_{n+1};$$
$$c_1, c_1, c_3, c_3, c_5, c_5, \ldots, c_{n-2}, c_n, c_n)\theta d_i$$

and

$$d_{i-1}\theta G_i(c_1, c_1, c_3, c_3, c_5, c_5, \dots, c_{n-2}, c_n, c_n;$$
$$c_0, c_2, c_2, c_4, c_4, c_6, \dots, c_{n-1}, c_{n-1}, c_{n+1})\varphi d_i$$

for $i = 1, \dots, n$, so that

$$\begin{aligned}(a, b) \in (\varphi\theta \cap \theta\varphi)^n &\subseteq \left((\varphi\theta)(\theta\varphi)\right)^{(n-1)/2}(\varphi\theta)\\ &= \varphi\theta\varphi\theta\dots\varphi\theta \qquad (n+1 \ \text{terms}).\end{aligned}$$

□

1.3. Corollary. *Let K be a variety of algebras. Then* $\mathrm{tn}(K) = n$ *if and only if n is the least positive integer such that K is congruence $(n+1)$-permutable.*

Proof. ($\Leftarrow$) follows from Theorem 1.2 and Hagemann's result (see [8, p. 8]) to the effect that a variety K is congruence $(n+1)$-permutable if and only if for every $\mathbf{B} \in K$ and every reflexive subalgebra τ of $\mathbf{B}^2$, we have $\tau^{n+1} \subseteq \tau^n$.

($\Rightarrow$) follows from ($\Leftarrow$) and Theorem 1.2. □

1.4. Corollary [12]. *Every tolerance trivial algebra is congruence permutable.*

Proof. Set $n = 1$ in Theorem 1.2. □

1.5. Corollary [4], [6], [15]. *A variety of algebras is tolerance trivial if and only if it is congruence permutable.*

Proof. Set $n = 1$ in Corollary 1.3. □

1.6. Corollary. *The following conditions are equivalent:*

(i) $\mathrm{tn}(\mathbf{A}) \leqslant n$;

(ii) *for any $c_0, c_1, \dots, c_n, c_{n+1} \in A$, there exist $d_0, d_1, \dots, d_n \in A$ and $(2n+2)$-ary algebraic functions $G_1, \dots, G_n$ on $\mathbf{A}$ such that $d_0 = c_0$, $d_n = c_{n+1}$ and*

$$d_{i-1} = G_i(c_0, c_1, \dots, c_{n-1}, c_n; c_1, c_2, \dots, c_n, c_{n+1}),$$
$$d_i = G_i(c_1, c_2, \dots, c_n, c_{n+1}; c_0, c_1, \dots, c_{n-1}, c_n)$$

for $i = 1, \dots, n$.

Proof. (i) $\Rightarrow$ (ii) is implicit in the proof of Theorem 1.2; (ii) $\Rightarrow$ (i) follows easily. □

1.7. Corollary. [12] *The algebra* $\mathbf{A}$ *is tolerance trivial if and only if for any* $a, b, c \in A$, *there is a 4-ary algebraic function* G *on* $\mathbf{A}$ *such that* $a = G(a, c, c, b)$ *and* $b = G(c, b, a, c)$.

Proof. Set $n = 1$ in Corollary 1.6. □

The converse of Corollary 1.4 is false: see [4] and [12, Remark 2.18 a].

II. BCK-ALGEBRAS

We now fix the type $\mathscr{F} = (F, ar)$ with $F = \{., 0\}$, $ar(\cdot) = 2$ and $ar(0) = 0$. We make standard use of the symbols $\mathbf{H}, \mathbf{I}, \mathbf{S}$ and $\mathbf{P}$ to denote class operators acting on classes K of $\mathscr{F}$-algebras (see e.g., [2, Chapter II, §9]). If $\mathbf{A} = (A; ., 0)$ is an $\mathscr{F}$-algebra, we write $\mathbf{H}(\mathbf{A})$ for $\mathbf{H}(\{\mathbf{A}\})$ (the class of all $\mathscr{F}$-homomorphic images of $\mathbf{A}$) and for $a, b \in A$, we abbreviate $a.b$ as ab, except where this may cause confusion.

A BCK-*algebra* is a $\mathscr{F}$-algebra satisfying the axioms:

$$\begin{array}{ll} \text{BCK (I)} & ((xy)(xz))(zy) = 0, \\ \text{BCK (II)} & (x(xy))y = 0, \\ \text{BCK (III)} & xx = 0, \\ \text{BCK (IV)} & 0x = 0, \\ \text{BCK (V)} & xy = yx = 0 \Rightarrow x = y. \end{array}$$

We denote by BCK the class of all BCK-algebras. (We assume some familiarity with these algebras: see survey articles [7], [11].) Clearly BCK is a quasi-variety of type $\mathscr{F}$. By a *BCK-variety*, we mean a variety V of type $\mathscr{F}$ such that $V \subseteq$ BCK. BCK itself is not a BCK-variety [16]; moreover, no nontrivial BCK-variety is congruence permutable [7, Theorem 4.3]. In view of the results of Section I, this makes BCK an interesting case study with respect to tolerances.

Henceforth $\mathbf{A} = (A; \cdot, 0)$ shall denote a given BCK-algebra. The relation $\leqslant$ on A, defined by $x \leqslant y$ iff $xy = 0$, is a partial order on A with least element 0, and $\mathbf{A}$ satisfies $xy \leqslant x$ (see [11]). An *ideal* of $\mathbf{A}$ is a subset I of A with $0 \in I$ such that $a \in I$ whenever $ab, b \in I$. The ideals of $\mathbf{A}$ are hereditary subsets of A and form a complete lattice, denoted $\operatorname{Id}\mathbf{A}$ (ordered by set inclusion).

Let a, b, $b_1, \ldots, b_n$, $b_{n+1} \in A$. We define inductively:

$$ab_1 \ldots b_n b_{n+1} = (ab_1 \ldots b_n)b_{n+1} \tag{2}$$

and we abbreviate this expression as $a \prod_{i=1}^{n+1} b_i$. The order of the b_i is immaterial in (2) however, in view of the BCK-identity $xyz = xzy$ [11, Theorem 1]. More generally, let $B = (b_j\,;\, j \in J)$ be a finite family in A. (Recall that a *family* is just another name for the mapping $j \mapsto b_j$ $(j \in J)$. The *range* of B is $\{b_j : j \in J\}$. We say that B is a family *in* a set C if its range is a subset of C. The family B is said to be *finite* if J is a finite set.) We may now define (without ambiguity):

$$a \amalg B = \begin{cases} a \prod\limits_{j \in J} b_j & \text{if } J \neq \emptyset; \\ a & \text{if } J = \emptyset. \end{cases}$$

We also define $ab^0 = a$; $ab^{n+1} = (ab^n)b$ $(n \in \omega)$.

If $C \subseteq A$, we denote by $\langle C \rangle$ or $\langle C \rangle_{\mathbf{A}}$ the ideal of $\mathbf{A}$ generated by C, i.e., $\langle C \rangle_{\mathbf{A}} = \bigcap\{I : C \subseteq I \in \operatorname{Id} \mathbf{A}\}$. Recall that $\langle \emptyset \rangle_{\mathbf{A}} = \{0\}$ and that for $C \neq \emptyset$, we have

$$\text{(3)} \qquad \langle C \rangle_{\mathbf{A}} = \{a \in A : a \amalg D = 0 \text{ for some finite family } D \text{ in } C\}$$

[10, Theorem 3]. If $C = \{c_1, \ldots, c_n\}$, we write $\langle c_1, \ldots, c_n \rangle$ for $\langle C \rangle$.

For $\eta \in \operatorname{Tol} \mathbf{A}$, we call $0/\eta := \{a \in A : (a, 0) \in \eta\}$ the *kernel* of η. We have $0/\eta \in \operatorname{Id} \mathbf{A}$, by [12, Theorem 2.2 c]. On the other hand, for $I \in \operatorname{Id} \mathbf{A}$ we define

$$\varphi_I = \{(a, b) \in A^2 : ab, ba \in I\},$$
$$\tau_I = \bigcap\{\eta \in \operatorname{Tol} \mathbf{A} : 0/\eta = I\},$$
$$\theta_I = \bigcap\{\eta \in \operatorname{Con} \mathbf{A} : 0/\eta = I\}.$$

It is known that $\varphi_I \in \operatorname{Con} \mathbf{A}$ and is the greatest tolerance on $\mathbf{A}$ whose kernel is I. Of course τ_I (resp. θ_I) is the least tolerance (resp. congruence) on $\mathbf{A}$ whose kernel is I. We recall from [12, Remark 2.5 b] that

$$\text{(4)} \qquad \theta_I = \bigcup_{n \in \omega} \tau_I^n.$$

The following characterization of τ_I was obtained in [12]; the notation has been changed to suit our present purposes.

2.1. Theorem. [12, Theorem 2.4]. *Let $I \in \operatorname{Id} \mathbf{A}$ and $a, b \in A$. Then $(a, b) \in \tau_I$ if and only if there exist $m \geqslant 1$, a $\{\cdot\}$-term $t = t(x_1, \ldots, x_m)$ elements $c_1, \ldots, c_m \in A$ and finite families $B_1, \ldots, B_m, D_1, \ldots, D_m$ in I such that*

$$a = t(c_1 \amalg B_1, \ldots, c_m \amalg B_m),$$
$$b = t(c_1 \amalg D_1, \ldots, c_m \amalg D_m).$$

2.2. Corollary. *Let $I \in \mathrm{Id}\,\mathbf{A}$ and $a, b \in A$. Then $(a,b) \in \theta_I$ if and only if for some positive n, m there exist m-ary $\{\cdot\}$-terms $t_1, \ldots, t_n$, elements $c_{ij} \in A$ and finite families B_{ij}, D_{ij} in I (for $i = 1, \ldots, n$ and $j = 1, \ldots, m$) such that*

$$a = t_1(c_{11} \amalg B_{11}, \ldots, c_{1m} \amalg B_{1m}),$$
$$b = t_n(c_{n1} \amalg D_{n1}, \ldots, c_{nm} \amalg D_{nm})$$

and in case $n > 1$, also:

$$t_i(c_{i1} \amalg D_{i1}, \ldots, c_{im} \amalg D_{im}) = t_k(c_{k1} \amalg B_{k1}, \ldots, c_{km} \amalg B_{km})$$

for $i = 1, \ldots, n-1$ and $k = i+1$.

Proof. The result follows immediately from Theorem 2.1 and (4); the requirement that all t_i have the same arity is merely a notational convenience. □

2.3. Corollary. *Let $I, J \in \mathrm{Id}\,\mathbf{A}$ and $\jmath \subseteq \mathrm{Id}\,\mathbf{A}$. Let $L = V_{\mathrm{Id}\,\mathbf{A}}\jmath$ $(= \langle \cup \jmath \rangle_{\mathbf{A}})$. Then*
(i) $I \subseteq J \Leftrightarrow \tau_I \subseteq \tau_J \Leftrightarrow \theta_I \subseteq \theta_J$;
(ii) $V_{\mathrm{Tol}\,\mathbf{A}}\{\tau_N : N \in \jmath\} = \tau_L$ *and* $V_{\mathrm{Con}\,\mathbf{A}}\{\theta_N : N \in \jmath\} = \theta_L$.

Proof. (i) If $I \subseteq J$, it follows immediately from Theorem 2.1 and Corollary 2.2 that $\tau_I \subseteq \tau_J$ and $\theta_I \subseteq \theta_J$. If $\tau_I \subseteq \tau_J$ and $a \in I$ then $(a, 0) \in \tau_I$, hence $(a,0) \in \tau_J$, i.e. $a \in 0/\tau_J = J$. So $\tau_I \subseteq \tau_J$ implies $I \subseteq J$. Similarly $\theta_I \subseteq \theta_J$ implies $I \subseteq J$.

(ii) From (i) we have $\tau_N \subseteq \tau_L$ for all $N \in \jmath$. Let $\eta \in \mathrm{Tol}\,\mathbf{A}$ with $\bigcup_{N \in \jmath} \tau_N \subseteq \eta$ and let $M = 0/\eta$. If $N \in \jmath$ and $a \in N$ then $(a,0) \in \tau_N$ so $(a,0) \in \eta$, i.e., $a \in M$. We therefore have $\cup\jmath \subseteq M$ hence $L \subseteq M$, (since $M \in \mathrm{Id}\,\mathbf{A}$). Now (i) implies that $\tau_L \subseteq \tau_M \subseteq \eta$. This proves that $\tau_L = V_{\mathrm{Tol}\,\mathbf{A}}\{\tau_N : N \in \jmath\}$. The second assertion may be proved similarly. □

We remark that the condition "**A** is a member of some BCK-variety" is used frequently in the literature, where in many cases "$\mathbf{H}(\mathbf{A}) \subseteq$ BCK" would suffice. The latter condition is strictly weaker than the former: see Example 2.12. The following simple result is therefore of interest.

2.4. Proposition. *The following conditions on a BCK-algebra* **A** *are equivalent:*
(i) $\mathbf{H}(\mathbf{A}) \subseteq \mathrm{BCK}$;
(ii) $(\forall \varrho, \sigma \in \mathrm{Con}\,\mathbf{A})(0/\varrho = 0/\sigma \Rightarrow \varrho = \sigma)$;
(iii) $(\forall I \in \mathrm{Id}\,\mathbf{A})(\theta_I = \varphi_I)$;
(iv) $(\forall \eta \in \mathrm{Tol}\,\mathbf{A})(\varphi_{0/\eta} = \cup\{\eta^n : n \in \omega\})$;
(v) $(\forall a, b \in A)((a,b) \in \theta_{\langle ab, ba \rangle})$.

Proof. (i) $\Rightarrow$ (ii) is proved in [7, p. 108] (under the unnecessarily strong assumption that $\mathbf{A}$ is a member of a BCK-variety).

(ii) $\Rightarrow$ (iii) follows since $0/\theta_I = 0/\varphi_I = I$.

(iii) $\Rightarrow$ (iv) follows easily from (4) and the fact that $\tau_I^n \subseteq \eta^n \subseteq \varphi_I$, where $I = 0/\eta$.

(iv) $\Rightarrow$(v) Set $I = \langle ab, ba\rangle$. From $(a,b) \in \varphi_I$ and (iv), we obtain $(a,b) \in \bigcup\{\tau_I^n : n \in \omega\} = \theta_I$.

(v)$\Rightarrow$(i) Let $\sigma \in \operatorname{Con}\mathbf{A}$. It suffices to check that $\mathbf{A}/\sigma$ satisfies the axiom BCK (V), so suppose for some $a, b \in A$, we have $ab, ba \in 0/\sigma$. If $I = \langle ab, ba\rangle$ then $I \subseteq 0/\sigma$. By (v) we have $(a,b) \in \theta_I \subseteq \theta_{0/\sigma}$ (by Corollary 2.3.(i)) $\subseteq \sigma$, as required. □

2.5 Remark. In [9, Theorem 1], Idziak states without proof the following necessary condition (due to Komori) for a class K of $\mathscr{F}$-algebras to be a BCK-variety: Let $\mathbf{T}$ be the absolutely free $\mathscr{F}$-algebra freely generated by two distinct variables x and y, and let $\mathbf{B} = (B; \cdot, 0)$ be the $\mathscr{F}$-algebra with $B = \{0, a, b\}$ such that $a0 = a$, $b0 = b$ and $cd = 0$ in all remaining cases. Let $\mu\colon \mathbf{T} \to \mathbf{B}$ be the unique homomorphism satisfying $\mu(x) = a$ and $\mu(y) = b$. If K is a BCK-variety then there exist binary $\mathscr{F}$-terms $t = t(x,y)$ and $s = s(x,y)$ such that $\mu(t) = a$, $\mu(s) = b$ and K satisfies $t = s$. This result is important since it is essential to Idziak's proof that BCK-varieties are congruence 3-permutable [9, Theorem 2]. As far as we know, however, no proof of Komori's theorem has been published. We feel it is of interest to show that a description of BCK-varieties (our Theorem 2.7 and its Corollary 2.8), very similar to Komori's, may be derived from our characterisation of θ_I (Corollary 2.2). Our (tolerance-based) approach is presumably quite different from Komori's methods. The next lemma, which will be needed in our argument, may also be used as a tool for deriving Komori's result from our Theorem 2.7 and conversely.

2.6. Lemma. *Let $t = t(x_1, \ldots, x_n)$ and $s = s(x,y)$ be $\mathscr{F}$-terms with $n \geqslant 1$.*

(i) *There is a $\{\cdot\}$-term $u = u(x_1, \ldots, x_n)$ such that* BCK *satisfies $t = u$.*

(ii) *There exist $i \in \{1, \ldots, n\}$ and $m \in \omega$ and $\{\cdot\}$-terms $u_j = u_j(x_1, \ldots, x_n)$ for $0 < j \leqslant m$, such that* BCK *satisfies $t = x_i u_1 \ldots u_m$.*

(iii) *If $w \in \{x_1, \ldots, x_n\}$ and $w_1, \ldots, w_n \in \{0, w\}$ then* BCK *satisfies $t(w_1, \ldots, w_n) = 0$ or* BCK *satisfies $t(w_1, \ldots, w_n) = w$.*

(iv) BCK *satisfies $s(x,x) = 0$ iff* BCK *satisfies $s(x,y)(xy)^p(yx)^q = 0$ for some $p, q \in \omega$.*

Proof. (i), (ii) and (iii) are easily proved by induction on the complexity of t, using BCK (III), BCK (IV) and the well-known fact (see [11, Theorem 2]) that BCK satisfies

$$x0 = x. \tag{5}$$

(iv) Let BCK satisfy $s(x,x) = 0$. Let $\mathbf{F} = (F\,;\,.,0)$ be a BCK-free $\mathscr{F}$-algebra freely generated by two distinct generators $a, b \in F$. Of course, $\mathbf{F} \in$ BCK, since BCK is an $\mathscr{F}$-quasi-variety. Let $J = \langle ab, ba\rangle_{\mathbf{F}}$ and let $\theta = \Theta((a,b))$ (i.e. θ is the least congruence on $\mathbf{F}$ identifying a and b). Clearly $(a,b) \in \varphi_J$, and so $\theta \subseteq \varphi_J$. Now since $(s(a,b),0) = (s(a,b), s(a,a)) \in \theta$, we have $s(a,b) \in 0/\varphi_J = J$. By (3) and the BCK-identity $(xy)z = (xz)y$, there exist $p, q \in \omega$ such that $s(a,b)(ab)^p(ba)^q = 0$. It follows that BCK satisfies $s(x,y)(xy)^p(yx)^q = 0$. The converse follows easily from BCK (III) and (5). □

2.7. Theorem. *Let K be a BCK-variety. Then there exist $n, m \in \omega$, and $\{\cdot\}$-terms $u_i = u_i(x,y)$ and $v_j = v_j(x,y)$ satisfying the equivalent conditions of Lemma* 2.6 (iv) *(for $0 < i \leqslant n$ and $0 < j \leqslant m$) such that K satisfies*

$$xu_1(x,y)\ldots u_n(x,y) = yv_1(x,y)\ldots v_m(x,y).$$

Proof. We may assume that K is nontrivial (for if not, take $n = m = 0$). Let $\mathbf{F} = (F;.,0)$ be a K-free $\mathscr{F}$-algebra freely generated by generators $a, b \in F$. Let $J = \langle ab, ba\rangle_{\mathbf{F}}$. We have $\mathbf{H}(\mathbf{F}) \subseteq$ BCK, so by Proposition 2.4, $(a,b) \in \theta_J$. It follows that for some positive $n, m \in \omega$, some m-ary $\{\cdot\}$-terms $t_1, \ldots, t_n$, some elements $c_{ij} \in F$ and some finite families B_{ij}, D_{ij} in J (for $i = 1, \ldots, n$ and $j = 1, \ldots, m$), the identities displayed in Corollary 2.2 hold in $\mathbf{F}$. It follows (using Lemma 2.6(i)) that there exist $\{\cdot\}$-terms $s_{ij} = s_{ij}(x,y)$ and finite families U_{ij}, W_{ij} of $\{\cdot\}$-terms $g = g(x,y)$ (for $i = 1, \ldots, n$ and $j = 1, \ldots, m$) such that K satisfies

$$(6)_0 \qquad x = t_1(s_{11}\prod U_{11}, \ldots, s_{1m}\prod U_{1m}),$$

$$(6)_n \qquad y = t_n(s_{n1}\prod W_{n1}, \ldots, s_{nm}\prod W_{nm})$$

and, in the case $n > 1$, also:

$$(6)_i \qquad t_i(s_{i1}\prod W_{i1}, \ldots, s_{im}\prod W_{im}) = t_k(c_{k1}\prod U_{k1}, \ldots, c_{km}\prod U_{km})$$

for $i = 1, \ldots, n-1$ and $k = i+1$. It is also clear that since the families B_{ij} and D_{ij} are in J, each term $g = g(x,y)$ in the combined ranges of the U_{ij} and W_{ij} may be chosen such that for some integers $d = d(g)$ and $e = e(g)$, BCK satisfies:

$$(7)_g \qquad g(x,y)(xy)^d(yx)^e = 0.$$

Equivalently, by Lemma 2.6(iv), BCK satisfies:

$$(8)_g \qquad g(x,x) = 0.$$

Considering the form of $(6)_0$, $(6)_1$, ..., $(6)_n$, we may deduce from the $(8)_g$ that BCK satisfies:

$$(9) \qquad t_i(s_{i1}(x,x),\dots,s_{im}(x,x)) = x$$

for $i = 1, \dots, n$. Now let i be the least integer among $0, \dots, n$ such that the first occurrences of the variables on the left and right hand sides of $(6)_i$ are occurrences of different variables. Necessarily these are an x-occurrence on the left and a y-occurrence on the right (since the t_i and s_{ij} are $\{\cdot\}$-terms and (9) holds). Also the nontriviality of K forces $0 < i < n$. By Lemma 2.6(ii), the terms t_l ($l = i$, $i+1$) may be assumed to have the form $x_l \prod V_l$, where x_l is a variable occurring in t_l and V_l is a finite family of m-ary $\{\cdot\}$-terms. Let us assume that in $(6)_i$, x_i and x_{i+1} have been replaced, respectively, by $s_{i\alpha} \prod W_{i\alpha}$ and $s_{i+1\beta} \prod U_{i+1\beta}$ where $\alpha, \beta \in \{1, \dots, m\}$. Applying Lemma 2.6(ii) to the terms $s_{i\alpha}$ and $s_{i+1\beta}$ we may rewrite $(6)_i$ (setting $k = i+1$) as:

$$\begin{aligned} &((x \prod W_{i0}) \prod W_{i\alpha}) \prod_{j=1}^{r} h_{ij}(s_{i1} \prod W_{i1}, \dots, s_{im} \prod W_{im}) \\ &= ((y \prod U_{k0}) \prod U_{k\beta}) \prod_{j=1}^{r} h_{kj}(s_{k1} \prod U_{k1}, \dots, s_{km} \prod U_{km}) \end{aligned}$$

for some $r \in \omega$, some m-ary $\{\cdot\}$-terms h_{lj} ($l = i$, $i+1$ and $j = 1, \dots, r$) and some finite families W_{l0} and U_{i+10} of $\{\cdot\}$-terms $g = g(x,y)$. (The use of a uniform r loses no generality, since BCK satisfies $x = x(xx)$.) It follows readily from (9) and Lemma 2.6(iii) that BCK satisfies $(8)_g$ for all g in the combined ranges of W_{i0} and U_{i+10}, as well as

$$h_{lj}(s_{l1}(x,x),\dots,s_{lm}(x,x)) = 0.$$

This reduces $(6)_i$ to an equation of the form described in the statement of the theorem. □

An $\mathscr{F}$-identity will be called an *xy-identification* if it has the form

$$(10) \qquad xu_1(x,y)\dots u_n(x,y) = yv_1(x,y)\dots v_m(x,y)$$

where $n, m \in \omega$, and there exist integers p_i, q_i, k_j, $l_j \in \omega$ such that BCK satisfies

$$(10)_{ij} \qquad u_i(x,y)(xy)^{p_i}(yx)^{q_i} = 0 = v_j(x,y)(xy)^{k_j}(yx)^{l_j}$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$.

2.8. Corollary. *Let K be any class of $\mathcal{F}$-algebras. Then the varietal closure* $\mathbf{HSP}(K)$ *is a BCK-variety if and only if K satisfies the identities* BCK(I), BCK(IV) *and* (5), *as well as some xy-identification.*

Proof. Necessity is clear. Conversely, suppose that K satisfies BCK(I), BCK(IV), (5) and the xy-identification given by (10) and $(10)_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, where $n, m \in \omega$. Then $\mathbf{HSP}(K)$ also satisfies these identities, and therefore satisfies BCK(II) and BCK(III); the calculations are:

$$(x(xy))y = ((x0)(xy))(y0) = 0, \quad \text{and}$$

$$xx = (xx)0 = ((x0)(x0))(00) = 0.$$

To establish BCK(V), let $\mathbf{C} = (C\,;\,\cdot,0) \in \mathbf{HSP}(K)$ and let $a, b \in C$ with $ab = 0 = ba$. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$ we have $u_i(a,b) = 0 = v_j(a,b)$ by $(10)_{ij}$, BCK(III) and (5). Thus we have $a = b$ by (10) and (5). This shows that $\mathbf{HSP}(K) \subseteq BCK$. □

2.9. Corollary. *If K is any nontrivial BCK-variety then* $\mathrm{tn}(K) = 2$.

Proof. Let K satisfy the xy-identification given by (10) and $(10)_{ij}$, where $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $n, m \in \omega$. Let $\mathbf{A} = (A; \cdot, 0) \in K$ and $\tau \in \mathrm{Tol}\,\mathbf{A}$ with $I = 0/\tau$. We show that $\tau^3 \subseteq \tau^2$. Observe the $\tau^3 \subseteq \varphi_I^3 = \varphi_I$ so $0/\tau^3 = I$. Now if $(a,b) \in \tau^3$ then $ab, ba \in 0/\tau^3$, so $(ab, 0), (ba, 0) \in \tau$. By $(10)_{ij}$, we have $(u_i(a,b), 0)$, $(v_j(a,b), 0) \in \tau$ for each i, j, and hence by (5), $(a, au_1(a,b)\ldots u_n(a,b))$, $(bv_1(a,b)\ldots v_m(a,b), b) \in \tau$. By (10), we have $(a,b) \in \tau^2$, as claimed. Thus $\mathrm{tn}(K) \leqslant 2$, and no nontrivial BCK-variety is congruence permutable (equivalently, tolerance trivial) [7, Theorem 4.3] so $\mathrm{tn}(K) = 2$. □

2.10. Corollary. *Let K be a BCK-variety and $\mathbf{A} \in K$. For any integer $m \geqslant 2$ and any $\tau_1, \ldots, \tau_m \in \mathrm{Tol}\,\mathbf{A}$ with the same kernel $I \in \mathrm{Id}\,\mathbf{A}$, we have*

$$\tau_1 \ldots \tau_m = \theta_I = \varphi_I.$$

Proof. From $\tau_I \subseteq \tau_j \subseteq \varphi_I (j = 1, \ldots, m)$, Proposition 2.4 and the previous result, we have:

$$\theta_I = \tau_I^2 \subseteq \tau_I^m \subseteq \tau_1 \ldots \tau_m \subseteq \varphi_I^m = \varphi_I = \theta_I.$$

□

2.11. Remarks. a. Corollary 2.9. could alternatively be deduced from Corollary 1.3. and Idziak's result that every BCK-variety is congruence 3-permutable [9, Theorem 2].

b. For any $i, j, p, q \in \omega$, the class of all BCK-algebras satisfying

$$(C_{p,q}^{i,j}): \qquad x(xy)^i(yx)^j = y(yx)^p(xy)^q$$

is a BCK-variety [7]. Such varieties are called *quasicommutative*. Clearly the above identity yields an xy-identification. Every finite BCK-algebra satisfies $(C_{p,q}^{i,j})$ for some $i, j, p, q \in \omega$ [7], hence *the tolerance number of a finite BCK-algebra is* 1 *or* 2.

c. In contrast with Corollary 2.9, we observe that *for every positive $n \in \omega$, there is a BCK-algebra* $\mathbf{A}$ *with* $\mathrm{tn}(\mathbf{A}) = n$. *Also there is a BCK-algebra* $\mathbf{B}$ *with* $\mathrm{tn}(\mathbf{B}) = \omega$. *Consequently* $\mathrm{tn}(\mathrm{BCK}) = \omega$. For $n = 1, 2$, examples may be found in [12]. We now assume the terminology and notation of [17].

Let $\mathscr{N}$ denote the BCK-algebra on the set ω where the BCK-operation is defined by $a.b = \max\{0, a-b\}$. Let $R(\mathscr{N})$ be the reflection of ω in the sense of [17] and $R(\omega)$ its distensible subset $\{r_n : n \in \omega\}$. For $n \geqslant 2$, let $\mathscr{N}_n$ denote the distension of $R(\mathscr{N})$ induced by the triple $(R(\omega), n, \delta_n)$ where $\delta_n(i,j) = |i-j|$ for $i, j \in n$. It is known that $\mathscr{N}_n$ is a BCK-algebra and that the nontrivial congruences of $\mathscr{N}_n$ are in one-to-one correspondence with the partitions of the set n: a partition π of n induces a partition $\pi' = \{p \times R(\omega) : p \in \pi\} \cup \{\omega\}$ of the base set of $\mathscr{N}_n$, the equivalence relation corresponding to π' is a congruence on $\mathscr{N}_n$ and all congruences on $\mathscr{N}_n$ arise in this way. It is also easily checked that $\mathscr{N}_n$ has exactly three ideals, the nontrivial one being ω. We claim that $\mathrm{tn}(\mathscr{N}_n) = n+1$ for $n \geqslant 2$.

Let $n \geqslant 2$ and $\eta \in \mathrm{Tol}\,\mathscr{N}_n$ with $0/\eta = I$. If $I = \{0\}$ then η is the identity congruence on $\mathscr{N}_n$. If $I = \mathscr{N}_n$ then η^2 is the total congruence on $\mathscr{N}_n$. So we may assume that $I = \omega$. We note that

$$(11) \qquad \omega^2 \cup \bigcup_{i \in n} (\{i\} \times R(\omega))^2 \subseteq \eta.$$

Indeed if $j, k \in \omega$, say $j = k.m$ $(m \in \omega)$, then from $m\eta 0$, we obtain $j\eta k$, as well as $(i, r_j)\eta(i, r_k)$ for any $i \in n$. Since the left-hand side of (11) is a congruence on $\mathscr{N}_n$, it must be $\theta_I = \tau_I$. Next, by [12, Theorem 2.2 b], we have

$$(12) \qquad \eta \subseteq \omega^2 \cup (n \times R(\omega))^2.$$

The expression on the right of (12) is the congruence φ_I. Also observe that if $k, \ell, m, q \in \omega$ with $m \geqslant k$ and $q \geqslant \ell$ then for any $i, j \in n$,

$$((i, r_k), (j, r_\ell)) \in \eta \Rightarrow ((i, r_m), (j, r_q)) \in \eta.$$

Now suppose $(a,b) \in \Theta(\eta)\backslash\eta$ and choose $h \in \omega$ minimal such that $(a,b) \in \eta^h$. Necessarily we have $a = (i, r_k)$, $b = (j, r_\ell)$ for some $i,j \in n$ $(i \neq j)$ and some $k, \ell \in \omega$. From the constraints on η established above it is not difficult to see that $h \leqslant n+1$. The case $h = n+1$ may be achieved by taking $\eta = T(\{((i,r_1),(i+1,r_1)) \colon i \in n\})$ and $a = (0, r_0)$, $b = (n-1, r_0)$. Thus $\mathrm{tn}(\mathscr{N}_n) = n+1$.

Finally, the BCK-algebra $\mathscr{N}_\infty$, constructed in [17] from the distending triple $D_\infty = (R(\omega), \omega, \delta_\infty)$ where $\delta_\infty(i,j) = |i-j|$ for $i,j \in \omega$, has the property that for no $n \geqslant 2$ are the congruences of $\mathscr{N}_\infty$ n-permutable [17, Theorem 6]. By Theorem 1.2, $\mathrm{tn}(\mathscr{N}_\infty) = \omega$.

2.12. Example. *The condition "*$\mathbf{H}(\mathbf{A}) \subseteq$ BCK*" does not imply that* $\mathbf{A}$ *is an element of some BCK-variety.* (This answers a question raised in [12].) To see this, recall that Wroński and Kabziński [18] have constructed a sequence D_n $(0 < n \in \omega)$ of finite BCK-algebras such that no BCK-variety contains all of the D_n. Now every finite BCK-algebra is in some BCK-variety, so if K is a finite subset of $\omega\backslash\{0\}$ then $\mathbf{H}(\prod_{n\in K} D_n) \subseteq BCK$ and there is a natural embedding $g_K \colon \prod_{n\in K} D_n \to \mathbf{A}$, where

$$A := \bigoplus_{0<n\in\omega} D_n = \Big\{a \in \prod_{0<n\in\omega} D_n : a(n) = 0 \text{ for almost all } n \in \omega\backslash\{0\}\Big\}$$

Note that $\mathbf{A} \in$ BCK, but since each D_n is embeddable in $\mathbf{A}$, it follows that $\mathbf{A}$ is in no BCK-variety. However $\mathbf{H}(\mathbf{A}) \subseteq$ BCK. For if $f \colon A \to B$ is an $\mathscr{F}$-homomorphism, where $\mathbf{B}$ is some $\mathscr{F}$-algebra, and $f(a)f(a') = 0_{\mathbf{B}} = f(a')f(a)$ for some $a, a' \in A$, we may consider the finite set

$$K = \{n \in \omega\backslash\{0\} : a(n) \neq 0 \text{ or } a'(n) \neq 0\}$$

and the $\mathscr{F}$-homomorphism $fg_K \colon \prod_{n\in K} D_n \to \mathbf{B}$.

We have $f(a), f(a') \in fg_K(\prod_{n\in K} D_n) \in BCK$, so $f(a) = f(a')$. It follows that $\mathbf{B} \in$ BCK, as claimed.

References

[1] *H-J. Bandelt*: Tolerance relations on lattices, Bull. Austral. Math. Soc. *23* (1981), 367–381.

[2] *S. Burris and H.P. Sankappanavar*: "A Course in Universal Algebra", Graduate Texts in Mathematics, vol. 78, Springer-Verlag, New York, 1981.

[3] *I. Chajda*: Distributivity and modularity of lattices of tolerance relations, Algebra Universalis *12* (1981), 247–255.

[4] *I. Chajda*: Tolerance trivial algebras and varieties, Acta Sci. Math (Szeged) *46* (1983), 35–40.

[5] *I. Chajda*: Tolerances in congruence permutable algebras, Czechoslovak Math. J. *38* (1988), 218–225.
[6] *I. Chajda and J. Rachůnek*: Relational characterizations of permutable and n-permutable varieties, Czechoslovak Math. J. *33* (1983), 505–508.
[7] *W.H. Cornish*: On Iséki's BCK-algebras, Lecture Notes in Pure and Applied Mathematics *74* (1982), 101–122, Marcel Dekker, New York.
[8] *J. Hagemann and A. Mitschke*: On n-permutable congruences, Algebra Universalis *3* (1973), 8–12.
[9] *P. M. Idziak*: On varieties of BCK-algebras, Mathematica Japonica *28* (1983), 157–162.
[10] *K. Iséki*: On ideals of BCK-algebras, Math. Seminar Notes Kobe Univ. *3* (1975), 1–12.
[11] *K. Iséki and S. Tanaka*: An introduction to the theory of BCK-algebras, Mathematica Japonica *23* (1978), 1–26.
[12] *J.G. Raftery, I.G. Rosenberg and T. Sturm*: Tolerance relations and BCK-algebras, Mathematica Japonica *36* (1991), 399–410.
[13] *J.G. Raftery and T. Sturm*: On ideal and congruence lattices of BCK-semilattices, Mathematica Japonica *32* (1987), 465–474.
[14] *I.G. Rosenberg and D. Schweigert*: Compatible orderings and tolerances on lattices, Annals of Discrete Mathematics *23* (1984), 119–150.
[15] *H. Werner*: A Mal'cev condition for admissible relations, Algebra Universalis *3* (1973), 263.
[16] *A. Wroński*: BCK-algebras do not form a variety, Mathematica Japonica *28* (1983), 211–213.
[17] *A. Wroński*: Reflections and distensions of BCK-algebras, Mathematica Japonica *28* (1983), 215–225.
[18] *A. Wroński and J.K. Kabziński*: There is no largest variety of BCK-algebras, Mathematica Japonica *29* (1984), 545–549.

Authors' addresses: J.G. Raftery, Department of Mathematics & Applied Mathematics, University of Natal, P. O. Box 375, Pietermaritzburg 3200, South Africa; T. Sturm, Department of Mathematics & Applied Mathematics, University of Natal, King George V. Avenue, Durban 4001, South Africa.