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TWO THEOREMS ON MEASURABLE SETS AND SETS HAVING THE BAIRE PROPERTY

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J. C. Oxtoby in his monograph "Measure and category" [2] presents a lot of analogies between measurable sets and sets having the Baire property. In our paper, another such analogy is shown.

Let $\mathbf{N}$ be the set of positive integers, $\mathbf{R}_+$—the set of positive reals and $\mathbf{R}$—the real line. If $A \subset \mathbf{R}$, $B \subset \mathbf{R}$, then $A \triangle B$ denotes the symmetric difference of A and B; $xA = \{xy\colon y \in A\}$. For any Lebesgue measurable set A, $|A|$ denotes its Lebesgue measure.

A point $x \in \mathbf{R}$ is said to be a *density point* of a measurable set $A \subset \mathbf{R}$ if

$$d(A, x) = \lim_{h \to 0+} \frac{|A \cap (x - h, x + h)|}{2h} = 1;$$

a *right density point* if

$$d^+(A, x) = \lim_{h \to 0+} \frac{|A \cap (x, x + h)|}{h} = 1.$$

If $d(A, x) = 0 (d^+(A, x) = 0)$, then we say that x is a *dispersion point (right dispersion point)* of A. $\Phi(A)$ denotes the set of all density points of A.

The terminology and definitions concerning topology and measure come from "Measure and category" by J. C. Oxtoby.

Lemma 1. *Let $A \subset \mathbf{R}_+$ be a measurable set such that $|A \cap (0, \delta)| > 0$ and $|(0, \delta) - A| > 0$ for any $\delta > 0$, and $(\lambda_n)_{n \in \mathbf{N}}$—a one-to-one sequence converging to* 1. *There exists a natural number n_0 such that*

$$\Phi(\lambda_{n_0} \cdot A) - \Phi(A) \neq \emptyset.$$

Proof. Suppose that $\Phi(A) \supset \Phi(\lambda_n A)$ for any natural number n. The sequence $(\lambda_n)_{n \in \mathbf{N}}$ contains a monotone subsequence. We can assume that $(\lambda_n)_{n \in \mathbf{N}}$ is an increasing or a decreasing sequence. In the first part of the proof we assume that this sequence is increasing.

Let x be an arbitrary density point of A and let $\alpha = \frac{|A\cap(0,x)|}{x}$. Obviously, $0 < \alpha < 1$ and, if we take any $\beta \in (\alpha, 1)$ and put

$$I = \bigcup\left\{(b,x)\colon \frac{|A\cap(c,x)|}{|(c,x)|} \geqslant \beta \text{ for any } c \in (b,x)\right\} = (b_0, x),$$

then $0 < b_0 < x$.

There exists a natural number n_0 such that $\lambda_{n_0}x \in (b_0, x)$. Let c be an arbitrary point from $(\lambda_{n_0}b_0, b_0)$. Since $\Phi(A\cap(c,\lambda_{n_0}x)) \supset \Phi(\lambda_{n_0}A\cap(c,\lambda_{n_0}x))$, therefore

$$|A\cap(c,\lambda_{n_0}x)| \geqslant |\lambda_{n_0}A\cap(c,\lambda_{n_0}x)|.$$

□

Moreover,

$$\frac{c}{\lambda_{n_0}} \in \left(b_0, \frac{b_0}{\lambda_{n_0}}\right) \subset (b_0, x),$$

thus

$$\frac{|A\cap(c,\lambda_{n_0}x)|}{|(c,\lambda_{n_0}x)|} \geqslant \frac{|\lambda_{n_0}A\cap(c,\lambda_{n_0}x)|}{|(c,\lambda_{n_0}x)|} = \frac{|A\cap(\frac{c}{\lambda_{n_0}},x)|}{|(\frac{c}{\lambda_{n_0}},x)|} \geqslant \beta.$$

On the other hand,

$$\frac{|A\cap(\lambda_{n_0}x,x)|}{|\lambda_{n_0}x,x)|} \geqslant \beta,$$

so $\frac{|A\cap(c,x)|}{(c,x)} \geqslant \beta$ for any $c \in (\lambda_{n_0}b_0, b_0)$. For $c \in \langle b_0, x)$, the same inequality is obvious by the definition of b_0. Finally, $b_0 = \inf I \leqslant \lambda_{n_0}b_0$, which gives a contradiction because $\lambda_{n_0}b_0 < b_0$.

If the sequence $(\lambda_n)_{n\in\mathbf{N}}$ is decreasing, the proof is analogous to the argument presented above. This time, we consider a dispersion point y of A and a density point x such that $0 < x < y$. Then we put $\alpha = \frac{|A\cap(x,y)|}{(x,y)}$ take any $\beta \in (\alpha, 1)$, set

$$I = \bigcup\left\{(x,b)\colon \frac{|A\cap(x,c)|}{|(x,c)|} \geqslant \beta \text{ for any } c \in (x,b)\right\} = (x, b_0)$$

and choose n_0 such that $\lambda_{n_0}x \in (x, b_0)$. It is not difficult to show that $b_0 = \sup I \geqslant \lambda_{n_0}b_0$, which is impossible.

In fact, we have proved that if $A \subset \mathbf{R}$ is a measurable set, x is a density point of A, y is a dispersion point of this set and $x > y$ $(x < y)$, then, for any increasing (decreasing) sequence $(\lambda_n)_{n\in\mathbf{N}}$ with $\lim_{n\to\infty} \lambda_n = 1$, there exists a natural number n_0 such that

$$\Phi(\lambda_{n_0}A) - \Phi(A) \neq \emptyset.$$

Theorem 1. *Let $A \subset \mathbf{R}_+$ be a measurable set such that $|A \cap (0,\delta)| > 0$ and $|(0,\delta) - A| > 0$ for any $\delta > 0$. Then the set*

$$\Lambda = \{\lambda > 0 \colon |(\lambda A \triangle A) \cap (0,\delta)| = 0 \ \ \textit{for some} \ \ \delta > 0\}$$

has cardinality less or equal to χ_0.

Proof. Suppose that the set Λ is uncountable. For any $\lambda \in \Lambda$ one can find the smallest natural number n_λ for which

$$\left|(\lambda A \triangle A) \cap \left(0, \frac{1}{n_\lambda}\right)\right| = 0.$$

Let $\Lambda_n = \{\lambda \in \Lambda \colon n_\lambda = n\}$ for any $n \in \mathbf{N}$. There exists n_0 such that λ_{n_0} is uncountable. The set Λ_{n_0} has a condensation point $\lambda_0 \in \Lambda_{n_0}$ ([1], p. 140). We have

$$\left|(\lambda_0 A \triangle A) \cap \left(0, \frac{1}{n_0}\right)\right| = 0,$$

therefore

$$\left|(\lambda A \triangle A) \cap \left(0, \frac{1}{n_0}\right)\right| = 0$$

if and only if

$$0 = \left|(\lambda A \triangle \lambda_0 A) \cap \left(0, \frac{1}{n_0}\right)\right| = \lambda_0 \left|\left(\frac{\lambda}{\lambda_0} A \triangle A\right) \cap \left(0, \frac{1}{\lambda_0 \cdot n_0}\right)\right|.$$

□

The above shows that there is a set $\Lambda' = \frac{1}{\lambda_0}\Lambda_{n_0}$ such that

$$\left|(\lambda' A \triangle A) \cap \left(0, \frac{1}{\lambda_0 n_0}\right)\right| = 0$$

for any $\lambda' \in \Lambda'$, and 1 is a point of condensation of Λ'.

The set $A' = A \cap (0, \frac{1}{2}\frac{1}{\lambda_0 n_0})$ and an arbitrary one-to-one sequence $(\lambda_n)_{n \in \mathbf{N}}$ such that $\lim_{n \to \infty} \lambda_n = 1$, $\lambda_n \in \Lambda'$ and $\lambda_n < 2$ for $n \in \mathbf{N}$ satisfy the conditions of Lemma 1, and

$$\lambda_n A' \subset \left(0, \frac{1}{\lambda_0 n_0}\right)$$

for any natural number n.

Hence there exists a natural number n_1 such that $\Phi(\lambda_{n_1} \cdot A') - \Phi(A') \neq \emptyset$. It is easy to see that $|\lambda_{n_1} \cdot A' - A'| > 0$. On the other hand,

$$(\lambda_{n_1} A \triangle A) \cap \left(0, \frac{1}{\lambda_0 n_0}\right) \supset (\lambda_{n_1} A' - A'),$$

and since $\lambda_{n_1} \in \Lambda'$, therefore

$$\left|(\lambda_{n_1} A \triangle A) \cap \left(0, \frac{1}{\lambda_0 n_0}\right)\right| = 0.$$

This contradiction completes the proof.

Now, let us make an attempt to prove the same theorem for sets having the Baire property.

Lemma 2. *Let $A \subset \mathbf{R}_+$ be a set having he Baire property, $(\lambda_n)_{n \in \mathbf{N}}$—a one-to-one sequence converging to 1. If $A \cap (0, \delta)$ and $(0, \delta) - A$ are sets of the second category for any $\delta > 0$, then there exists a natural n_0 such that $(\lambda_{n_0} A - A)$ is a set of the second category.*

P r o o f. A is a set of the second category and has the Baire property, thus ([2], p. 20, Theorem 4.6) there are a nonempty regular open set U and a first category set P such that $A = U \triangle P$.

Without loss of generality, like in the proof of Lemma 1, we may assume that the sequence $(\lambda_n)_{n \in \mathbf{N}}$ is monotone. Assume first that it is increasing.

Let x be an arbitrary point of U. We denote by (y, z) the component of U which contains x. Obviously, $y > 0$.

There exists a natural number n_0 such that $\lambda_{n_0} x \in (y, z)$. Thus $(\lambda_{n_0} x, x) \subset (y, z) \subset U$. As the set $(y, x) - A$ is of the first category, therefore $\lambda_{n_0}(y, x) - \lambda_{n_0} A$ is of the first category, too, but $(\lambda_{n_0} y, \lambda_{n_0} x) - A$ is a set of the second category (since $(\lambda_{n_0} y, y) - \bar{U}$ is a nonempty open set).

Finally, $((\lambda_{n_0} y, \lambda_{n_0} x) \cap \lambda_{n_0} A) - A$ and $\lambda_{n_0} A - A$ are sets of the second category.

If the sequence $(\lambda_n)_{n \in \mathbf{N}}$ is decreasing, we choose a point x from a bounded component of U and a natural number n_0 with $\lambda_{n_0} x \in (x, z)$ and repeat the first part of the proof.

Without substantial changes (taking sets of the first category instead of sets of measure zero and Lemma 2 instead of Lemma 1), the proof of Theorem 1 can be used to establish the following result: □

Theorem 2. *Let $A \subset \mathbf{R}_+$ be a set having the Baire property and such that, for any $\delta > 0$, both $A \cap (0, \delta)$ and $(0, \delta) - A$ are of the second category. Then the set $\Lambda = \{\lambda < 0 \colon (\lambda A \triangle A) \cap (0, \delta)$ is of the first category for some $\delta > 0\}$ has cardinality less or equal to χ_0.*

References

[1] *C. Kuratowski*: Topologie, vol. I, 1952.
[2] *J. C. Oxtoby*: Measure and Category, Springer-Verlag, 1971.

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