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ON INTEGRAL INCLUSIONS OF VOLTERRA TYPE IN BANACH SPACES

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1. INTRODUCTION

In this paper, we study Volterra integral inclusions defined in a Banach space. The necessity of studying such mathematical objects, comes from control theory and engineering problems. Recall that every control system (finite or infinite dimensional), under minimal hypotheses on its data, has an equivalent formulation in which the dynamics are described by an inclusion (differential, integral or functional inclusion). In this inclusion description, the control variable does not appear explicitly ("deparametrization" of the system). This equivalent inclusion description of the system, plays an important role when studying the relaxed (i.e. "convexified") system (see [24]). On the other hand, recently Glashoff-Sprekels [8], [9], studied the problem of thermostatic regulation, in which the heating devices controlling the temperature of the system, are governed by a relay switch, and established that the system dynamics can be modeled via an integral inclusion. Finally, recently Leitmann and his coworkers [15], [16], advocated a nonstochastic approach to the robustness of uncertain control systems, which is based on differential and integral inclusions.

The results in the paper, extend the single-valued works of Szufla [26] and Vaughn [29] and the multivalued ones by Ragimkhanov [25], Lyapin [17] (who studied integral inclusions in $\mathbf{R}^n$) and by Papageorgiou [20], [21] (who considered integral inclusions in Banach spaces).

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2. Preliminaries

In this section we recall some basic definitions and results about the measurability and continuity properties of multifunctions (set-valued functions), that we will need in the sequel.

So let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{ nonempty, closed, (convex)}\}$$
$$\text{and} \quad P_{(w)k(c)} = \{A \subseteq X\colon \text{ nonempty, (weakly-)compact, (convex)}\}.$$

A multifunction $F\colon \Omega \to P_f(X)$ is said to be measurable, if for every $x \in X$, the $\mathbf{R}_+$-valued function $\omega \to d(x, F(\omega)) = \inf\{\|x - z\|\colon z \in F(\omega)\}$ is measurable. In fact, this is equivalent to saying that for every $U \subseteq X$ open, $F^-(U) = \{\omega \in \Omega\colon F(\omega) \cap U \neq \emptyset\} \in \Sigma$ or that there exists a sequence $\{f_n\}_{n\geqslant 1}$ of measurable functions $f_n : \Omega \to X$ *s.t.* $F(\omega) = \overline{\{f_n(\omega)\}}_{n\geqslant 1}$ for all $\omega \in \Omega$. For details we refer to the survey paper of Wagner [30]. For a multifunction $F\colon \Omega \to 2^X \setminus \{\emptyset\}$, the graph of $F(\cdot)$ is defined by $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\}$. We will say that $F(\cdot)$ is "graph-measurable" if and only if $GrF \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. For a $P_f(X)$-valued multifunction, we know that measurability implies graph measurability, while the converse is true if there exists a σ-finite measure $\mu(\cdot)$ on Σ, with respect to which Σ is complete. A multifunction $F\colon \Omega \to 2^X \setminus \{\emptyset\}$ is said to be weakly (or scalarly) measurable, if for all $x^* \in X^*$, $\omega \to \sigma(x^*, F(\omega)) = \sup\{(x^*, z)\colon z \in F(\omega)\}$ is a measurable function. Again measurability implies weak measurability and the converse is true if there exists a complete, σ-finite measure $\mu(\cdot)$ on Σ and the multifunction is $P_{wkc}(X)$-valued.

Suppose (Ω, Σ, μ) is a finite measure space and $F : \Omega \to 2^X \setminus \{\emptyset\}$ a multifunction. By S_F^1 we will denote the set of integrable selectors of $F(\cdot)$; i.e. $S_F^1 = \{f \in L^1(X)\colon f(\omega) \in F(\omega)\mu\text{- a.e.}\}$. This set may be empty. For a graph measurable multifunction, it is nonempty if and only if $\omega \to \inf\{\|z\|\colon z \in F(\omega)\} \in L_+^1$. This is the case if $\omega \to |F(\omega)| = \sup\{\|z\|\colon z \in F(\omega)\} \in L_+^1$ and such a multifunction is usually called "integrably bounded". For a graph measurable multifunction S_F^1 is closed in $L^1(X)$ if and only if $F(\cdot)$ is $P_f(X)$-valued and convex if and only if $F(\cdot)$ is convex valued. For details we refer to [22]. Using the set S_F^1, we can define a set valued integral for $F(\cdot)$ by setting $\int_\Omega F(\omega)\, \mathrm{d}\mu(\omega) = \left\{\int_\Omega f(\omega)\, \mathrm{d}\mu(\omega)\colon f \in S_F^1\right\}$. The vector-valued integrals involved in this definition are understood in the sense of Bochner. A detailed study of this set-valued integral can be found in [12].

Next let Y, Z be Hausdorff topological space and $F\colon Y \to 2^Z \setminus \{\emptyset\}$. We say that $F(\cdot)$ is upper semicontinuous (*u.s.c.*) (resp. lower semicontinuous (*l.s.c.*)), if for every $U \subseteq Z$ open, the set $F^+(U) = \{y \in Y : F(y) \subseteq U\}$ (resp. $F^-(U) = \{y \in Y : F(y) \cap U \neq \emptyset\}$), is open in Y. If $F(\cdot)$ is both *u.s.c.* and *l.s.c.* then we say that $F(\cdot)$ is continuous. In fact, continuity is equivalent to saying that $F(\cdot)$ is continuous from Y into $2^Z \setminus \{\emptyset\}$, the latter equipped with the Vietoris topology. If Z is a metric space, on $P_f(Z)$ we can define a (generalized) metric, known as the Hausdorff metric, by setting $h(A, B) = \max[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)]$, $A, B \in P_f(Z)$. It is an elementary, yet rather technical exercise, to verify that completeness of Z implies completeness of the metric space $(P_f(Z), h)$, A multifunction $F: Y \to P_f(Z)$ is said to be Hausdorff continuous (h-continuous) if it is continuous from Y into the metric space $(P_f(Z), h)$. If $F(\cdot)$ is $P_k(Z)$-valued, then continuity and h-continuity coincide. This follows from the fact that on $P_k(Z)$, the Vietoris and Hausdorff topologies coincide (see Klein-Thompson [14], corollary 4.2.3, p. 41).

Let X be a Banach space and $\mathcal{B}$ its family of bounded set. Then the Hausdorff (ball)-measure of noncompactness $\beta: \mathcal{B} \to \mathbf{R}_+$ is defined by

$$\beta(B) = \inf\{r > 0 : B \text{ can be covered by finitely many balls of radius } r\}.$$

A comprehensive introduction to the subject of measures of noncompactness and their applications, can be found in the book of Banas-Goebel [2].

Finally, if $\{A_n\}_{n \geqslant 1} \subseteq 2^X \setminus \{\emptyset\}$, we set

$$\underline{\lim}\, A_n = \{x \in \ : \lim d(x, A_n) = 0\} = \{x \in X : x_n \xrightarrow{s} x,\ x_n \in A_n,\ n \geqslant 1\}$$

and

$$\begin{aligned}\overline{\lim}\, A_n &= \{x \in \ : \underline{\lim}\, d(x, A_n) = 0\} \\ &= \{x \in X : x_{n_k} \xrightarrow{s} x,\ x_{n_k} \in A_{n_k},\ n_1 < n_2 < \ldots < n_k < \ldots\}.\end{aligned}$$

It is clear from the above definitions that we always have

$$\underline{\lim}\, A_n \subseteq \overline{\lim}\, A_n$$

and both sets are closed in X. We say that the A_n's converges in the Kuratowski sense to A (denoted by $A_n \xrightarrow{K} A$) if and only if $\underline{\lim}\, A_n = \overline{\lim}\, A_n = A$.

Let $T = [0, b]$ and X a separable Banach space. We will be studying the following integral inclusion of the Volterra type:

$$(*) \qquad x(t) \in p(t) + \int_0^t K(t,s) F\big(s, x(s)\big)\,\mathrm{d}s, \quad t \in T$$

where $p \in C(T, X)$. By a solution of $(*)$, we understand a function $x(\cdot) \in C(T, X)$ *s.t.* $x(t) = p(t) + \int_0^t K(t,s) f(s)\,\mathrm{d}s$, $t \in T$ with $f \in S^1_{F(\cdot, x(\cdot))}$ (i.e., $f \in L^1(X)$, $f(t) \in F\big(t, x(t)\big)$ a.e.).

We start with an existence result, for the case where the orientor field $F(t, x)$ is convex-valued. Our hypotheses on the data of $(*)$ are following:

$H(F)$: $F\colon T \times X \to P_{fc}(X)$ is a multifunction *s.t.*
  (1) $(t, x) \to F(t, x)$ is measurable,
  (2) $x \to F(t, x)$ is *u.s.c.* from X into X_w (where X_w denotes the Banach space X endoved with the weak topology),
  (3) $|F(t,x)| = \sup\{\|v\|\colon v \in F(t,x)\} \leqslant a(t) + b(t)\|x\|$ a.e., with $a(\cdot), b(\cdot) \in L^2_+$,
  (4) $\beta\big(F(t, B)\big) \leqslant k(t)\beta(B)$ a.e. for all $B \subseteq X$ bounded and with $k(\cdot) \in L^1_+$.

Remark. Note that hypothesis $H(F)(4)$ implies that for all $t \in T \setminus N$, N being a Lebesgue-null subset of T, and for all $x \in X$, we have $F(t,x) \in P_k(X)$. Just let $B = \{x\}$ and recall that $\beta(\{x\}) = 0$ so that $\beta\big(F(t,x)\big) = 0$ for all $(t, x) \in (T \setminus N) \times X$ (see $H(F)(4)$) $\Rightarrow F(t,x) \in P_k(X)$ for all $(t,x) \in (T \setminus N) \times X$.

$H(K)$: $K\colon \Delta = \{(t,s)\colon 0 \leqslant s \leqslant t \leqslant b\} \to \mathscr{L}(X)$ is a strongly continuous kernel *s.t.* $\|K(t', s) - K(t,s)\|_{\mathscr{L}} \leqslant \frac{c(t'-t)}{t-s}$ for all $(t,s), (t', s) \in \Delta$, $t' > t$ (here $\mathscr{L}(X)$ denotes the Banach space of all bounded, linear operators from X into itself, and "strong continuity", refers to continuity of $K(\cdot,\cdot)$ into $\mathscr{L}(X)$ equiped with the strong operator topology).

Remark. Suppose that $\{A(t)\}_{t \in T}$ is a family of closed, densely defined linear operators *s.t.* $D\big(A(t)\big) = D$ (i.e., independent of $t \in T$), $R\big(\lambda; A(t)\big) = \big(\lambda I - A(t)\big)^{-1}$ exists for all $t \in T$ and all $\lambda \in \mathbf{C}$ with $\mathrm{Re}\lambda \leqslant 0$ (i.e. for all $t \in T$, the resolvent set $\varrho\big(A(t)\big)$ contains the half-plane $\mathrm{Re}\lambda \leqslant 0$), $\|R\big(\lambda; A(t)\big)\|_{\mathscr{L}} \leqslant \frac{d}{1+|\lambda|}$ and for all $t, s \in T$. $\|A(t)A(0)^{-1} - A(s)A(0)^{-1}\|_{\mathscr{L}} \leqslant c|t-s|^\gamma$ $0 < \gamma \leqslant 1$ (in fact this last condition is equivalent to saying that $\|A(t)A(\tau) - A(s)A(\tau)^{-1}\|_{\mathscr{L}} \leqslant c|t-s|^\gamma$ for all $t, s \in T$). Then this family of unbounded operators, generates a strongly continuous evolution operator (fundamental solution) $K\colon \Delta \to \mathscr{L}(X)$, which satisfies hypothesis $H(K)$. To see this, note that from the properties of the evolution operator $K(t,s)$ (see

Tanabe [27], chapter 5) and the mean value theorem, for any $x^* \in X^*$, any $x \in X$ and some $\tau \in [t, t']$ $t', t \in T$, $t' > t$, we have

$$\begin{aligned}\left|\left(x^*, K(t',s)x - K(t,s)x\right)\right| &\leqslant (t'-t)\left|\left(x^*, \frac{\partial}{\partial t}K(\tau,s)x\right)\right| \\ &\leqslant (t'-t)\left\|\frac{\partial}{\partial t}K(\tau,s)\right\|_{\mathscr{L}} \|x\| \, \|x^*\|.\end{aligned}$$

But from inequality 5.141, p. 149 in Tanabe [27] (see also Friedman [7], corollary, p. 127), we have

$$\left\|\frac{\partial}{\partial t}K(\tau,s)\right\| \leqslant \frac{c}{\tau - s} \leqslant \frac{c}{t-s}.$$

Thus we have

$$\|K(t',s)x - K(t,s)x\| \leqslant (t'-t)\frac{c\|x\|}{t-s} \;\Rightarrow\; \|K(t',s) - K(t,s)\|_{\mathscr{L}} \leqslant \frac{c(t'-t)}{t-s}.$$

Let H, X be separable Hilbert spaces *s.t.* X embeds into H continuously and densely. Identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$, with all embeddings being continuous and dense. Such a triple of spaces is usually known in the literature as an "evolution triple" (see Zeidler [31]). A typical example is $H = L^2(Z)$, $X = H_0^m(Z)$ and $X^* = H^{-m}(Z)$ with $m \in \mathbf{N}$ and Z a bounded domain in $\mathbf{R}^n$, with smooth boundary. Let $A\colon T \to \mathscr{L}(X, X^*)$ be a map *s.t.* $t \to A(t)x$ is measurable for all $x \in X$, $\langle A(t)x, x\rangle \geqslant c\|x\|_X^2$, where $\langle\cdot,\cdot\rangle$ denotes the duality brackets for the pair (X, X^*) and $\|A(t')x - A(t)x\|_{X^*} \leqslant \beta|t'-t|\|x\|_X$, $\beta > 0$. Then $\{A(t)\}_{t\in T}$ generates an evolution operator $K\colon \Delta \to \mathscr{L}(H)$ satisfying $H(K)$. For details, we refer to Tanabe [27], chapter 5, section 4.

Theorem 3.1. *If hypotheses $H(F)$ and $H(K)$ hold and $p \in C(T, X)$, then $(*)$ admits a solution.*

Proof. We start by deriving an a priori bound for the solutions of $(*)$. So let $x(\cdot) \in C(T, X)$ be such a solution. Then by definition, for some $f \in S^1_{F(\cdot,x(\cdot))}$ and for all $t \in T$, we have

$$x(t) = p(t) + \int_0^t K(t,s)f(s)\,\mathrm{d}s$$

$$\begin{aligned}\Rightarrow \|x(t)\| &\leqslant \|p\|_\infty + \int_0^t \|K(t,s)\|_{\mathscr{L}} \cdot \|f(s)\|\,\mathrm{d}s \\ &\leqslant \|p\|_\infty + \int_0^t M\big(a(s) + b(s)\|x(s)\|\big)\,\mathrm{d}s \qquad \Big(M = \sup_{(t,s)\in\Delta} \|K(t,s)\|_{\mathscr{L}}\Big).\end{aligned}$$

Invoking Gronwall's inequality, we get $M_1 > 0$ *s.t.* for all $t \in T$ and all solutions $x(\cdot)$ of $(*)$, we have

$$\|x(t)\| \leqslant M_1.$$

Let $\hat{F}(t,x) = F(t,x)$ if $\|x\| \leqslant M_1$ and $\hat{F}(t,x) = F\big(t, \frac{M_1 x}{\|x\|}\big)$ if $\|x\| > M_1$, then $\hat{F}(t,x) = F\big(t, p_{M_1}(x)\big)$, with $p_{M_1}\colon X \to X$ being the M_1-radial retraction. Recalling that $p_{M_1}(\cdot)$ is Lipschitz continuous, we can easily see that $(t,x) \to \hat{F}(t,x)$ is measurable, $x \to \hat{F}(t,x)$ is *u.s.c.* form X into X_w and $|\hat{F}(t,x)| = \sup\{\|v\|\colon v \in \hat{F}(t,x)\} \leqslant a(t) + b(t)M_1 = \varphi(t)$ a.e. with $\varphi(\cdot) \in L^2_+$. Finally, if $B \subseteq X$ is bounded since $p_{M_1}(B) \subseteq \overline{\mathrm{conv}}(B \cup \{0\})$, we have using the properties of β:

$$\begin{aligned}\beta(\hat{F}(t,B)) &= \beta\big(F\big(t, p_{M_1}(B)\big)\big) \leqslant k(t)\beta\big(p_{M_1}(B)\big)\\ &\leqslant k(t)\beta\big(\overline{\mathrm{conv}}(B \cup \{0\})\big) \leqslant k(t)\beta(B) \quad \text{a.e.}\end{aligned}$$

Next let $W \subseteq C(T,X)$ be defined by

$$W = \Big\{y \in C(T,X)\colon y(t) = p(t) + \int_0^t K(t,s)g(s)\,\mathrm{d}s,\ t \in T,\ \|g(t)\| \leqslant \varphi(t) \text{ a.e.}\Big\}.$$

Clearly W is nonempty, closed, convex and bounded subset of $C(T,X)$. We also claim that it is equicontinuous. To this end let $t, t' \in T$, with $t < t'$, $t' = t + h$.

$$\begin{aligned}\|y(t') - y(t)\| &\leqslant \|p(t') - p(t)\| + \Big\|\int_0^{t'} K(t',s)g(s)\,\mathrm{d}s - \int_0^t K(t,s)g(s)\,\mathrm{d}s\Big\|\\ &\leqslant \|p(t') - p(t)\| + \int_{t-h}^{t'} 2M\varphi(s)\,\mathrm{d}s + \Big\|\int_0^{t-h} \big(K(t',s) - K(t,s)\big)g(s)\,\mathrm{d}s\Big\|\\ &\leqslant \|p(t') - p(t)\| + \int_{t-h}^{t'} 2M\varphi(s)\,\mathrm{d}s + \int_0^{t-h} \frac{c(t'-t)}{t-s}\varphi(s)\,\mathrm{d}s.\end{aligned}$$

Note that $s \to \frac{1}{\varphi(s)}$ belongs in $L^2[0,b]$. So applying the Cauchy-Schwartz inequality, we get that

$$\begin{aligned}\int_0^{t-h} \frac{c(t'-t)}{t-s}\varphi(s)\,\mathrm{d}s &\leqslant ch\|\varphi\|_2\Big[\int_h^t \frac{1}{r^2}\,\mathrm{d}r\Big]^{1/2}\\ &\leqslant M_2 v(h) \qquad (v(h) \to 0 \text{ as } h \to 0^+,\ M_2 > 0).\end{aligned}$$

Therefore we have

$$\|y(t') - y(t)\| \leqslant \|p(t') - p(t)\| + \int_t^{t'} 2M\varphi(s)\,\mathrm{d}s + M_2 v(t'-t)$$

for $t \geqslant h$. For $t < h$ the estimation of $\|y(t') - y(t)\|$ is clear. Therefore W is indeed equicontinuous.

Next let $R\colon W \to P_{fc}(W)$ be defined by

$$R(x) = \left\{y \in C(T,X)\colon y(t) = p(t) + \int_0^t K(t,s)f(s)\,\mathrm{d}s,\ t \in T,\ f \in S^1_{\hat{F}(\cdot,x(\cdot))}\right\}.$$

Let $B \subseteq W$ be nonempty, closed. We have:

$$\beta\big(R(B)(t)\big) \leqslant \beta\Big[\int_0^t K(t,s)u(s)\,\mathrm{d}s\colon u \in S^1_{\hat{F}(\cdot,x(\cdot))},\ x \in B\Big].$$

Note that for every $z \in X$, $d\big(z, \hat{F}\big(s, B(s)\big)\big) = \inf_{v\in B(s)} d\big(z, \hat{F}(s,v)\big)$, where $B(s) = \{x(s)\colon x \in B\}$. Clearly then $s \to B(s)$ is a measurable multifunction, while $(s,v) \to d\big(z, \hat{F}(s,v)\big)$ is measurable. Therefore theorem 6.1 of [13] tells us that $s \to d\big(z, \hat{F}\big(s, B(s)\big)\big)$ is measurable. $\Rightarrow s \to H(s) = \overline{\hat{F}\big(s, B(s)\big)}$ is measurable. Then we can find functions $h_n\colon T \to X$ $n \geqslant 1$ measurable *s.t.* $H(s) = \overline{\{h_n(s)\}}_{n\geqslant 1}$. We have:

$$\beta\big(R(B)(t)\big) \leqslant \beta\Big[\int_0^t K(t,s)h_n(s)\,\mathrm{d}s\colon n \geqslant 1\Big].$$

Invoking proposition 1.6 of Mönch [18], we get

$$\beta\Big[\int_0^t K(t,s)h_n(s)\,\mathrm{d}s\colon n \geqslant 1\Big] \leqslant \int_0^t M\beta\big(h_n(s)\colon n \geqslant 1\big)\,\mathrm{d}s \leqslant \int_0^t Mk(s)\beta\big(B(s)\big)\,\mathrm{d}s.$$

Define $\psi(B) = \sup_{t\in T}\big[\mathrm{e}^{-r\int_0^t k(s)\mathrm{d}s}\beta\big(B(t)\big)\big]$, for $r > 0$, $B \subseteq W$. Since W is an equicontinuous, closed, convex and bounded subset of $C(T,X)$ and exploiting the properties of $\beta(\cdot)$, we can easily check that $\psi(\cdot)$ is a sublinear measure of noncompactness in the sense of Banas-Goebel [2]. Then we have

$$\begin{aligned}
\beta\big(R(B)(t)\big) &\leqslant \int_0^t Mk(s)\mathrm{e}^{r\int_0^s k(\tau)\mathrm{d}\tau}\mathrm{e}^{-r\int_0^s k(\tau)\mathrm{d}\tau}\beta\big(B(s)\big)\,\mathrm{d}s\\
&\leqslant \psi(B)\frac{M}{r}\mathrm{e}^{r\int_0^t k(s)\mathrm{d}s}\\
&\Rightarrow \psi\big(R(B)\big) \leqslant \frac{M}{r}\psi(B).
\end{aligned}$$

So if we choose $r > M$, we get that $R(\cdot)$ is a ψ-contradiction.

Next we will show that GrR is closed in $W \times W$. So let $[x_n, y_n] \in GrR$, $n \geqslant 1$ and assume that $x_n \to x$, $y_n \to y$ in $C(T,X)$. We have

$$y_n(t) = p(t) + \int_0^t K(t,s)f_n(s)\,\mathrm{d}s, \quad t \in T$$

with $f_n \in S^1_{\hat{F}(\cdot,x_n(\cdot))}$. Since $\hat{F}(t,x)$ is for almost all $t \in T$, $P_{kc}(X)$-valued and $\hat{F}(t,\cdot)$ is *u.s.c.* from X into X_w, theorem 7.4.2, p. 90 of Klein-Thompson [14], tells us that $\overline{\text{conv}} \bigcup_{n\geqslant 1} \hat{F}(t,x_n(t)) \in P_{wkc}(X)$ μ-a.e. Also because of the measurability of $\hat{F}(\cdot,\cdot)$, $t \to G(t) = \overline{\text{conv}} \bigcup_{n\geqslant 1} \hat{F}(t,x_n(t))$ is measurable and $|G(t)| \leqslant \varphi(t)$ a.e. Then proposition 3.1 of [23] tells us that S^1_G is w-compact in $L^1(X)$. Since $\{f_n\}_{n\geqslant 1} \subseteq S^1_G$, by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(X)$, then $\int_0^t K(t,s)f_n(s)\,\mathrm{d}s \xrightarrow{w} \int_0^t K(t,s)f(s)\,\mathrm{d}s$ in X and from theorem 3.1 of [19], we get $f(t) \in \overline{\text{conv}} w - \overline{\lim}\{f_n(t)\}_{n\geqslant 1} \subseteq \overline{\text{conv}} w - \overline{\lim}\,\hat{F}(t,x_n(t)) \subseteq w - \overline{\lim}\,\hat{F}(t,x(t))$ a.e. (the last inclusion following from the upper-semicontinuity of $\hat{F}(t,\cdot)$ from X into X_w, from the convexity of the values of $\hat{F}(t,x)$ and from the fact that $x_n \to x$ in $C(T,X)$). Thus $f \in S^1_{\hat{F}(\cdot,x(\cdot))}$. Therefore in the limit as $n \to \infty$, we get

$$y(t) = p(t) + \int_0^t K(t,s)f(s)\,\mathrm{d}s, \quad t \in T, \quad f \in S^1_{\hat{F}(\cdot,x(\cdot))}$$
$$\Rightarrow [x,y] \in GrR$$
$$\Rightarrow R(\cdot) \quad \text{has a closed graph} \quad W \times W.$$

Applying theorem 4.1 of Tarafdar-Výborný [28] to get $x \in W$ *s.t.* $x \in R(x)$. Then $x \in C(T,X)$ solves the integral inclusion $(*)$ with the orientor field $\hat{F}(t,x)$. But working as in the beginning of the proof and using the definition of $\hat{F}(t,x)$, we get via Gronwall's inequality that $\|x(t)\| \leqslant M_1$ and so $\hat{F}(t,x(t)) = F(t,x(t)) \Rightarrow x(\cdot) \in C(T,X)$ is the desired solution of $(*)$. □

We can weaken the measurability hypothesis on the orientor field $F(t,x)$ if we assume that X^* is separable. So our hypothesis on the orientor field $F(t,x)$ is now the following:

$H(F)_1$: $F\colon T \times X \to P_{kc}(X)$ is a multifunction *s.t.*

(1) $(t,x) \to F(t,x)$ is weakly measurable,
(2) $x \to F(t,x)$ is *u.s.c.* from X into X_w,
(3) $|F(t,x)| \leqslant a(t) + b(t)\|x\|$ a.e., with $a(\cdot), b(\cdot) \in L^2_+$,
(4) $\beta(F(t,B)) \leqslant k(t)\beta(B)$ a.e. for all $B \subseteq X$ bounded and with $k(\cdot) \in L^1_+$.

Theorem 3.2. *If X^* is separable, hypotheses $H(F)_1$ and $H(K)$ hold and $p(\cdot) \in C(T,X)$, then $(*)$ admits a solution.*

Proof. The proof is the same of theorem 3.1. It only changes, when we prove the measurability of $s \to \overline{\text{conv}} H(s) = \overline{\text{conv}} \hat{F}(s,B(s))$. Note that $\hat{F}(\cdot,\cdot)$ is weakly measurable, since $F(\cdot,\cdot)$ is. Then for every $x^* \in X^*$, we have

$$\sigma(x^*,H(s)) = \sup_{v\in B(s)} \sigma(x^*,\hat{F}(s,v))$$

and from theorem 6.1 of [13], we have that $s \to \sigma(x^*, H(s))$ is measurable. Then note that if $\{x_n^*\}_{n\geqslant 1}$ is dense in X^*, since $\sigma(., H(s))$ is continuous ($H(s)$ being bounded), we have

$$Gr(\overline{\mathrm{conv}}H) = \bigcap_{n\geqslant 1} \{(s,v) \in T \times X : (x_n^*, v) \leqslant \sigma(x_n^*, H(s))\} \in \mathscr{L}(T) \times B(X),$$

with $\mathscr{L}(T)$ being the Lebesgue σ-field of T (i.e., the Lebesgue completion of $B(T)$). Hence $t \to \overline{\mathrm{conv}}H(t)$ is Lebesgue measurable, and so we can find $h_n : T \to X$ $n \geqslant 1$ Lebesgue measurable functions *s.t.* $\overline{\mathrm{conv}}H(t) = \overline{\{h_n(t)\}}_{n\geqslant 1}$ for all $t \in T$. Then we proceed as in the proof of theorem 3.1. Note that in a similar way, we get $t \to G(t) = \overline{\mathrm{conv}} \bigcup_{n\geqslant 1} \hat{F}(t, x_n(t))$ is weakly measurable and since $G(t) \in P_{wkc}(X)$, $t \in T$, it is measurable and the arguments in the proof of theorem 3.1 apply. □

We can relax our hypothesis on the kernel $K(t,s)$ if we strengthen further our growth hypothesis on the orientor field $F(t,x)$. So our hypothesis on $F(t,x)$ is now the following:

$H(F)_2$: $F: T \times X \to P_{fc}(X)$ is a multifunction *s.t.* hypotheses $H(F)(1)$ (or $H(F)_1(1)$ with X^* separable) and $H(F)(2)$ hold and

- (3) $|F(t,x)| \leqslant a(t) + b(t)\|x\|$ a.e., with $a(\cdot), b(\cdot) \in L^1_+$,
- (4) there exists Lebesgue-null set $N \subseteq T$ *s.t.* for all $B \subseteq X$ bounded $F((T \setminus N), B)$ is bounded,
- (5) $\beta(F(t,B)) \leqslant k(t)\beta(B)$ *a.e.* for all $B \subseteq X$ bounded and with $k(\cdot) \in L^1_+$.

Remark. Note that hypothesis $H(F)_2$ is satisfied if in $H(F)_2(3)$ $a(\cdot), b(\cdot) \in L^\infty_+$.

The weakened hypothesis on the kernel $K(t,s)$ is now the following:

$H(K)_1$: $K: \Delta \to \mathscr{L}(X)$ is a strongly continuous kernel.

Theorem 3.3. *If hypotheses $H(F)_2$ and $K(K)$ hold, then $(*)$ admits a solution.*

Proof. As in the beginning of the proof of theorem 3.1, we get that if $x \in C(T,X)$ solves $(*)$, then for all $t \in T$, $\|x(t)\| \leqslant M_1$, $M_1 > 0$. Let $B_{M_1}(0) = \{x \in X : \|x\| \leqslant M_1\}$ and $V = F(T \setminus N, B_{M_1}(0))$. Because of hypothesis $H(F)_2(4)$, $V \subseteq X$ is bounded. Let

$$W = \left\{ y \in C(T,X) : y(t) = p(t) + \int_0^t K(t,s)g(s)\,\mathrm{d}s,\ t \in T,\ g(s) \in \overline{\mathrm{conv}}V \text{ a.e.} \right\}.$$

Clearly $W \subseteq C(T,X)$ is nonempty, closed, convex and bounded. We claim that it is also equicontinuous. To this end, note that for all $(t,s) \in \Delta$, $s \notin N \subseteq T$ (see hypothesis $H(F)_2(4)$) and all $\|y\| \leqslant M_1$, we have

$$K(t,s)F(s,y) \in B_{M|V|}(0) = \hat{B}$$

where $B_{M|V|}(0) = \{w \in X : \|w\| \leqslant M|V|\}$, $|V| = \sup\{\|v\| : v \in V\} < \infty$ (since V is bounded). Hence if $y \in W$, we have for $t', t \in T$, $t' > t$ and $g \in S^1_{\overline{\text{conv}} V}$

$$\|y(t') - y(t)\| \leqslant \|p(t') - p(t)\| + \left\| \int_0^{t'} K(t',s)g(s)\,\mathrm{d}s - \int_0^{t} K(t,s)g(s)\,\mathrm{d}s \right\|.$$

Observe that by the "mean value theorem" for Bochner integrals (see Diestel-Uhl [5], corollary 8, p. 48), we have

$$\int_0^{t'} K(t',s)g(s)\,\mathrm{d}s \in t'\hat{B} \quad \text{and} \quad \int_0^{t} K(t,s)g(s)\,\mathrm{d}s \in t\hat{B}.$$

Therefore we get

$$\int_0^{t'} K(t',s)g(s)\,\mathrm{d}s - \int_0^{t} K(t,s)g(s)\,\mathrm{d}s \in t'\hat{B} - t\hat{B} = (t'-t)\hat{B}.$$

Hence we finally have that

$$\|y(t') - y(t)\| \leqslant \|p(t') - p(t)\| + (t'-t)M|V|$$

which establishes the equicontinuity.

The rest of the proof is the same as in theorem 3.1 (see also theorem 3.2 for the case where $H(F)_1(1)$ holds, with X^* separable). □

We can also have an existence result for the case where the orientor field $F(t,x)$ is not necessarily convex-valued. We will need the following hypothesis on $F(t,x)$.

$H(F)_3$: $F : T \times X \to P_f(X)$ is a multifunction *s.t.*

(1) $(t,x) \to F(t,x)$ is measurable,

(2) $x \to F(t,x)$ is *l.s.c.*,

(3) $|F(t,x)| \leqslant a(t) + b(t)\|x\|$ a.e., with $a(\cdot), b(\cdot) \in L^2_+$,

(4) $\beta(F(t,B)) \leqslant k(t)\beta(B)$ a.e. for all $B \subseteq X$ nonempty bounded and with $k(\cdot) \in L^1_+$.

As in the "convex" case an alternative set of hypotheses on $F(t,x)$ is the following:

$H(F)_3'$: $F : T \times X \to P_f(X)$ is a multifunction *s.t.* $H(F)_3$ holds and in addition

(5)′ there exist $N \subseteq T$ Lebesgue-null *s.t.* for all $B \subseteq X$ bounded $F(T \setminus N, B)$ is bounded.

As before this hypothesis on the orientor field will correspond to a weaker hypothesis on the kernel $K(t,s)$.

Theorem 3.4. *If hypotheses $H(F)_3$ had $H(K)$ (or $H(F)_3'$ and $H(K)_1$) hold, and $p \in C(T,X)$, then $(*)$ admits a solution.*

Proof. Let $\hat{F}(t,x)$ and $W \subseteq C(T,X)$ be as in the proof of theorem 3.1. Note that $\hat{F}(\cdot,\cdot)$ is measurable, $\hat{F}(t,\cdot)$ is *l.s.c.*, $|\hat{F}(t,x)| \leqslant \varphi(t)$ a.e. with $\varphi(\cdot) \in L^2_+$ and $\beta(\hat{F}(t,B)) \leqslant k(t)\beta(B)$ a.e. for all $B \subseteq X$ bounded. Define $L\colon W \to P_f(L^1(X))$ by $L(x) = S^1_{\hat{F}(\cdot,x(\cdot))}$. From theorem 4.1 of [19], we have that $L(\cdot)$ is *l.s.c.* and clearly has decomposable values (i.e., if $A \in \mathscr{L}(T)$ = Lebesgue σ-field of T and $f_1, f_2 \in S^1_{\hat{F}(\cdot,x(\cdot))}$, then $f = \chi_A f_1 + \chi_{A^c} f_2 \in S^1_{\hat{F}(\cdot,x(\cdot))}$). So we can apply theorem 3 of Bressan-Colombo [4] and get $u\colon W \to L^1(X)$ continuous *s.t.* $u(x) \in L(x)$ for all $x \in W$. Then let v: $W \to W$ be defined by

$$v(x)(t) = p(t) + \int_0^t K(t,s)u(x)(s)\,\mathrm{d}s, \quad t \in T.$$

Clearly, because $u(\cdot)$ is continuous, so is $v(\cdot)$. Also as we did with the multifunction $R(\cdot)$ in the proof of theorem 3.1, we establish that v is a ψ-contradiction. Applying the Sadovski-Darbo fixed point theorem, we get $x \in W$ *s.t.* $v(x) = x$. We can easily check that $\|x(t)\| \leqslant M_1 \Rightarrow \hat{F}(t,x(t)) = F(t,x(t)) \Rightarrow x(\cdot)$ solves $(*)$. □

4. The solution set

In the previous section, we obtained conditions on the data that guaranteed that the solution set of $(*)$ is nonempty. In this section we examine the properties of this solution set.

We start with a continuous dependence result that examines the changes in the solution set as we vary the function $p(t)$ and the orientor field $F(t,x)$.

So let Λ be a compact metric space and consider the following of integral inclusions, parametrized by elements in Λ.

$$(*)_\lambda \qquad x(t) \in p(t,\lambda) + \int_0^t K(t,s)F(s,x(s),\lambda)\,\mathrm{d}s.$$

Denote the solution set of $(*)_\lambda$ by $S(\lambda)$. Our goal is to investigate the continuity properties of the multifunction $\lambda \to S(\lambda)$.

For this we will need the following hypotheses:

$H(F)_4$: $F\colon T \times X \times \Lambda \to P_{kc}(X)$ is a multifunction *s.t.*

(1) $t \to F(t,x,\lambda)$ is measurable,
(2) $h\big(F(t,x,\lambda), F(t,x',\lambda)\big) \leqslant \eta(t)\|x-x'\|$ a.e. for all $\lambda \in \Lambda$ and with $\eta(\cdot) \in L^1_+$,
(3) $\lambda \to F(t,x,\lambda)$ is h-continuous,
(4) $|F(t,x,\lambda)| \leqslant a(t) + b(t)\|x\|$ a.e. for all $\lambda \in \Lambda$ and with $a(\cdot).b(\cdot) \in L^2_+$,
(5) $\beta\big(F(t,B,\lambda)\big) \leqslant k(t)\beta(B)$ a.e. for all $\lambda \in \Lambda$ and with $k(\cdot) \in L^1_+$.

$H(p)$: $\lambda \to p(.,\lambda)$ is continuous from Λ into $C(T,X)$.

Theorem 4.1. *If X is a separable, reflexive, strictly convex Banach space and hypotheses $H(F)_4$, $H(K)$ and $H(p)$ hold, then $S\colon \Lambda \to P_k\big(C(T,X)\big)$ is continuous and h-continuous.*

Remark. From Asplund's renorming theorem, we know that every reflexive Banach space can be equivalently renormed so that both X and X^* are strictly convex.

Proof. First we will show that for every $\lambda \in \Lambda$, $S(\lambda) \in P_k\big(C(T,X)\big)$. The nonemptiness of $S(\lambda)$ follows from theorem 3.1. Also let $\{x_n\}_{n\geqslant 1} \subseteq S(\lambda)$. Then by definition we have

$$x_n(t) \in p(t,\lambda) + \int_0^t K(t,s) f_n(s)\,\mathrm{d}s, \quad t \in T$$

with $f_n \in S^1_{F(\cdot,x_n(\cdot),\lambda)}$. Applying proposition 1.6 of Mönch [18], we get

$$\begin{aligned} \beta\big(\{x_n(t)\}_{n\geqslant 1}\big) &\leqslant \int_0^t M\beta\big(\{f_n(s)\}_{n\geqslant 1}\big)\,\mathrm{d}s \\ &\leqslant \int_0^t Mk(s)\beta\big(\{x_n(s)\}_{n\geqslant 1}\big)\,\mathrm{d}s \\ &\Rightarrow \beta\big(\{x_n(t)\}_{n\geqslant 1}\big) = 0 \quad \text{(Gronwall's inequality)}, \\ &\Rightarrow \overline{\{x_n(t)\}}_{n\geqslant 1} \quad \text{is compact for every} \quad t \in T. \end{aligned}$$

Also from the proof of theorem 3.1 we know that it is equicontinuous. Hence the Arzela-Ascoli theorem tells us that $\{x_n(\cdot)\}_{n\geqslant 1}$ is relatively compact in $S(\lambda) \Rightarrow S(\lambda)$ is relatively compact in $C(T,X)$. So we may assume that $x_n \to x$ in $C(T,X)$. Next note that $\|f_n(t)\| \leqslant a(t) + b(t)M_1 = \varphi(t)$ a.e. and because X is reflexive, from Dunford's theorem (see Diestel-Uhl [5], theorem 1, p. 101), we have that $\{f_n\}_{n\geqslant 1}$ is relatively weakly compact in $L^1(X)$. So by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(X)$. Then from hypothesis $H(F)_4$ and theorem 3.1 of [19] we get $f \in S^1_{F(\cdot,x(\cdot),\lambda)}$. Hence in the limit as $n \to \infty$ we have that

$$x(t) \in p(t,\lambda) + \int_0^t K(t,s) f(s)\,\mathrm{d}s, \quad t \in T$$

with $f \in S^1_{F(\cdot,x(\cdot),\lambda)}$. Thus $x(\cdot) \in S(\lambda) \Rightarrow S(\lambda)$ is closed, hence compact in $C(T,X)$.

Next let $\lambda_n \to \lambda$ in Λ and take $x \in \overline{\lim} S(\lambda_n)$. Then by definition and by denoting for economy in the notation, subsequences with the same index as sequences, we know (see section 2), that we can find $x_n \in S(\lambda_n)$ *s.t.* $x_n \xrightarrow{s} x$ in $C(T,X)$. Then by definition

$$x_n(t) = p(t,\lambda_n) + \int_0^t K(t,s) f_n(s)\,\mathrm{d}s$$

for all $t \in T$ and with $f_n \in S^1_{F(\cdot,x_n(\cdot),\lambda_n)}$.

Note that because of hypothesis $H(F)_4$ and since $x_n \to x$, we have that

$$\overline{\bigcup_{n\geqslant 1} F\big(t,x_n(t),\lambda_n\big)} \in P_k(X) \qquad \text{a.e.}$$

(see Klein-Thompson [14], theorem 7.4.2, p. 90)

$$\Rightarrow H(t) = \overline{\mathrm{conv}} \bigcup_{n\geqslant 1} F\big(t,x_n(t),\lambda_n\big) \in P_{kc}(X)$$

(by Mazur's theorem; see Diestel-Uhl [5], theorem 12, p. 51). Also because of hypothesis $H(F)_4(1)$, $t \to F\big(t,x_n(t),\lambda_n\big)$ $n \geqslant 1$ is measurable $\Rightarrow t \to \overline{\bigcup_{n\geqslant 1} F\big(t,x_n(t),\lambda_n\big)}$ is measurable $\Rightarrow t \to H(t)$ is measurable (see Himmelberg [10], theorem 9.1). Furthermore $|H(t)| \leqslant a(t) + b(t)\hat{M} = \varphi_1(t)$ a.e. $\varphi_1(\cdot) \in L^2_+$, with $\hat{M} > 0$ being such that $\|x_n\|_\infty \leqslant \hat{M}$ for all $n \geqslant 1$. Then proposition 3.1 of [23] tells us that $S^1_H \in P_{wkc}\big(L^1(X)\big)$. Observe that $\{f_n\}_{n\geqslant 1} \subseteq S^1_H$. So by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(X)$. Then for every $v \in L^\infty(X^*) = L^1(X)^*$, we have

$$\langle v, f_n\rangle = \int_0^b \big(v(t), f_n(t)\big)\,\mathrm{d}t \leqslant \sigma\big(v, S^1_{F(\cdot,x_n(\cdot),\lambda_n)}\big) = \int_0^b \sigma\big(v(t), F\big(t,x_n(t),\lambda_n\big)\big)\,\mathrm{d}t.$$

But note that because of hypothesis $H(F)_4$,

$$\sigma\big(v(t), F\big(t,x_n(t),\lambda_n\big)\big) \to \sigma\big(v(t), F\big(t,x(t),\lambda\big)\big).$$

So in the limit as $n \to \infty$, we get

$$\langle v, f\rangle = \int_0^b \big(v(t), f(t)\big)\,\mathrm{d}t \leqslant \int_0^b \sigma\big(v(t), F\big(t,x(t),\lambda\big)\big)\,\mathrm{d}t = \sigma\big(v, S^1_{F(\cdot,x(\cdot),\lambda)}\big).$$

Since $v \in L^\infty(X^*)$ was arbitrary, we deduce $f \in S^1_{F(.,x(\cdot),\lambda)}$. Also note that $\int_0^t K(t,s) f_n(s)\,\mathrm{d}s \xrightarrow{w} \int_0^t K(t,s) f(s)\,\mathrm{d}s$ in X. Hence in the limit as $n \to \infty$, we get

$$(\bar{1}) \qquad x(t) = p(t) + \int_0^t K(t,s) f(s)\,\mathrm{d}s, \quad t \in T, \quad f \in S^1_{F(.,x(\cdot),\lambda)}$$
$$\Rightarrow x \in S(\lambda)$$
$$\Rightarrow \overline{\lim}\, S(\lambda_n) \subseteq S(\lambda).$$

Next let $x \in S(\lambda)$. Then by definition, we have

$$x(t) = p(t,\lambda) + \int_0^t K(t,s) f(s)\,\mathrm{d}s, \quad t \in T, \quad f \in S^1_{F(.,x(\cdot),\lambda)}.$$

Set $m_n(t) = \operatorname{proj}\big[f(t); F\big(t, x(t), \lambda_n\big)\big]$ and $u_n(t,z) = \operatorname{proj}[m_n(t); F(t,z,\lambda_n)]$. Since X is strictly convex, reflexive and $F(\cdot,\cdot,\cdot)$ is convex valued, $m_n(\cdot)$ and $u_n(\cdot,\cdot)$ are both well-defined, single valued functions. Furthermore from theorem 4.2 of [11], we know that $m_n(\cdot)$ and $u_n(.,z)$ are measurable functions, while from theorem 3.33, p. 322 of Attouch [1], we have that $u_n(t,\cdot)$ is continuous. Then consider the following integral equation:

$$x_n(t) = p(t,\lambda_n) + \int_0^t K(t,s) u_n\big(s, x_n(s)\big)\,\mathrm{d}s.$$

From theorem 3.1 we know that this has a solution $x_n(\cdot) \in C(T,X)$. Also we have:

$$\begin{aligned}
&\|x_n(t) - x(t)\| \\
&\quad \leqslant \left\| \int_0^t K(t,s)\big[u_n\big(s,x_n(s)\big) - f(s)\big]\,\mathrm{d}s \right\| \\
&\quad \leqslant \int_0^t M\big[\big\|u_n\big(s,x_n(s)\big) - m_n(s)\big\| + \|m_n(s) - f(s)\|\big]\,\mathrm{d}s \\
&\quad \leqslant \int_0^t M\big[h\big(F\big(s,x_n(s),\lambda_n\big), F\big(s,x(s),\lambda_n\big)\big) + h\big(F\big(s,x(s),\lambda_n\big), F\big(s,x(s),\lambda\big)\big)\big]\,\mathrm{d}s \\
&\quad \leqslant \int_0^t M\eta(s)\|x_n(s) - x(s)\|\,\mathrm{d}s + \int_0^t Mh\big(F\big(s,x(s),\lambda_n\big)\big)F\big(s,x(s),\lambda\big)\,\mathrm{d}s.
\end{aligned}$$

But by hypothesis $H(F)_4(3)$, $h\big(F\big(s,x(s),\lambda_n\big), F\big(s,x(s),\lambda\big)\big) \to 0$ as $n \to \infty$. So given $\varepsilon > 0$ for $n \geqslant 1$ large enough, we will have:

$$\|x_n(t) - x(t)\| \leqslant \varepsilon + M \int_0^t \eta(s)\|x_n(s) - x(s)\|\,\mathrm{d}s.$$

Applying Gronwall's inequality, we get

$$\|x_n - x\|_\infty \leqslant \varepsilon \exp(M\|\eta\|_1)$$

for $n \geqslant 1$ large enough. So $x_n \to x$ in $C(T, X)$. Note that $x_n \in S(\lambda_n)$, $n \geqslant 1$. Thus we have that

$$S(\lambda) \subseteq \underline{\lim}\, S(\lambda_n). \tag{2}$$

From (1) and (2) above, we get that

$$S(\lambda_n) \xrightarrow{K} S(\lambda) \quad \text{as} \quad n \to \infty.$$

We claim that $V = \overline{\bigcup_{n\geqslant 1} S(\lambda_n)}$ is compact in X. Indeed let $\{x_m\}_{m\geqslant 1} \subseteq V$. By definition, we have

$$x_m(t) = p(t, \lambda_n) + \int_0^t K(t,s) f_m(s)\, \mathrm{d}s, \quad f_m \in S^1_{F(\cdot, x_m(\cdot), \lambda_m)}, \quad m \geqslant 1.$$

Set $B(t) = \{x_m(t)\}_{m\geqslant 1}$. As before, we get

$$\begin{aligned} \beta(B(t)) &\leqslant \int_0^t Mk(s)\beta(B(s))\, \mathrm{d}s \\ &\Rightarrow \beta(B(t)) = 0, \quad t \in T. \\ &\Rightarrow \overline{B(t)} \quad \text{is compact for all} \quad t \in T. \end{aligned}$$

Furthermore since $\{x_m(\cdot)\}_{m\geqslant 1}$ is equicontinuous, from the Arzela-Ascoli theorem, we have that $\{x_m\}_{m\geqslant 1}$ is relatively compact in $C(T, X) \Rightarrow V$ is compact in $C(T, X)$.

Then from (3) and since $S(\lambda_n), S(\lambda) \subseteq V$, from Klein-Thompson [14] theorems 7.1.10 and 7.1.16, we deduce that $S(\cdot)$ is continuous in the Vietoris topology. Since $S(\cdot)$ is $P_k(C(T, X))$-valued, we then conclude that $S(\cdot)$ is also h-continuous (see section 2). □

Remark. The compactness of the values of $S(\cdot)$ is true with X being only a separable Banach space.

Next we ask the question of whether the solution set of $(*)$ is connected (Kneser type theorem). We have a partial answer to this problem. Namely for a particular type of orientor fields, which appear in control problems, we have that property. So we will assume that $F(t, x)$ has the following special form: $F(t, x) = f(t, x)U(t)$,

$t \in T$. We will need the following hypotheses. Assume that Y is another separable Banach space. In the context of control systems, this will be the control space.

$H(f)$: $f: T \times X \to \mathscr{L}(Y, X)$ is a map *s.t.*

(1) $t \to f(t, x)u$ is measurable for all $u \in Y$,

(2) $\|f(t,x)u - f(t,y)u\| \leqslant \eta(t)\|x-y\|$ for all $t \in T \setminus N$, $\lambda(N) = 0$ and $u \in U(t)$,

(3) $\|f(t,x)\|_{\mathscr{L}} \leqslant a(t)\|x\|$ a.e. with $a(\cdot) \in L^2_+$,

(4) $\beta\big(f(t,B)U(t)\big) \leqslant k(t)\beta(B)$ a.e., for all $B \subseteq X$ bounded and with $k(\cdot) \in L^1_+$.

$H(U)$: $U \to P_{wkc}(Y)$ is measurable multifunction *s.t.* $|U(t)| = \sup\{\|u\| : u \in U(t)\} \leqslant N$ a.e.

Theorem 4.2. *If hypotheses $H(f)$, $H(U)$, $H(K)$ hold and $p \in C(T,X)$ then the solution set S of $(*)$ with $F(t,x) = f(t,x)U(t)$, is nonempty, compact and path connected in $C(T,X)$.*

Proof. The nonemptiness and compactness of S in $C(T,X)$ follows from theorems 3.1 and 4.1 (see also the remark following that theorem). We only need to establish path connectedness. Let $W \subseteq C(T,X)$ and $R: W \to P_{fc}(W)$, be as in the proof of theorem 3.1. Let $y \in R(x)$. By definition we have

$$y(t) = p(t) + \int_0^t K(t,s)g(s)\,\mathrm{d}s$$

for all $t \in T$ and $g \in S^1_{f(\cdot,x(\cdot))U(\cdot)}$. A simple application of Aumann's selection theorem (see Wagner [30], theorem 5.10), gives us $u \in S^1_U$ *s.t.* $g(t) = f\big(t, x(t)\big)u(t)$ a.e. Then let $v_{x,y}: C(T,X) \to C(T,X)$ be defined by

$$v_{xy}(z)(t) = p(t) + \int_0^t K(t,s) f\big(s, z(s)\big) u(s)\,\mathrm{d}s.$$

We have:

$$\begin{aligned}\|v_{xy}(z')(t) - v_{xy}(z)(t)\| &\leqslant \int_0^t M \big\| f\big(s, z'(s)\big)u(s) - f\big(s, z(s)\big)u(s)\big\|\,\mathrm{d}s \\ &\leqslant \int_0^t M\eta(s)\|z'(s) - z(s)\|\,\mathrm{d}s.\end{aligned}$$

Introduce on $C(T,X)$ the following equivalent norm (the Bielecki norm):

$$\|x\|_0 = \sup_{t\in T}\left[\mathrm{e}^{-r\int_0^t k(s)\mathrm{d}s} x(t)\right], \quad r > 0.$$

Then we have

$$\|v_{xy}(z')(t) - v_{xy}(z)(t)\| \leqslant \int_0^t Mk(s)e^{r\int_0^s k(s)ds}e^{-r\int_0^s k(s)ds}\|z'(s) - z(s)\|\, ds$$
$$\Rightarrow \|v_{xy}(z')(t) - v_{xy}(z)(t)\| \leqslant \|z' - z\|_0 \frac{M}{r} e^{r\int_0^t k(s)ds}$$
$$\Rightarrow \|v_{xy}(z') - v_{xy}(z)\|_0 \leqslant \frac{M}{r}\|z' - z\|_0.$$

So if we choose $r > M$, we have that $v_{xy}(\cdot)$ is $\|.\|_0$-contractive. Also for all $z \in W$, $v_{xy}(z) \in R(z)$ and $v_{xy}(x) = y$. Thus, we can apply theorem 1.1 of Bogatyrev [3] and get that $S = \{x \in C(T, X) : x \in R(x)\}$ is path-connected by the theorem. □

Remark. Since every path-connected set is connected (see Dugundji [6], theorem 5.3, p. 115), we see that the conclusion of our theorem is stronger than the usual Kneser-type theorems about differential and integral equations and inclusions.

5. An example

In this section we present an example of a partial differential inclusion for which we can establish the existence of solutions using the results of this paper.

So let $T = [0, b]$ and let $Z \subseteq \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial Z = \Gamma$. Let $z = (z_1, \ldots, z_N)$ and $D_i = \frac{\partial}{\partial z_i}$. By a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$, we understand a tuple of nonnegative integers $\alpha_1, \ldots \alpha_N$. The length of the multi-index is defined by $|\alpha| = \sum_{k=1}^{N} |\alpha_k|$. Also we set $D^\alpha u = D_1^{\alpha_1} \ldots D_N^{\alpha_N} u$. For $\alpha = 0$, we set $D^0 u = u$. We consider the following partial differential inclusion:

$$\frac{\partial x(t,z)}{\partial t} + \sum_{|\alpha|,|\beta| \leqslant m} (-1)^{|\alpha|} D^\alpha \big(a_{\alpha\beta}(t,z) D^\beta x(t,z)\big) \in F\big(t, z, x(t,z)\big) \text{ on } T \times Z$$
$$D^\gamma x(t,z) = 0 \quad \text{on} \quad T \times \Gamma, \quad |\gamma| \leqslant m-1$$
$$(**) \qquad x(0,z) = x_0(z) \quad \text{on} \quad Z.$$

We will make the following hypotheses concerning the data on (**).

$H(a)$: $a_{\alpha\beta} \in L^\infty(T \times Z)$, $|a_{\alpha\beta}(t', z) - a_{\alpha\beta}(t, z)| \leqslant \theta(z)|t' - t|$ a.e. with $\theta(\cdot) \in L^\infty$ and

$$\sum_{|\alpha|,|\beta| \leqslant m} a_{\alpha\beta}(t,z)\xi_\alpha \xi_\beta \geqslant c\|\xi\|^2$$

for all $\xi \in \mathbf{R}^{N_m}$ $\left(N_m = \frac{(N+m)!}{m!N!}\right)$ and with $c > 0$.

$H(F)_5$: $F: T\times Z\times \mathbf{R} \to P_{fc}(\mathbf{R})$ is defined by $F(t,z,x) = \overline{\text{conv}}\{f_n(t,z,x)\}_{n\geqslant 1}$, where for each $n \geqslant 1$ $(t,z) \to f_n(t,z,x)$ is measurable, $\sup\limits_{n,m>1} |f_n(t,z,x)| - f_m(t,z,x)| \leqslant \eta(t,z)|x'-x|$ a.e. with $\eta(\cdot,\cdot) \in L^\infty(T\times Z)$ and $|f_n(t,z,x)| \leqslant a(t,z) + b(t,z)|x|$ a.e. with $a(\cdot,\cdot) \in L^2(T\times Z)$ and $b(\cdot,\cdot) \in L^\infty(T\times Z)$.

Let $X = H_0^m(Z)$, $H = L^2(Z)$ and $X^* = H_0^m(Z)^* = H^{-m}(Z)$. Then from the Sobolev embedding theorem, we know that $X \hookrightarrow H \hookrightarrow X^*$, with all embeddings being dense, continuous and compact. So (X,H,X^*) is an evolution triple. Consider the time dependent Dirichlet form $u: T\times H_0^m(Z)\times H_0^m(Z) \to \mathbf{R}$ defined by

$$u(t,x,y) = \sum_{|\alpha|,|\beta|\leqslant m} \int_Z a_{\alpha\beta}(t,z) D^\beta x(z) D^\alpha y(z)\,\mathrm{d}z.$$

Using the Cauchy-Schwartz inequality, we can easily get that

$$|u(t,x,y)| \leqslant \hat{c}\|x\|_{H_0^m(Z)}\|y\|_{H_0^m(Z)}$$

for some $\hat{c} > 0$. Also from the "strong ellipticity" condition (see hypothesis $H(a)$) and the Poincare inequality (see Zeidler [31]), we get

$$u(t,x,x) \geqslant \hat{c}_1\|x\|^2_{H_0^m(Z)}$$

with $\hat{c}_1 > 0$. Furthermore from hypothsis $H(a)$ and the Cauchy-Schwartz inequality, we get

$$|u(t',x,y) - u(t,x,y)| \leqslant \|\theta\|_\infty |t'-t| \|x\|_{H_0^m(Z)}\|y\|_{H_0^m(Z)}.$$

Let $A: T \to \mathscr{L}(X,X^*)$ be defined by

$$\langle A(t)x, y\rangle = u(t,x,y)$$

where $\langle\cdot,\cdot\rangle$ denotes the duality brackets for the pair (X,X^*).

Next let $\hat{F}: T\times H \to P_{wkc}(H)$ be defined by

$$\hat{F}(t,z) = \left\{y \in L^2(Z) : y(z) \in \overline{\text{conv}}\left\{f_n\left(t,z,x(z)\right)\right\}_{n\geqslant 1} \text{ a.e.}\right\}.$$

Note that for every $v \in L^2(Z)$, we have

$$\begin{aligned}\sigma\left(v,\hat{F}(t,x)\right) = \sup_{y\in\hat{F}(t,z)} (v,y)_{L^2(Z)} = \sup_{n\geqslant 1}\int_Z v(z) f_n\left(t,z,x(z)\right)\mathrm{d}z \\ \Rightarrow t \to \sigma\left(v,\hat{F}(t,x)\right) \\ \Rightarrow t \to \hat{F}(t,x) \quad \text{is measurable.}\end{aligned}$$

Next we claim that $x \to \hat{F}(t,x)$ is *u.s.c.* from H into H_w, where H_w denotes the Hilbert space H endowed with the weak topology. For this we will need the following lemma:

Lemma 5.1. *If X is a separable Banach space, $F: X \to P_{wkc}(X)$ is a multifunction s.t. for every $K \in P_{wk}(X)$, $F|_K$ is u.s.c. from X into X_w, then $F(\cdot)$ is u.s.c. from X into X_w.*

Proof. We know (see section 2), that $F(\cdot)$ will be *u.s.c.* from X into X_w, if for every $U \subseteq X$ weakly open $V = \{x \in X : F(x) \subseteq U\}$ is open in X. This is equivalent to saying that for every $D \subseteq X$ weakly closed, the set $C = \{x \in X : F(x) \cap D \neq \emptyset\}$ must be closed. So let $K = \{x_n\}_{n\geqslant 1}$, $x_n \to x$. By hypothesis $F|_K$ is *u.s.c* from X into X_w. Hence, since $F(\cdot)$ is $P_{wkc}(X)$-valued, we get that $\overline{\bigcup_{n\geqslant 1} F(x_n)}^w \in P_{wk}(X)$. So by passing to a subsequence if necessary, we may assume that $y_n \xrightarrow{w} y$ in X. Then since $F|_K$ is *u.s.c.* from X into X_w, $y \in F(x)$ and $y \in D$, since D is weakly closed. Therefore C is closed, establishing the desired upper semicontinuity of $F(\cdot)$. □

Continuing with the analysis of (**), let $[x_m, y_m] \in Gr\hat{F}(t,\cdot)$ $m \geqslant 1$ and assume that $x_m \xrightarrow{s} x$ in $L^2(Z)$, while $y_m \xrightarrow{w} y$ in $L^2(Z)$. By a passing to a subsequence if necessary, we may assume that $x_m(z) \to x(z)$ a.e. Invoking theorem 3.1 of [19], we get that

$$y(z) \in \overline{\text{conv}}\, \overline{\lim_{m\to\infty}} \{y_m(z)\}_{m\geqslant 1} \subseteq \overline{\text{conv}}\, \overline{\lim_{m\to\infty}} \left[\overline{\text{conv}}\{f_n(t,z,x_m(z))\}_{m\geqslant 1}\right] \quad \text{a.e.}$$

But from proposition 3.1 and 4.1 of [19], we get

$$\overline{\lim_{m\to\infty}} \left[\overline{\text{conv}}\{f_n(t,z,x_m(z))\}_{n\geqslant 1}\right] \subseteq \overline{\text{conv}}\{f_n(t,z,x(z))\}_{n\geqslant 1} \quad \text{a.e.}$$

So we get $y(z) \in \overline{\text{conv}}\{f_n(t,z,x(z))\}_{n\geqslant 1}$ a.e. $\Rightarrow [x,y] \in Gr\hat{F}(t.,)$. Therefore for every $B \subseteq L^2(Z)$ bounded (hence relatively weakly compact), we have that $\hat{F}(t,\cdot)|_{\overline{B}}$ has a graph that is closed in $H \times H_w$, so theorem 7.1.16, p. 78 of Klein-Thompson [14], tells us that $\hat{F}(t,\cdot)|_{\overline{B}}$ is *u.s.c.* from H into H_w. Finally using lemma 5.1 above, we conclude that $\hat{F}(t,\cdot)$ is *u.s.c.* from H into H_w, as claimed.

Now let $\hat{f}_n : T \times H \to H$ be the Nemitski (superposition) operator corresponding to function $f_n(t,z,x)$, $n \geqslant 1$; i.e.,

$$\hat{f}_n(t,x)(z) = f_n(t,z,x(z)) \quad \text{a.e.}$$

for every $x \in L^2(Z)$ and every $n \geqslant 1$.

Because of hypothesis $H(f)$, we have that

$$\sup_{n,m\geqslant 1} \|\hat{f}_n(t,x') - \hat{f}_m(t,x)\|_{L^2(Z)} \leqslant M'\|\eta\|_\infty \|x' - x\|_{L^2(Z)}$$

for all $x', x \in L^2(Z)$ and some $M' > 0$.

Recalling the definitions of the multifunction $\hat{F}(t,x)$ and of the measure of non-compactness $\beta(\cdot)$, we get immediately for every $B \subseteq L^2(Z)$ bounded

$$\beta(\hat{F}(t,B)) \leqslant \overline{M}\beta(B) \quad \text{a.e. with} \quad \overline{M} > 0.$$

Now rewrite $(**)$ in the following equivalent evolution inclusion form:

$$(**)' \qquad \begin{cases} \dot{x}(t) + A(t)x(t) \in \hat{F}(t,x(t)) & \text{a.e.} \\ x(0) = x_0(\cdot) \in L^2(Z). \end{cases}$$

From proposition 5.5.1, p. 153 of Tanabe [27], we know that a solution $x(\cdot) \in W(T) = \{x \in L^2(X) \colon x \in L^2(X^*)\} \subseteq C(T,H)$ (see also Zeidler [31]), has the form

$$x(t) = K(t,0)x_0 + \int_0^t K(t,s)f(s)\,\mathrm{d}s$$

with $t \in T$, $f \in S^1_{\hat{F}(\cdot,x(\cdot))}$. Here $K(t,s)$ is the evolution operator (fundamental solution), generated by $\{A(t)\}_{t\in T}$. From Tanabe [27], p. 149, relation 5.141, we have that $K(\cdot,\cdot)$ satisfies hypotesis $H(K)$.

So evolution inclusion $(**)'$ (equivalently problem $(**)$), is equivalent to the following Volterra integral inclusion in $H = L^2(Z)$

$$(**)'' \qquad x(t) \in p(t) + \int_0^t K(t,s)\hat{F}(s,x(s))\,\mathrm{d}s, \quad t \in T$$

with $p(t) = K(t,0)x_0$, $p(\cdot) \in C(T,H)$.

We have already checked that the data of $(**)''$ satisfy the hypotheses of theorem 3.1. So using that result, together with theorem 23.A, p. 424 of Zeidler [31], we get:

Theorem 5.2. *If hypoheses $H(a)$, $H(f)$ hold and $x_0 \in L^2(Z)$, then $(**)$ has a solution $x(\cdot,\cdot) \in L^2(T, H_0^m(Z)) \cap C(T, L^2(Z))$, with*

$$\frac{\partial x}{\partial t} \in L^2(T, H^{-m}(Z)).$$

Furthermore, the solution set of $(**)$ is compact in $C(T, L^2(Z))$.

R e m a r k . System (**) incorporates distributed parameter control systems. Indeed, let $Y = L^2(Z)$ be the control space, $U(t,z) = \{u \in \mathbf{R}\colon |u| \leqslant r(t,z)\}$, with $r(\cdot,\cdot) \in L^\infty(T \times Z)$ is the control constraint set and $f(t,z,x)u$ the control vector field. Assume that $(t,z) \to f(t,z,x)$ measurable, $\sup\limits_{u,v \in U(t,z)} |f(t,z,x')u - f(t,z,x)v| \leqslant \eta(t,z)|x' - x|$ a.e. with $\eta(\cdot,\cdot) \in L^1(T \times Z)$, and $|f(t,z,x)| \leqslant a(t,z) + b(t,z)|x|$ a.e. with $a(\cdot,\cdot) \in L^2(T \times Z)$, $b(\cdot,\cdot) \in L^\infty(T \times Z)$. Clearly $U(\cdot,\cdot)$ is measurable. So we can find $u_n\colon T \times Z \to \mathbf{R}$, $n \geqslant 1$ measurable functions *s.t.* $U(t,z) = \mathrm{cl}\{u_n(t,z)\}_{n \geqslant 1}$. Then $f(t,z,x)U(t,z) = \overline{\{h_n(t,z,x)\}_{n \geqslant 1}}$, with $h_n(t,z,x) = f(t,z,x)u_n(t,z)$. Then those functions $h_n(\cdot,\cdot,.,)$ satisfy hypothesis $H(f)$. Hence by theorem 5.2, the distributed parameter control system has a set of trajectories that is compact in $C(T, L^2(Z))$. So if we are given to minimize a cost functinal $\varphi(x(t,\cdot))$ where $\varphi\colon C(T, L^2(Z)) \to \overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$ is *l.s.c.*, then the optimal control problem admits a solution.

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