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ON THE STABILITY OF SOLUTIONS OF LINEAR DIFFERENTIAL SYSTEMS WITH SLOWLY VARYING COEFFICIENTS

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INTRODUCTION

Consider the solutions $y(t)$ of the system of equations

$$y'(t) = A(t)y(t) \tag{1.1}$$

where $A(t)$ is an $n \times n$ matrix for each fixed $t \geqslant 0$. It is well known that we cannot assure the stability of solutions of (1.1) by merely supposing $A(t)$ to be a stability matrix for each fixed t. (See, for examples, [5, p. 310] or [6, p. 494]).

However, if we assume this to be the case in conjunction with a condition that in some sense asserts that the coefficient matrix $A(t)$ varies slowly, then it is possible to obtain stability results for solutions of (1.1). For example, if we suppose that the matrices $A(t)$ vary slowly in the sense that either

$$\sup_{0 \leqslant t < \infty} \|A'(t)\| \text{ is sufficiently small;} \tag{1.2}$$

$$\text{or} \quad \int_0^\infty \|A'(t)\| \mathrm{d}t < \infty, \tag{1.3}$$

then theorems of this type have been obtained by Hale and Stokes [4] under (1.2) and by Cesari [2] under (1.3).

The methods used to obtain these results involved careful estimation procedures based on suitably interpreting (1.1) as a preturbation of a constant coefficient situation. It is the purpose of this paper to study this question from a different perspective; namely by constructing an appropriate Liapounof like function. In fact, if A is a constant stability matrix, it is known that the quadratic form (Py, y), where P is the unique solution of the matrix equation

$$PA + A^{*}P = -I, \tag{1.4}$$

serves as a Lyapunoff function for the equation $\frac{dy}{dt} = Ay$. This suggest that in the variable coefficient matrix case, the function $(P(t)y(t), y(t))$, where $P(t)$ is the solution of (1.4) corresponding to $A = A(t)$, might serve in a similar capacity for the system (1.1). And we will show that under suitable circumstances it does so serve. Namely, assuming that for each $t \geqslant 0$, $A(t)$ belongs to the class of stability matrices whose eigenvalues have real parts $\leqslant -\delta < 0$, with $A(t)$ continuously differentiable and uniformly bounded by M for $t \geqslant 0$, then for $(P(t)y(t), y(t))$ we will obtain the estimate

$$(P(t)y(t), y(t)) \leqslant \mathrm{e}^{-[\frac{t}{\beta} - \frac{K}{\alpha}\int_0^t \|A'(s)\|\mathrm{d}s]}(P(0)y(0), y(0)) \tag{1.5}$$

for $t \geqslant 0$, where α, β and K are constants depending on δ and M.

From (1.5) it will be easy to derive the stability results described above that follow form conditions (1.2) and (1.3) together with a further stability result which follows from the condition

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|A'(s)\|\mathrm{d}s = 0$$

that generalizes (1.3).

The plan of the paper is as follows. In Section 2 we will derive various properties of the matrix solution P of equation (1.4) needed in the sequel. Then in Section 3 we will obtain the estimate (1.5) as well as all the stability results that ensue form it.

2. The Matrix Equation $PA + A^{*}P = -I$

In this section we gather together the basic facts that we will need concerning the solution P of the matrix equation $PA = A^{*}P = -I$. For this purpose it will be convenient for us to first study the solution Y of the somewhat more general equation

$$YA + BY = C \tag{2.1}$$

where A, B and C are given $n \times n$ matrices.

We begin by describing a technique for solving (2.1) when A and B are stability matrices, that is, matrices whose eigenvalues all have negative real parts. We denote the set of all stability matrices by $\mathcal{S}$. Of particular interest to us will be the compact subset of $\mathcal{S}$ denoted by $\mathcal{A}(\delta, M)$ consisting of those matrices A

(i) whose eigenvalues all have negative real parts $\leqslant -\delta$ with $\delta > 0$; and
(ii) which are bounded by M: $||A|| \leqslant M$.

(Here, by the matrix norm in (ii) we mean the one induced by using the Euclidean norm $|x| = [x_1^2 + \ldots + x_n^2]^{\frac{1}{2}}$ for vectors in $x = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$.) The method for solving (2.1) as well as the method for studying the solutions of this equation as functions of its coefficients and right side depends on the following estimate for the exponential matrix e^{tA} when $A \in \mathcal{A}(\delta, M)$.

Proposition 2.1. *For $A \in \mathcal{A}(\delta, M)$ we have the uniform estimate*

$$||e^{tA}|| \leqslant J e^{-\mu t} \quad (t \geqslant 0) \tag{2.2}$$

where for μ we can choose any positive number in the interval $(0, \delta)$, with J then depending on μ, δ and M: $J = J(\mu, \delta, M)$.

For the sake of completeness, we will give a proof of this result in the appendix. We are now in position to solve (2.1) and we do so in the theorem that follows.

Theorem 2.2. *Let A and B denote fixed stability matrices and C an arbitrary matrix, then the integral*

$$Y = -\int_0^\infty e^{tB} C e^{tA} dt \tag{2.3}$$

exists and represents the unique solution of the matrix equation (2.1):

$$YA + BY = C.$$

Proof. Since a fixed stability matrix belongs to the set $\mathcal{A}(\delta, M)$ for some suitable δ and M, the estimate (2.2) is applicable to A and B; and hence the integral defining Y exists.

Now to show that Y satisfies (2.1), we differentiate the integrand and use the fact that A commutes with e^{tA} to obtain

$$\frac{d}{dt}(e^{tB} C e^{tA}) = e^{tB} C A e^{tA} + B e^{tB} C e^{tA} = (e^{tB} C e^{tA})A + B(e^{tB} C e^{tA}).$$

Integrating from $t = 0$ to $t = \infty$, we find, in view of $e^{tB} C e^{tA} \to 0$ as $t \to \infty$ (because of (2.2)), that

$$-C = \Big(\int_0^\infty e^{tB} C e^{tA} dt\Big) A + B\Big(\int_0^\infty e^{tB} C e^{tA} dt\Big) = (-Y)A + B(-Y),$$

the desired result.

Having shown that (2.1) always has a solution for arbitrary right sides C, we can then conclude that this solution is unique by reasoning as follows: We re-interpret (2.1) as a system of n^2 linear equations for the n^2 entries y_{jk} of Y. Since this system has a solution for an arbitrary right side, it follows from well-known theorems of linear algebra, that this solution must be unique. □

R e m a r k . This method of solving equations (2.1) is described by Bellman [1, p. 175], who is, however, unable to attribute it to an original source.

Our next result concerns the dependence of the solution of (2.1) on the coefficients and right side of this equation.

Theorem 2.3. *Let $Y = Y(A, B, C)$ denote the unique solution of (2.1):*

$$YA + BY = C$$

for given A, $B \in \mathcal{S}$ and C arbitrary. Then Y depends continuously on A, B and C.

P r o o f . Replacing A, B and C in (2.1) by $A + \Delta A$, $B + \Delta B$ and $C + \Delta C$ respectively, call the corresponding solution $Y + \Delta Y$:

$$(Y + \Delta Y)(A + \Delta A) + (B + \Delta B)(Y + \Delta Y) = C + \Delta C. \tag{2.4}$$

This solution wil exist by Theorem 2.2, provided that $A + \Delta A$ and $B + \Delta B$ are both stability matrices, and this will surely be the case if ΔA and ΔB are sufficiently small; moreover, it is clear that for a suitably determined δ and M, $A + \Delta A$ as well as $B + \Delta B$ well belong to $\mathcal{A}(\delta, M)$ for ΔA and ΔB sufficiently small and we will assume this to be the case. Now subtract equation (2.1) satisfied by Y from equation (2.4) satisfied by $Y + \Delta Y$. Then, after transposing some terms, we have

$$\Delta Y(A + \Delta A) + (B + \Delta B)\Delta Y = \Delta C - Y(\Delta A) - (\Delta B)Y.$$

According to Theorem 2.2, the solution ΔY of this equation is given by

$$\Delta Y = \int_0^\infty \mathrm{e}^{t(B+\Delta B)}\left[Y(\Delta A) + (\Delta B)Y - \Delta C\right]\mathrm{e}^{t(A+\Delta A)}\mathrm{d}t.$$

Since both $A + \Delta A$ and $B + \Delta B$ belong to $\mathcal{A}(\delta, M)$, we may apply (2.2) to estimate the integral on the right, thereby obtaining

$$||\Delta Y|| \leqslant \left[(||\Delta A|| + ||\Delta B||)||Y|| + ||\Delta C||\right] \int_0^\infty J^2\mathrm{e}^{-2\mu t}\mathrm{d}t;$$

from which the asserted contunuity of Y follows immediately. □

We now turn our attention to the matrix equation

$$PA + A^*P = -I. \tag{2.5}$$

If A is a stability matrix, the same is true for A^*; it follows then by Theorem 2.2 that (2.5) has a unique solution P given by

$$P = P(A) = \int_0^\infty e^{tA^*} e^{tA} dt \tag{2.6}$$

Our next objective will be to show that when A is confined to $\mathcal{A}(\delta, M)$, $P(A)$ has the properties described in the two propositions below.

Proposition 2.4. *For $P = P(A)$ with $A \in \mathcal{A}(\delta, M)$, the quadratic form (Px, x) is positive definite and uniformly bounded:*

$$\beta(x, x) \leqslant (Px, x) \leqslant \alpha(x, x), \tag{2.7}$$

where α and β are positive constants depending only on δ and M.

Proposition 2.5. *Regarding $P = P(A)$ as a function of the matrix entries a_{jk} of A with $A \in \mathcal{S}$, the derivatives $\frac{\partial P}{\partial a_{jk}}$ exist, depend continuously on A; and consequently, for $A \in \mathcal{A}(\delta, M)$ they are bounded:*

$$\left\| \frac{\partial P}{\partial a_{jk}} \right\| \leqslant L \quad (A \in \mathcal{A}(\delta, M)), \tag{2.8}$$

with L depending on δ and M.

Proof of Proposition 2.4. Clearly, as the upper and lower bounds for $\dfrac{(P(A)x, x)}{(x, x)}$ we can take

$$\alpha = \sup_{\substack{|x|=1 \\ A \in \mathcal{A}(\delta, M)}} (P(A)x, x) \quad \text{and} \quad \beta = \inf_{\substack{|x|=1 \\ A \in \mathcal{A}(\delta, M)}} (P(A)x, x)$$

respectively. Since $P(A)$ depends continuously on A for $A \in \mathcal{A}(\delta, M)$ (because of Theorem 2.3) and $\mathcal{A}(\delta, M)$ is a compact set, it follows that α and β are achieved maxima and minima respectively, and so are finite. It remains only to show that β is positive, and to do this it is enough to establish that $(P(A)x, x)$ is positive, definite for any fixed stability matrix A.

To accomplish this we use (2.6) together with the bi-linearity of the inner product and the observation that e^{tA^*} is the adjoint of e^{tA} to conclude that

$$\begin{aligned}(P(A)x, x) &= \left(\left[\int_0^\infty e^{tA^*}e^{tA}dt\right]x, x\right) = \int_0^\infty (e^{tA^*}e^{tA}x, x)dt \\ &= \int_0^\infty (e^{tA}x, e^{tA}x)dt = \int_0^\infty |e^{tA}x|^2 dt > 0\end{aligned}$$

if $x \neq 0$ due to the fact that $e^{tA}x \neq 0$ if $x \neq 0$. The latter following from the uniqueness theorem for solutions of $\frac{dy}{dt} = Ay$. □

P r o o f of Prooposition 2.5. Consider the effect on P, the solution of (2.5):

$$PA + A^*P = -I,$$

when A is changed by just changing a_{jk}, the entry in the jth row and kth column, by the amount Δa_{jk} without changing any other entry. Let ΔA denote the resulting change in the matrix A, and let ΔP denote the corresponding change in P induced by this change in A; the matrix $P + \Delta P$ is thus a solution of the equation

$$(P + \Delta P)(A + \Delta A) + (A + \Delta A)^*(P + \Delta P) = -I.$$

Subtracting the defining equation (2.5) for P from this equation, transposing the term $P(\Delta A) + (\Delta A)^*P$, and then dividing by Δa_{jk}, we obtain

$$\begin{aligned}\frac{\Delta P}{\Delta a_{jk}}(A + \Delta A) + (A + \Delta A)^*\frac{\Delta P}{\Delta a_{jk}} &= -P\frac{(\Delta A)}{\Delta a_{jk}} - \frac{(\Delta A)^*}{\Delta a_{jk}}P = \\ &= -PI_{jk} - I_{jk}^*P\end{aligned} \tag{2.9}$$

where I_{jk} denotes the $n \times n$ matrix with entry equal to 1 in the jth row and kth column and zero entries everywhere else.

Now send $\Delta a_{jk} \to 0$, then according to Theorem 2.3, $\frac{\Delta P}{\Delta a_{jk}}$, as the solution of (2.9), will converge to Z, the unique matrix solution of the equation

$$ZA + A^*Z = -PI_{jk} - I_{jk}^*P. \tag{2.10}$$

In other words, the partial derivative $\frac{\partial P}{\partial a_{jk}}$ exists and is equal to Z.

Next we note that as $P = P(A)$ is already known to depend continuously on A (because of Theorem 2.3), the right side of (2.10) depends continuously on A. Thus, by another application of Theorem 2.3, $\frac{\partial P}{\partial a_{jk}} = Z$, as the solution of (2.10), must also depend continuously on A. Finally, the compactness of $\mathcal{A}(\delta, M)$ then gives us the bound (2.8). □

3. The Basic Estimate and Stability Results

In this section we will derive the estimate (1.5) mentioned in the introduction and the stability results which follow from this estimate.

We will be considering solutions $y = y(t)$ of th initial value problem

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = A(t)y \quad (t > 0), \\ y(0) = y_0 \end{cases} \tag{3.1}$$

in which $A(t)$ will be assumed to be a stability matrix for each $t \geqslant 0$. Now let $P(t)$ denote the unique solution of the matrix equation

$$P(t)A(t) + A^*(t)P(t) = -I.$$

More precisely, let $P = P(A)$ denote the unique solution of the matrix equation $PA + A^*P = -I$ shown to exist in Section 2 whenever A is a given stability matrix; then when $A = A(t)$, we consider, corresponding to it, $P = P(A(t))$ and this is what we mean by $P(t)$, i.e. $P(t) = P(A(t))$.

Theorem 3.1. *Let $y = y(t)$ be a solution of (3.1) in which $A(t)$ has a continuous derivative $A'(t)$ for all $t \geqslant 0$ with*

$$A(t) \in \mathcal{A}(\delta, M) \quad \text{for } t \geqslant 0. \tag{3.2}$$

Let $P(t)$ be as defined above. Then for $(P(t)y(t), y(t))$ we have the following estimate.

$$(P(t)y(t), y(t)) \leqslant \mathrm{e}^{-[\frac{t}{\beta} - \frac{K}{\alpha}\int_0^t \|A'(s)\|\mathrm{d}s]}(P(0)y(0), y(0)) \quad \text{for } t \geqslant 0 \tag{3.3}$$

where α, β and K are constants depending only on δ and M.

Proof. Differentiating $(P(t)y(t), y(t))$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(P(t)y(t), y(t)) = (P'y, y) + (Py', y) + (Py, y') \tag{3.4}$$

where the prime denotes differentiation with respect to t. Because of the equation (3.1) satisfied by y, we can transform the last two terms on the right as follows:

$$\begin{aligned} (Py', y) + (Py, y') &= (PAy, y) + (Py, Ay) = (PAy, y) + (A^*Py, y) \\ &= ([PA + A^*P]y, y) = -(y, y) \leqslant -\frac{1}{\beta}(Py, y) \end{aligned} \tag{3.5}$$

where in the steps at the end we have used the defining equation (2.5) for P: $PA + A^*P = -I$, together with the upper bound (2.7): $(Py, y) \leqslant \beta(y, y)$ for the quadratic form (Py, y).

Next, to estimate $(P'y, y)$, the remaining term on the right of (3.4), we first estimate P' by using the chain rule:

$$P'(t) = \sum_{1 \leqslant j,k \leqslant n} \frac{\partial P}{\partial a_{jk}} a'_{jk}(t)$$

(where the a_{jk} denote the entries of A), in conjunction with the bound $\left\| \frac{\partial P}{\partial a_{jk}} \right\| \leqslant L$ provided by Proposition 2.5 whenever $A \in \mathcal{A}(\delta, M)$. This leads to $\|P'(t)\| \leqslant K\|A'(t)\|$ with K, like L, depending on δ and M. An application of Schwartz's inequality then yields

$$(P'y, y) \leqslant |P'y|\,|y| \leqslant \|P'\|\,|y|^2 \leqslant K\|A'(t)\|\,|y|^2 \leqslant \frac{K}{\alpha}\|A'(t)\|(Py, y),$$

in view of the lower bound (2.7): $\alpha|y|^2 \leqslant (Py, y)$, for (Py, y). Inserting (3.5) and (3.6) into (3.4), we arrive at the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}(P(t)y(t), y(t)) \leqslant \left[\frac{K}{\alpha}\|A'(t)\| - \frac{1}{\beta}\right](P(t)y(t), y(t));$$

and solving this differential inequality we obtain the estimate (3.3). □

Another application of (2.7) then immediately yields.

Corollary 3.2. *Under exactly the same assumptions regarding $A(t)$ as in Theorem 3.1, the solutions $y(t)$ of (3.1) satisfy the estimate*

$$|y(t)|^2 \leqslant \left(\frac{\beta}{\alpha}\right) \mathrm{e}^{-\left[\frac{t}{\beta} - \frac{K}{\alpha}\int_0^t \|A'(s)\|\mathrm{d}s\right]}|y(0)|^2 \quad (t \geqslant 0) \tag{3.7}$$

From Corollary 3.2 we can readily obtain the following stability results mentioned in the introduction.

Theorem 3.3. *Assume that the coefficient matrix $A(t)$ in (3.1) belongs to $\mathcal{A}(\delta, M)$ for all $t \geqslant 0$ and that it has a continuous derivative $A'(t)$ for $t \geqslant 0$. Then the solutions $y(t)$ of (3.1) are asymptotically stable provided that $A(t)$ is slowly varying* *in any of the following senses:*

$$\sup_{0\leqslant s<\infty} \|A'(s)\| \text{ is sufficiently small;} \tag{3.8}$$

$$\int_0^\infty \|A'(s)\|\mathrm{d}s < \infty; \tag{3.9}$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|A'(s)\|\mathrm{d}s = 0. \tag{3.10}$$

P r o o f. Asymptotic stability for solutions $y(t)$ of (3.1) means that $y(t) \to 0$ as $t \to \infty$; and it is easy to show that this follows from the estimate (3.7) under any of the conditions (3.8), (3.9) and (3.10). For example, if (3.8) holds, we have from (3.7)

$$|y(t)|^2 \leqslant \left(\frac{\beta}{\alpha}\right) \mathrm{e}^{-(\frac{1}{\beta}-\frac{Km}{\alpha})t}|y(0)|^2$$

where $m = \sup_{0\leqslant s<\infty} \|A'(s)\|$; and so if m is so small that $\frac{1}{\beta} - \frac{Km}{\alpha} > 0, y(t) \to 0$ as $t \to \infty$. Next, suppose that (3.10) holds, then if t is sufficiently large, say $t > T$, we can arrange for

$$\frac{1}{t}\int_o^t \|A'(s)\|\mathrm{d}s \leqslant \frac{1}{2}\frac{\alpha}{K}\frac{1}{\beta};$$

and because of (3.7) this will imply that

$$\begin{aligned}|y(t)|^2 &\leqslant \left(\frac{\beta}{\alpha}\right) \mathrm{e}^{-t(\frac{1}{\beta}-\frac{K}{\alpha}\frac{1}{t}\int_0^t \|A'(s)\|\mathrm{d}s)}|y(0)|^2\\ &\leqslant \left(\frac{\beta}{\alpha}\right) \mathrm{e}^{-\frac{t}{2\beta}}|y(0)|^2 \quad \text{for } t > T;\end{aligned}$$

from which we conclude that $y(t) \to 0$ as $t \to \infty$. Finally, since condition (3.9) implies condition (3.10), we also get the asymptotic stability under (3.9). □

R e m a r k 1. Here we want to note that we can reformulate the stability result stated above which follows from condition (3.9) in a slightly different way. The reformulation is based on the observation that by virtue of (3.9)

$$A(\infty) = \lim_{t\to\infty} A(t) = \lim_{t\to\infty} \int_0^t A'(s)\mathrm{d}s + A(0) = \int_0^\infty A'(s)\mathrm{d}s + A(0)$$

exists. Because of this it is not necessary to assume explicitly that coefficient matrices $A(t) \in \mathcal{A}(\delta, M)$ for $t \geqslant 0$; rather we may assume that $A(\infty)$ is a stability matrix and this will automatically assure that for t sufficiently large $A(t) \in \mathcal{A}(\delta, M)$ for suitable δ and M. Formulated this way, the resulting stability theorem, under the assumption that (3.9) holds and that $A(\infty)$ is a stability matrix, is essentially due to Cesari [2].

R e m a r k 2. Finally, let us observe that in framing the conditions of Theorem 3.3, the role played by the inital time $t = 0$, may in an obvious way, be replaced by any later time $t = \tau > 0$, with the same conclusions regarding the asymptotic stability still obtaining.

APPENDIX

Here we will give the

P r o o f of Proposition 2.1. The proof is based on the contour integral representation

$$e^{tA} = \frac{1}{2\pi i} \int_\Gamma e^{tz}(zI - A)^{-1} dz \tag{5.1}$$

where Γ is any contour in the complex plane that contains all the eigenvalues of A in its interior.

Our choice of Γ is based on the observation that the eigenvalues λ of the matrices A with $||A|| \leqslant M$ are uniformly bounded. This follows either by noting that the eigenvalues of a matrix depend continuously on the matrix entries or by making use of an explicit bound for the eigenvalues in terms of the matrix entries, such as, for example the estimate $|\lambda| \leqslant \max_{1\leqslant j\leqslant n} \sum_{k=1}^{n} |a_{jk}|$ which is a consequence of Gershgorin's theorem [3]. Thus, there exists an $R = R(M)$ so that the eigenvalues λ of the matrices A with $||A|| \leqslant M$ are all contained in the interior of the circle of radius R centered at the origin: $|\lambda| < R$. Consider now the set of matrices $\mathcal{A}(\delta, M)$; if $\delta \geqslant R(M)$ this set will be empty. Thus only the situation $\delta < R(M)$ is of interest, and in this case it is clear that the eigenvalues of the matrices $A \in \mathcal{A}(\delta, M)$ will all lie in the interior of the rectangle bounded on top and bottom by the lines $y = \pm R$ respectively, on the left by the line $x = -R$ and on the right by the line $x = -\mu$ with $\mu \in (0, \delta)$. For the contour Γ in the representation (5.1) we take the just described rectangle.

By the standard estimate for complex integrals, we find from (5.1) that

$$||e^{tA}|| \leqslant \frac{1}{2\pi} \sup_{z\in\Gamma} |e^{tz}| \sup_{z\in\Gamma} ||(zI - A)^{-1}|| L(\Gamma), \tag{5.2}$$

where $L(\Gamma)$ denotes the length of Γ. We now note that

$$\sup_{z\in\Gamma} |e^{tz}| = \sup_{z\in\Gamma} e^{t\Re z} = e^{-\mu t} \quad (t \geqslant 0)$$

by virtue of the fact that $\Re z$ assumes its maximum on the rectangle Γ along the right side $x = -\mu$. Hence (5.2) will immediately yield the desired estimate (2.2): $\|e^{tA}\| \leqslant Je^{-\mu t}(t \geqslant 0)$ for $A \in \mathcal{A}(\delta, M)$ with

$$J = \sup_{\substack{z \in \Gamma \\ A \in \mathcal{A}(\delta,M)}} \|(zI - A)^{-1}\| \frac{L(\Gamma)}{2\pi}$$

provided that we can show that the supremum on the right is finite. And this will be done by showing that $(zI - A)^{-1}$ depends continuously on z and A for $z \in \Gamma$ and $A \in \mathcal{A}(\delta, M)$; so that, on account of the comnpactness of Γ and $\mathcal{A}(\delta, M)$, it will follow that the supremum in question is actually an achieved maximum and hence finite.

To establish the continuity of $(zI - A)^{-1}$ we use the well-known Neumann series representation for the inverse of an operator which is close to an invertible operator:

$$(T + \Delta T)^{-1} = \sum_{k=0}^{\infty} (-1)^k (T^{-1}\Delta T)^k T^{-1}. \tag{5.3}$$

Here T and ΔT denote bounded linear operators of a Banach space into itself, with T being assumed to have a bounded inverse T^{-1}; from which it follows that $T + \Delta T$ also has a bounded inverse given by the series (5.3) above provided that $\|\Delta T\| < \dfrac{1}{\|T^{-1}\|}$. (See [7, pp. 164–165].)

Taking norms in (5.3), after having transposed the first term in the series, we obtain

$$\|T + \Delta T)^{-1} - T^{-1}\| \leqslant \sum_{k=1}^{\infty} \|T^{-1}\|^{k+1} \|\Delta T\|^k = \frac{\|T^{-1}\|^2 \|\Delta T\|}{1 - \|T^{-1}\|\,\|\Delta T\|}$$

for $\|\Delta T\| < \dfrac{1}{\|T^{-1}\|}$. Applying this with $T = zI - A$ and $\Delta T = \Delta zI - \Delta A$ (under the assumption that $z, z + \Delta z \in \Gamma$ and $A, A + \Delta A \in \mathcal{A}(\delta, M)$), we find that

$$\|([z + \Delta z]I - [A + \Delta A])^{-1} - (zI - A)^{-1}\| \leqslant \frac{c^2(|\Delta z| + \|\Delta A\|)}{1 - c(|\Delta z| + \|\Delta A\|)}$$

where $c = \|(zI - A)^{-1}\|$, provided that $|\Delta z|$ and $\|\Delta A\|$ are sufficiently small; from which the desired continuity follows immediately. □

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