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John Dauns

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MODULE CLASSIFYING FUNCTORS

JOHN DAUNS, New Orleans

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INTRODUCTION

It is shown that there exist three contravariant functors G^1, G^2, and G^3 applicable to any associative ring $R(1 \in R)$—where $G^i(R)$ is a partially ordered class that is a complete join semi-lattice with zero; $G^3(R)$ is a set. The functor G^i classifies the class of all right R-modules $\{A, B, \ldots\}$ into a class of equivalence classes $G^i(R) = \{[A]^i, [B]^i, \ldots\}$, where $A \in [A]^i$, and $[A]^i$ consists of a class of modules that are similar to A, or are like A. Let $ZA \subset Z_2A \subset A$ denote the singular, and the second singular submodules. Define the torsion free and torsion parts of G^i by $G^i_F(R) = \{[A] \mid ZA = 0\}$, and $G^i_T = \{[A] \mid A = Z_2A\}$. Then $G^3(R)$ is a lattice direst sum $G^3(R) = G^3_T(R) \oplus G^3_F(R)$ of convex and complete sublattices $G^3_T(R)$, $G^3_F(R) \subset G^3(R)$.

Above and throughout, here L is called a join *semi-lattice* if L is a partially ordered class (po-class) any two of whose elements x and y have a least upper bound $x \vee y$ in L. It is *complete* if every nonempty subset $S \subset L$ has a supremum $\bigvee S \in L$. Note that $\bigwedge S$ need not exist and that subclasses of L are not required to have a supremum. If L_1 and L_2 are semi-lattices with $0 \in L_i$, so is their direct sum $L_1 \oplus L_2$ where $x_1 \vee x_2 = (x_1, x_2) \leqslant (y_1, y_2) = y_1 \vee y_2$ iff both $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, and where L_1, $L_2 \subset L_1 \oplus L_2$ are convex. Here a category refers to a large category in the sense of MacLane [18, p. 23]. For those semi-lattice and category concepts which carry over to classes, the customary terminology is used, e.g. semi-lattice homomorphism, convex semi-sublattice, direct sum, and dense. The term "lattice" here is applied to sets only.

The main emphasis is on $i = 3$ because $G^3(R)$ is a set and a complete lattice for all R. It is a consequence of Göbel and Wald [11] that $G^1_F(\mathbf{Z})$ is a class. Also, $G^2(R)$ is a set if and only if R is right Noetherian. The proof of the latter is heavily based on a result of Matlis ([19, p. 512, Proposition 1.2]).

Some functors in ring theory (such as various radicals) are defined on the category $\mathbf{A}$ of all rings with identity and identity preserving ring homomorphisms that are onto. Here, $G^i: \mathbf{A} \to \mathbf{B}$ where $\mathbf{B}$ is an appropriate semi-lattice category. Surjective ring homomorphisms $\varphi: R \to S$ induce semi-lattice homomorphism $G^i(\varphi): G^i(S) \to G^i(R)$. Always $G^i(\varphi)G^i_T(S) \subseteq G^i_T(R)$, that is G^i_T is a subfunctor of G^i. In general, G^i_F is not. However, there is a natural subcategory $\mathbf{A}^* \subset \mathbf{A}$ such that when G^i is restricted to $\mathbf{A}^*$, then G^i_F is also a subfunctor of G^i. On all of $\mathbf{A}$, there are natural transformations η^j_i of functors $\eta^j_i : G^i \to G^j$ for $1 \leqslant i < j \leqslant 3$. The kernels of the functorial semi-lattice homomorphisms $\eta^j_i(R): G^i(R) \to G^j(R)$ possibly could be related to algebraic ring, or module category theoretic properties of R. E.g. $\eta^3_2(R)$: $G^2(R) \to G^3(R)$ is bijective if and only if R is right Noetherian. Or, use of some results of Teply, [22, p. 442] and [23, p. 451, Theorem 2.1], shows that if R/Z_2R is of finite Goldie dimension, that then $\eta^3_2(R): G^2_F(R) \to G^3_F(R)$ is one to one.

In section 4 a module ΛT is constructed which represents the infimum $[A^{(1)}]^3 \wedge \ldots \wedge [A^{(n)}]^3 = [\Lambda T]^3$. (Theorem III). The module ΛT is characterized by a universal mapping property (4.4). For an infinite set $\{A^{(\gamma)}\}$, we know that $\bigwedge [A^{(\gamma)}]^3 = [M] \in G^3(R)$ exists and is represented by some module M. So far it is still an open question to find a concrete representation of M. It is beyond the limits of this paper to determine additional lattice structural properties of the lattice $G^3_T(R)$; however this latter problem was one of the reasons for constructing the infimum module ΛT.

Besides G^1, G^2, and G^3 there are other such functors which can be constructed by the present techniques. The primary aim of this paper is to present a general technique for constructing functors from ring (or other) categories to categories of po-classes. Although, it is beyond the scope of this paper to go into restricted specialized technical applications of these functors, as an illustration, section 5 develops some techniques for computing $G^3_F(R)$, and for using lattice concepts in $G^3_F(R)$ to obtain algebraic information about R, for a class of rings which are certain subdirect products.

The results of Dauns [5] were applicable only to $\mathbf{A}^*$, G^3_F, and nonsingular, that is torsion free modules. Among other things, this article removes the torsion free restriction. For this paper, a knowledge of [5] is not assumed. There it was shown that $G^3_F(R)$ is a complete Boolean lattice. Hence in this case $R \to S$ functorially induces an ordinary ring homomorphism $G^3_F(S) \to G^3_F(R)$. It was also shown how G^3_F can be used to decompose and classify modules and rings. To repeat the obvious, what makes any functor, such as J (Jacobson radical), K_0, G^3_F, or G^3 important, is that it can be applied universally to all rings.

1. Prelimilaries

Six functors from rings to posets are defined, and three natural transformations between some of these.

1.1 Notation. A module M means a right unital module over an associative ring R. Let $<$ or $\leqslant$ denote submodules; and let $<<$ refer to essential or large submodules. The notation $P <\not< Q$ means that $P < Q$ but that P is not essential in Q. A submodule $P < Q$ is a *complement* if it has no proper essential extension inside Q, in which case P is said to be *closed* in Q. If $K < M$ and $x \in M$, then $x^{\perp} = \{r \in R \mid xr = 0\} < R$, and for $x + K \in M/K$, $(x+K)^{\perp} = x^{-1}K = \{r \in R \mid xr \in K\} < R$. For a subset $X \subset M$, set $X^{\perp} = \{r \in R \mid xr = 0 \text{ for all } x \in X\} = \{r \mid Xr = 0\}$. Then $M^{\perp} = \{r \mid Mr = 0\} \triangleleft R$, where "$\triangleleft$" denotes ideals in R and other rings.

The operation of taking injective hulls of right R-module M is denoted by both "$\widehat{\ }$" and "E" as $\widehat{M} \doteqdot E(M) = EM$. The set $Z(M) = ZM = \{x \in M \mid x^{\perp} << R\} < M$ is known as the *singular submodule* of M; and $Z[M/(ZM)] = (Z_2M)/ZM$ defines the *second singular submodule* which is also called the *torsion submodule* of M. Thus M is *torsion* if $M = Z_2M$, and *torsion free*, abbreviated t.f., if $Z_2M = 0$. Note that $Z_2M = 0$ iff $ZM = 0$. Throughout, the symbols $<$, $\leqslant$, $<<$, $<\not<$, $^{\perp}$, $^{-1}$, $\widehat{\ }$, E, Z, and Z_2 always refer to right R-modules and never to rings other than R.

The cardinality of any set X is denoted by $|X|$ and $\mathscr{P}(X)$ denotes the set of all subsets of X.

1.2. Definition. For any submodule $K < M$, define the *complement closure* $\overline{K}$ of K in M only in case $ZM \subseteq K$ by $\widehat{K}/K = Z(M/K)$. Then (a) $\overline{K} = \{x \in M \mid x^{-1}K << R\} = \{x \mid K << K + xR\}$. (b) $K << \overline{K}$; $\overline{K}$ is the intersection of all the complement submodules of M containing K. (c) $Z(M/\overline{K}) = 0$. (d) When $K < M$ is fully invariant, then $\overline{K} < M$ is also. In particular, if $ZR \subseteq K \triangleleft R$, then $\overline{K} \triangleleft R$. (e) If $K = ZR$, then $\overline{K} = Z_2R \triangleleft R$.

1.3. Module Types. Let A, B, $C \ldots$ denote arbitrary right R-modules. There are three quasi-orders "$\prec_i$", $i = 1, 2, 3$, on the class of all right R-modules, where $A \prec_i B$ can mean one of the following three depending upon i. There exists some index set J depending upon A and B such that

(1) $A \subset \oplus\{B \mid J\}$;

(2) $EA \subset \oplus\{EB \mid J\}$;

(3) $A \subset E(\oplus\{B \mid J\})$.

Having selected any one of the above three $\prec_i$, define an equivalence relation "$\sim_i$" on the class of all R-modules by $A \sim_i B$ if and only if $A \prec_i B$ and $B \prec_i A$. Each equivalence class $[A]^i = \{C \mid C \sim_i A\}$ is called a *type* because it consists of modules of the same king or type. Then define $[A]^i \leqslant_i [B]^i$ if $A' \prec_i B'$ for some (or equivalently for any or all) $A' \in [A]^i$ and $B' \in [B]^i$. The class $G^i(R)$ of all types $G^i(R) = \{[A]^i, [B]^i, [C]^i, \ldots\}$ becomes a partially ordered class under an order relation denoted as "$\leqslant_i$" where $[A]^i \leqslant [B]^i$ provided that $A \prec_i B$.Write $[A]^i < [B]^i$ if $A \prec_i B$, but *not* $B \prec_i A$. Note that $0 = [0]^i = [(0)]^i \leqslant [A]^i$ for all i, and that $[A]^i = [EA]^i$ for all A and $i = 2, 3$. The equivalence classes of torsion and torsion free submodules define two subclasses $G_T^i(R) = \{[A]^i \mid A = Z_2 A\}$, and $G_F^i(R) = \{[A]^i \mid ZA = 0\}$, called the *torsion* and *torsion free* types respectively. If the least upper bound, or the greatest lower bound of $[A]^i, [B]^i \in G^i(R)$ exists, they are denoted by $[A]^i \vee [B]^i$ and $[A]^i \wedge [B]^i$, and similarly for infinite suprema and infima.

For $1 \leqslant i \leqslant j \leqslant 3$, if $A \leqslant_i B$ then also $A \leqslant_j B$. Hence there is an order preserving surjective function $\eta_i^j(R)\colon G^i(R) \to G^j(R)$ defined by $\eta_i^j(R)[A]^i = [A]^j$. Note that $\eta_i^i(R) = 1$. If R is fixed and understood abbreviate $\eta_i^j = \eta_i^j(R)$.

2. Lattices

If A and B are abelian p-groups of different (ordinal) p-lengths (Fuchs [7, Vol. I, p. 154]), then $[A]^1 \neq [B]^1 \in G^1(\mathbf{Z})$. For every abelian p-group B, $[B]^1 \leqslant [\mathbf{Z}(p^\infty)]^1$. Hence both $\{[B]^1 \mid [B]^1 \leqslant [Z(p^\infty)]^1\} \subset G_T^1(\mathbf{Z})$ are not sets.

2.1. Lemma. *Let X be a set of representatives of isomorphy classes of injective hulls of cyclic R-modules. Then*

$$|G^3(R)| \leqslant |\mathscr{P}(X)| \leqslant 2^{|\mathscr{P}(R)|}.$$

Proof. Let $X \subset \{E(R/L) \mid L < R\}$. For an arbitrary $[M]^3 \in G^3(R)$ take any subset $T \subset M$ such that $\{E(R/x^\perp) \mid x \in T\} \subset X$ is a set of representatives of the isomorphy classes of injective hulls of cyclic submodules of M without repetitions. Then define $M_* = \oplus\{E(xR) \mid x \in T\}$. If $\oplus\{x_i R \mid i \in I\} << M$ is any essential direct sum of cyclics, then $\oplus_I x_i R \subset \oplus_I M_*$ and $M \subset E(\oplus_I M_*)$. Conversely $M_* \subset E(\oplus_T M)$. Thus $[M]^3 = [M_*]^3$. Consequently $f\colon G^3(R) \to \mathscr{P}(X), f([M]^3) = \{E(R/x^\perp) \mid x \in T\}$ is a one to one function.

The first objective of this section will be to prove the following theorem. □

2.2. Theorem I. *Let R be a ring with identity. For any $1 \leqslant i \leqslant j \leqslant 3$, let $G_T^i(R)$, $G_F^i(R) \subset G^i(R)$ and $\eta_i^j = \eta_i^j(R)$ be as in 1.3. Let $\{[A_\gamma]^i \mid \gamma \in \Gamma\} \subseteq G^i(R)$ be any nonempty subset. Then*

(1) (i) $\sup_\gamma[A_\gamma]^i = \bigvee_\gamma[A_\gamma]^i = [\oplus_\gamma A_\gamma]^i$.

(ii) *$G^i(R)$ is a complete semi-lattice.*

(2) *$G^i_T(R)$, $G^i_F(R) \subset G^i(R)$ are convex (and complete) semi-sublattices with $G^i_T(R) \oplus G^i_F(R) \subseteq G^i(R)$; for $i = 2$ or 3, $G^i(R) = G^i_T(R) \oplus G^i_F(R)$.*

(3) *$\eta^j_i : G^i(R) \to G^j(R)$ is a surjective semi-lattice homomorphism which preserves arbitrary suprema, and so are also its restrictions and corestrictions η^j_i : $G^i_T(R) \to G^j_T(R)$ and $\eta^j_i : G^i_F(R) \to G^j_F(R)$.*

2.3. Main Corollary 1 to Theorem I. *With the notation and hypotheses of the previous theorem,*

(4) *$G^3(R)$ is a set.*

(5) *$G^3(R)$ is a complete lattice with largest element $1 = \bigvee G^3(R) \in G^3(R)$.*

(6) *$G^3(R) = G^3_T(R) \oplus G^3_F(R)$ is a lattice direct sum of convex (and complete) sublattices $G^3_T(R)$, $G^3_F(R) \subset G^3(R)$.*

Proof of 2.2 and 2.3. (1) (i) and (ii) are clear. (2) Clearly $G^i_F(R)$ is convex. Let $[B^i] < [A]^i$ with $A = Z_2A$. Then for some $I, B \subset E(\oplus_I Z_2A) = Z_2E(\oplus_I A)$ by [3, p. 3, 1.2 (g)]. Hence $G^i_T(R)$ is convex. For any module M, let $Z_2M \oplus C << M$. Then $[Z_2M]^i \vee [C]^i \leqslant [M]^i \leqslant [Z_2M]^i \vee [M/Z_2M]^i$ where both ends of the inequality belong to $G^i_T(R) \oplus G^i_F(R)$. For $i = 2$ or 3, $[M]^i = [\widehat{M}]^i$ and the inequalities are equalities. (3) is clear, (4) was proved in 2.1.

(5). It will be shown more generally that $G^i(R)$ satisfies (5) whenever $G^i(R)$ is a locally small category, i.e. for any $[A]^i \in G^i(R)$, $\{[B]^i \in G^i(R) \mid [B]^i \leqslant [A]^i\}$ is a set. For any $\{[A_\gamma]^i \mid \gamma \in \Gamma\} \subset G^i(R)$, let $S = \{[B]^i \in G^i(R) \mid \forall\gamma \in \Gamma, [B]^i \leqslant [A_\gamma]^i\}$. Then $\sup S = \bigwedge_\gamma[A_\gamma]^i \in G^i(R)$. Thus $G^i(R)$ is a complete lattice with largest element. (6) The same argument establishes the same conclusion also for $G^3_T(R)$ and $G^3_F(R)$. Hence $G^3_T \oplus G^3_F(R) = G^3(R)$. □

2.4. Corollary 2 to Theorem I. *For $i = 1$, 2, or 3, suppose that all the definitions in 1.3 remain verbatim the same, except that torsion free modules A, B, … only are allowed. Let then $_F[A]^i$ denote the resulting equivalence class, and $_FG^i(R) = \{_F[A]^i, {_F[B]^i}, \ldots\}$. Then*

(i) $[A]^i = {_F[A]^i}$ *for all* t.f. A.

(ii) *$G^i_F(R) \to {_FG^i(R)}$, $[A]^i \to {_F[A]^i}$ is an isomorphism of partially ordered classes.*

(iii) *The analogues of* (i) *and* (ii) *hold if instead of* t.f. *modules, only torsion modules are used.*

2.5. Corollary 3 to Theorem I. *The ring R is right Noetherian $\Longleftrightarrow G^2(R) = G^3(R)$.*

Proof. $\Longrightarrow$: As a consequence of a result of Matlis ([19, p. 512, Proposition 1.2]) we have $\sim_2 = \sim_3$, $G^2(R) = G^3(R)$, and the latter is a set. $\Longleftarrow$: By 2.3(4) and by hypothesis, there exists a set $S = \{B\}$ of R-modules B such that $G^2(R) = \{[B]^2 \mid B \in S\}$. Let A be any injective R-module. Then $[A]^2 = [B]^2$ for some $B \in S$, and $A \subset \oplus_J \widehat{B}$ for some J. Define $\tau = \sup\{|\widehat{B}| \mid B \in S\}$. Kaplansky's lemma (see Anderson and Fuller [1, p. 295]) shows that A is a direct sum of τ- generated modules. By the Faith-Walker theorem [1, p. 293], R is right Noetherian. □

3. FUNCTORS

Categories $\mathbf{A}^* \subset \mathbf{A}$, and $\mathbf{B}$ are defined so that G^i, $G^i_T : \mathbf{A} \to \mathbf{B}$, and $G^i_F : \mathbf{A}^* \to \mathbf{B}$ become functors for all $i = 1$, 2, and 3.

3.1. Categories. Let $\mathbf{A}$ be the category of all associative rings with identity, and identity preserving ring homomorphisms which are onto. Then $\mathbf{A}^* \subset \mathbf{A}$ denotes the subcategory having the same objects as $\mathbf{A}$ but which contains only those morphisms $\varphi \in \mathbf{A}$ whose kernels $\varphi^{-1}0$ are closed right ideals.

Define $\mathbf{B}$ as the category of complete semi-lattices with smallest element 0; morphisms are zero preserving semi-lattice homomorphisms which are one to one, and which preserve arbitrary suprema of subsets.

3.2. Notation. For simplicity, for a typical $\varphi: R \to S$ in $\mathbf{A}$, set $\ker \varphi = I \triangleleft R$, and assume that $\varphi: R \to R/I = S$ is the natural projection. Right singular submodules and injective hulls with respect to S will be denoted by Z^S, Z_2^S, and E_S.

Throughout this section N is a right S-module (notation: $N = N_S$); N_φ denotes the induced right R-module. Since $(EN)_\varphi$ is meaningless, define $EN_\varphi = E(N_\varphi)$.

For any right R-module M, $\mathscr{L}_R(M)$ denotes the set (and lattice) of large submodules of M; and similarly for $N = N_S$ and $\mathscr{L}_S(N)$.

For $n \in N = N_S$, set $n^{-1}0 = \{s \in S \mid ns = 0\}$. When n is viewed as $n \in N_\varphi$, then $I \subseteq n^\perp$, and $\varphi n^\perp = n^\perp / I = n^{-1}0$.

3.3. Facts. For any right S-modules P, Q, and N the following hold.

(1) $\mathrm{Hom}_S(P, Q) = \mathrm{Hom}_R(P_\varphi, Q_\varphi)$.

(2) The set of S-submodules of N coincides with the set of R-modules of N_φ. Thus a submodule of N is large as an S-submodule if and only if it is large as an R-module. In particular, right S-complements of N are the same as right R-complements of N_φ.

(3) Consequently $N_\varphi << (E_S N)_\varphi$. Hence there is an inclusion $N_\varphi << (E_S N) \leqslant EN_\varphi$. Furthermore,

(4) $E_S N = \{x \in EN_\varphi \mid xI = 0\}$; $(E_S N)_\varphi$ is a quasi-injective R-module.

3.4. Lemma. *For all i and for φ as in 3.2, and any right S-submodules P and Q,*
(i) $[P]^i_S \leqslant [Q]^i_S \in G^i(S) \iff [P]^i \leqslant [Q]^i \in G^i(R)$.
(ii) *Hence in particular the assignment $\varphi^{*i}([N]^i_S) = [N_\varphi]^i$ is a well defined order preserving function $\varphi^{*i}\colon G^i(S) \to G^i(R)$.*

Proof. (i) Only $i = 2$ is proved; $i = 1$ is trivial, while $i = 3$ is easier than $i = 2$, so we prove only $i = 2$. For some Γ, suppose that $(x_\gamma)_{\gamma\in\Gamma} \in \widehat{P} \subset \oplus_\Gamma EQ_\varphi$. Then for every $\gamma \in \Gamma$, $x_\gamma I \subset PI = 0$, and hence $x_\gamma \in E_S Q \subset EQ_\varphi$. In view of the latter, and by several applications of 3.3 (3) and 3.4 (4) we get that

$$[P_\varphi]^2 \leqslant [Q_\varphi]^2 \iff \exists\Gamma, \quad \widehat{P}_\varphi \subset \bigoplus_\Gamma EQ_\varphi \iff \widehat{P} \subset \bigoplus_\Gamma E_S Q \iff [P]^2_S \leqslant [Q]^2_S$$

□

3.5. Lemma. *The maps φ^{*i}, $i = 1$, 2, and 3, above are one to one.*

Proof. (i) The case $i = 1$ is easy, and $i = 2$ can be patterned after $i = 3$. So suppose that $\varphi^{*3}[P]^3_S = \varphi^{*3}[Q]^3_S$. This means that $P_\varphi \subset E(\oplus\{Q_\varphi \mid \Gamma\})$ and $Q_\varphi \subset E(\oplus\{P_\varphi \mid \Delta\})$ for some sets Γ andΔ. View $E(\oplus_\Gamma Q_\varphi) \subset \Pi_\Gamma EQ_\varphi$. By 3.3 (4),

$$E_S(\oplus_\Gamma Q) = E(\oplus_\Gamma Q_\varphi) \cap \{\xi \in \Pi_\Gamma EQ_\varphi \mid \xi I = 0\}.$$

Since the latter set contains P, $P \subset E_S(\oplus_\Gamma Q)$. Similarly, $Q \subset E_S(\oplus_\Delta P)$, and hence $[P]^3_S = [Q]^3_S$. □

3.6. Proposition. *For $i = 1$, 2, or 3 and φ^{*i} as in 3.4, $\varphi^{*i}G^i(S) \subset G^i(R)$ is convex.*

Proof. Let $A = A_R$, $Q = Q_S$ and $[A]^i \leqslant \varphi^{*i}[Q]^i_S \in G^i(R)$. If $i = 1$, this means that $A \subset \oplus\{Q_\varphi \mid \Gamma\}$ for some Γ. Since $Q_\varphi I = 0$, also $AI = 0$. Hence $A = A_S$ is already an S-module with $\varphi^{*1}[A_S]^1_S = [A_\varphi]^1 = [A]^1$.

Next, let $i = 3$. then $A \subset E(\oplus\{Q_\varphi \mid \Gamma\})$. (Note that possibly $AI \neq 0$.) Set $N = N_S = \oplus\{Q_\varphi \mid \Gamma\}$. Thus $A \subset EN_\varphi$. Since $N_\varphi << EN_\varphi$ it follows that $A \cap N_\varphi << A$. The whole proof now hinges on the fact that $(A \cap N_\varphi)I = 0$ and that $[A \cap N_\varphi]^3 = [A]^3$. Hence $A \cap N = (A \cap N)_S$ is an S-module such that $\varphi^{*3}([A \cap N]^3_S) = [A \cap N_\varphi]^3 = [A]^3$, and φ^{*3} is onto. Omit $i = 2$, it is similar.

Aside from proving a certain minimal amount of later needed facts, the next proposition also explains why the category $\mathbf{A}^*$ is absolutely unavoidable, and how it arises naturally. □

3.7. Proposition. *For S, N, $\varphi\colon R \to S = R/I$, N_φ, Z^S, and Z_2^S as in 3.2, the following hold for all i:*

(1) $Z^S N \subseteq Z(N_\varphi)$;

(2) $Z_2^S N \subseteq Z_2(N_\varphi)$;

(3) $\varphi^{*i} G_T^i(S) \subseteq G_T^i(R)$.

(4) *If $I < R$ is a right complement, then*

(a) $\forall N = N_S$, $Z^S N = Z(N)_\varphi$; *in particular,* $Z^S(R/I) = Z(R/I)$.

(b) $\varphi^{*i} G_F^i(S) \subseteq G_F^i(R)$.

Proof. By 3.3 (2), $\mathscr{L}_S(N) = \mathscr{L}_R(N_\varphi)$. For $n \in N = N_S$, set $n^{-1} = \{s \in S \mid ns = 0\}$. When n is veiwed as $n \in N_\varphi$, then $I \subseteq n^\perp$, and $\varphi n^\perp = n^\perp / I = n^{-1}0$.

(1) For $n \in N$, it is easy to see that if $n^\perp \not<\!< R$, that then also necessarily $n^{-1}0 \notin \mathscr{L}_R(R/I) = \mathscr{L}_S(S)$. Hence $Z^S N \subseteq Z(N_\varphi)$.

(2) By 3.3 (2), any quotient S-module of N is also an R-module (and conversely). Since Z^S is a subfunctor of the identity functor, the natural projection π by restriction in the second line in the figure also corestricts to give a commutative diagram as follows.

$$\begin{array}{ccccc} \frac{N}{Z^S N} & \xrightarrow{\pi} & \frac{n}{ZN_\varphi} & & \\ \cup & & \cup & & \\ Z^S\left[\frac{N}{Z^S N}\right] & \xrightarrow{\pi} & Z^S\left[\frac{N}{ZN_\varphi}\right] & \subset & Z\left[\frac{N}{ZN_\varphi}\right] \end{array}$$

Hence $Z_2^S N \subseteq Z_2(N_\varphi)$.

(3) If $[N]_S^i \in G_T^i(S)$, then $Z_2^S N = N = Z_2 N_\varphi$ by (2). Hence $\varphi^{*i}[N]_S^i = [N_\varphi]^i \in G_T^i(R)$.

(4) (a) It suffices to show that for any $n \in Z(N_\varphi)$, also $n \in Z^S(N)$. But $nR \cong R/n^\perp$ with $n^\perp << R$. Since $I < R$ is a right complement, $n^{-1}0 = n^\perp / I << R/I$ remains large modulo I. Thus $n^{-1}0 \in \mathscr{L}_R(R/I) = \mathscr{L}_S(R/I)$. Hence $n \in Z^S(N)$.

(b) Let $[N]_S \in G_F^i(S)$ be arbitrary. By 4(a), $Z^S(N) = Z(N_\varphi) = 0$, and $\varphi^{*i}([N]_S^i) = [N_\varphi]^i \in G_F^i(R)$. thus $\varphi^{*i} G_F^i(S) \subseteq G_F^i(R)$. □

3.8. Theorem II. *Let* **A** *and* **B** *be the ring and semi-lattice categories of* 3.1. *Let $\varphi\colon R \to S$ be any surjective ring homorphism of rings with identity. For any $i \leqslant 3$, and any $1 \leqslant i \leqslant j \leqslant 3$, define $G_T^i(R)$, $G_F^i(R) \subset G^i(R)$ and $\eta_i^j(R)\colon G^i(R) \to G^j(R)$ is in* 1.3, *and define $G^i(\varphi) = \varphi^{*i}$ as in* 3.4. *Then*

(i) *$G^i\colon \mathbf{A} \to \mathbf{B}$ is a contravariant functor. In particular, $\varphi^{*i}\colon G^i(S) \to G^i(R)$ is a zero preserving, monic semi-lattice homomorphism which preserves arbitrary suprema of subsets.*

(ii) *$\eta_i^j\colon G^i \to G^j$ is a natural transformation of functors.*

(iii) *G_T^i is a subfunctor of G^i; i.e. for any φ and S, $\varphi^{*i} G_T^i(S) \subset G_T^i(R)$.*

(iv) $\varphi^{*i}G^i(S) \subset G^i(R)$ *is a convex and complete semi-sublattice for* $i = 1, 2, 3$.

(v) $\varphi^{*3}G^3(S) \subset G^3(R)$ *is a convex and complete sublattice;* $\varphi^{*3}: G^3(S) \to G^3(R)$ *is zero preserving monic lattice homomorphism which preserves arbitrary infima and suprema. Its corestriction* $\varphi^{*3}: G^3(S) \to \varphi^{*3}G^3(S)$ *is a lattice isomorphism; it and its inverse preserve arbitrary infima and suprema.*

Proof. (i) follows from 3.4, 2.2 (i), and 3.5 (ii). Let $[N]^i_S \in G^i(S)$. Then $\eta^j_i(R)(\varphi^{*i}([N]^i_S)) = \eta^j_i(R)([N_\varphi]^i) = [N_\varphi]^j$; whereas $\varphi^{*i}(\eta^j_i(S)([N]^i_S)) = \varphi^{*i}([N]^j_S) = [N_\varphi]^j$. Hence η^j_i is a natural transformation because the following diagram commutes:

$$\begin{array}{ccc} G^i(S) & \xrightarrow{\varphi^{*i}} & G^i(R) \\ \downarrow{\scriptstyle \eta^j_i(S)} & & \downarrow{\scriptstyle \eta^j_i(R)} \\ G^j(S) & \xrightarrow{\varphi^{*j}} & G^j(R) \end{array}$$

(iii): by 3.7 (3); (iv): by 3.6.

(v) By 2.2 (1) (i), $\bigvee \varphi^{*3}G^3(S) \in G^3(R)$ exists, and $\bigvee \varphi^{*3}G^3(S) \in \varphi^{*3}G^3(S)$. Any suprema closed poset with 0 is a complete lattice. Thus $\varphi^{*3}G^3(S)$ is a convex and complete sublattice of $G^3(R)$. Let f be the corestriction $f = \varphi^{*3}: G^3(S) \to \varphi^{*3}G^3(S)$. But any bijective map f of any lattices whatever is a lattice isomorphism if and only if f and its inverse f^{-1} preserve order. Now by 3.4, both $f = \varphi^{*3}$ and f^{-1} are lattice isomorphisms both of which preserve arbitrary suprema, and hence also infima. □

3.9. Main Corollary to Theorem II. *Let* $\mathbf{A}^* \subset \mathbf{A}$ *be the subcategory in* 3.1 *and assume that* $\varphi: R \to S$ *in* $\mathbf{A}^*$ *is a surjective ring homomorphism with* $\varphi^{-1}0 = I < R$ *a complement right ideal. Then* $G^i: \mathbf{A}^* \to \mathbf{B}$ *for* $i = 1, 2,$ *and* 3 *is a contravariant functor satisfying the above conclusions* 3.8 (i)-(v). *In particular*

(iii*) $\varphi^{*i}G^i_T(S) \subseteq G^i_T(R)$ *and* $\varphi^{*i}G^i_F(S) \subseteq G^i_F(R)$.

(vi) $G^3 = G^3_T \oplus G^3_F$ *is direct sum of subfunctors of* G^3, *i.e.* $G^3(R) = G^3_t(R) \oplus G^3_F(R)$ *and* $G^3(\varphi) = G^3_T(\varphi) \oplus G^3_F(\varphi)$.

4. Infimum Modules

First, for any finite set $[A^{(1)}]^3, \ldots, [A^{(n)}]^3 \in G^3(R)$, a module ΛT depending symmetrically on the modules $A^{(1)}, \ldots, A^{(n)}$ is constructed such that $[\Lambda T]^3 = [A^{(1)}]^3 \wedge \ldots \wedge [A^{(n)}]^3$, where ΛT is defined by a certain sets Λ and T. The following simple fact will be used repeatedly, and it also explains what is really going on.

4.1. Fact. Let $\{M_i \mid i \in I\}$ be any set of modules, and $0 \neq \xi \in E(\oplus M_i)$ arbitrary. Assume that $r_0 \in R$ is any element such that $0 \neq \xi r_0 = y_1 + \ldots + y_n \in M_{i(1)} \oplus \ldots \oplus M_{i(n)}$ with all $0 \neq y_k \in M_{i(k)}$, and such that the length n is minimal. Then
$(\xi R_0)^{\perp} = y_1^{\perp} = \ldots = y_n^{\perp}$; hence $\xi r_0 R \cong y_k R$ under the natural projection map for each $k \leqslant n$.

4.2. Construction. Let $A^{(1)}, \ldots, A^{(n)}$ be any modules. For any set $T = \{(a_\lambda^{(1)}, a_\lambda^{(2)}, \ldots, a_\lambda^{(n)}) \mid \lambda \in \Lambda\} \subset A^{(1)} \times \ldots \times A^{(n)}$ define $T^{(i)} = \sum\{a_\lambda^{(i)} R \mid \lambda \in \Lambda\}$ for $1 \leqslant i \leqslant n$. By Zorn's lemma select a T satisfying the following three conditions:

(1) $\forall \lambda \in \Lambda,\ a_\lambda^{(1)\perp} = a_\lambda^{(2)\perp} = \ldots a_\lambda^{(n)\perp}$.

(2) $T^{(i)} = \oplus\{a_\lambda^{(i)} R \mid \lambda \in \Lambda\} \leqslant A^{(i)}$ for all i.

(3) T is maximal with respect to properties (1) and (2).

Each such a set T defines a module ΛT by

$$\Lambda T = \oplus\{tR \mid t \in T\} \subseteq T^{(1)} \oplus \ldots T^{(n)} \leqslant A^{(1)} \oplus \ldots \oplus A^{(n)}.$$

(4) Each $y \in \Lambda T$ is a finite sum of the following form:

$$y = \sum_\lambda (a_\lambda^{(1)}, \ldots, a_\lambda^{(n)}) r_\lambda = (y^{(1)}, \ldots, y^{(n)}) \qquad r_\lambda \in R;$$
$$y^{(j)} = \sum_\lambda a_\lambda^{(j)} r_\lambda \in \oplus \lambda a_\lambda^{(j)} R,\ y^{(j)\perp} = \bigcap_\lambda (a_\lambda^{(j)} r_\lambda)^{\perp},$$
$$j = 1, \ldots, n;\ y^{\perp} = y^{(1)\perp} = \ldots = y^{(j)\perp} = \ldots = y^{(n)\perp}.$$

4.3. Theorem III. *With the previous notation and hypotheses*

$$[\Lambda T]^3 = \bigwedge_{i=1}^{n} [A^{(i)}]^3 = \inf_{1 \leqslant i \leqslant n} [A^{(i)}]^3 \in G^3(R).$$

Corollary to Theorem III. *If D is nay module having the property that for each $i = 1, \ldots, n$, this module D can be imbedded in the injective hull of some (arbitrary) direct sum of the $A^{(i)}$, then it follows that there is an embedding*

$$D \subset E\Big(\bigoplus_{|D|} \Lambda T\Big).$$

P r o o f o f 4.3 a n d 4.4. Clearly, $[\Lambda T]^3 \leqslant [A^{(}i)]^3$ for all i. It suffices to show that if $[D]^3 \leqslant [A^{(i)}]^3$ for all i, that then $[D]^3 \leqslant [\Lambda T]^3$. By assumption, there exist monic maps $f^{(i)}\colon D \to E(\oplus_\alpha A^{(i)}_\alpha)$ where all $A^{(i)}_\alpha = A^{(i)}$ and the ordinal α runs over some initial ordinal interval or segment, which may be taken to be the same one for all i, for simplicity.

First it will be shown that for any $0 \neq d \in D$ there is an $s \in R$ such that there is a monic map $g\colon dsR \to \Lambda T$. Choose a $p_1 \in R$ so that

$$0 \neq f^{(1)}dp_1 = a^{(1)}_1 + \ldots + a^{(1)}_{n(1)} \in A^{(1)}_{\alpha(1,1)} \oplus \ldots \oplus A^{(1)}_{\alpha(1,n(1))};$$

$$\alpha(1,1) < \ldots < \alpha\big(1, n(1)\big)\,;\; 0 \neq a^{(1)}_j \in A^{(1)}_{\alpha(1,j)},\; j = 1, \ldots,\; n(1);$$

where $n(1)$ is minimal with this property. Then $(f^{(1)}dp_1)^\perp = (dp_1)^\perp = a^{(2)\perp}_1 = \ldots = a^{(1)\perp}_{n(1)}$. Next, there exists a $p_2 \in R$ such that

$$0 \neq f^{(2)}dp_1p_2 = a^{(2)}_1 + \ldots + a^{(2)}_{n(2)} \in A^{(2)}_{\alpha(2,1)} \oplus \ldots \oplus A^{(2)}_{\alpha(2,n(2))};$$

$$\alpha(2,1) < \ldots < \alpha\big(2, n(2)\big)\,;\; 0 \neq a^{(2)}_j \in A^{(2)}_{\alpha(2,j)},\; j = 1,\; \ldots,\; n(2);$$

where $n(2)$ again is minimal with this property. Thus $(f^{(2)}dp_1p_2)^\perp = (dp_1p_2)^\perp = a^{(2)\perp}_1 = \ldots = a^{(2)\perp}_{n(2)}$. From $dp_1p_2 \neq 0$, it follows by the minimality of $n(1)$ that all $0 \neq a^{(1)}_j p_2 \in A^{(1)}_{\alpha(1,j)}$, and $(dp_1p_2)^\perp = (a^{(1)}_j p_2)^\perp$ for all j. Continue in this manner and obtain for all $i \in n$,

$$\begin{aligned}
&0 \neq f^{(i)}dp_1p_2\ldots p_i = a^{(i)}_1 + \ldots + a^{(i)}_{n(i)} \in A^{(i)}_{\alpha(i,1)} \oplus \ldots \oplus A^{(i)}_{\alpha(i,n(i))};\\
&0 \neq a^{(i)}_j \in A^{(i)}_{\alpha(i,j)}\,;\; n(i) \text{ minimal};\\
&(f^{(i)}dp_1p_2\ldots p_i)^\perp = (dp_1p_2\ldots p_i)^\perp = a^{(i)\perp}_1 = \ldots = a^{(i)\perp}_{n(i)}
\end{aligned}$$

for all $j = 1, \ldots, n(i)$.

Next, set $p_0 = p_1p_2\ldots p_n$ and

$$\begin{aligned}
x_1 &= a^{(1)}_1 p_2 \ldots p_n \in A^{(1)}_{\alpha(1,1)} = A^{(1)}, \qquad dp_0 \neq 0,\; x_i \neq 0, \qquad \text{all } i\\
x_2 &= a^{(2)}_1 p_3 \ldots p_n \in A^{2)}_{\alpha(2,1)} = A^{(2)},\\
&\vdots\\
x_{n-1} &= a^{(n-1)}_1 p_n \in A^{(n-1)}_{\alpha(n-1,1)} = A^{(n-1)},\\
x_n &= a^{(n)}_1 \in A^{(n)}_{\alpha(n,1)} = A^{(n)},\\
(dp_0)^\perp &= (f^{(i)}dp_0)^\perp = x^\perp_1 = x^\perp_2 = \ldots = x^\perp_i = \ldots = x^\perp_n, \qquad \text{all } i.
\end{aligned}$$

Suppose that $T^i \cap x_i R = 0$ for all $i = 1, \ldots, n$. Then the set $T \cup \{(x_1, \ldots, x_n)\}$ satisfies the conditions (1) and (2) in the Construction 4.2, and thus violates the maximality of T. Hence for some $q \in R$ and some i,

$$0 \neq x_i q \in T^{(i)} \cap x_i R, \ (x_i q)^\perp = (dp_0 q)^\perp, \ dp_0 q \neq 0.$$

As in 4.2 (4), there exists a $y = (y^{(1)}, \ldots, y^{(n)}) \in T$ with $y^{(i)} = x_i q$ and

$$y^\perp = y^{(j)\perp} = (x_i q)^\perp, \qquad 1 \leqslant j \leqslant n.$$

Thus $(dp_0 q)^\perp = y^\perp$. Set $s = p_0 q$ and define $g\colon dsR \to \Lambda T$ by $gds = y$.

Now let $\oplus\{d_\gamma R \mid \gamma \in \Gamma\} << D$ with $|\Gamma| \leqslant |D|$. For each $0 \neq d_\gamma \in D$, let $s_\gamma \in R$ and $g_\gamma : d_\gamma s_\gamma R \to (\Lambda T)_\gamma = \Lambda T$ be monic as above. Then there is an induced monomorphism $g = \oplus g_\gamma \colon \oplus d_\gamma s_\gamma R \to \oplus(\Lambda T)_\gamma$ which extends to a monic map $\hat{g}\colon D \to E(\oplus(\Lambda T)_\gamma) \subset E(\oplus_{|D|}\Lambda T)$. Thus $[D]^3 \leqslant [\Lambda T]^3$ and also 4.4 holds. □

5. Applications and Examples

Of the six functors G_T^i, G_F^i, the functor G_F^3 is the most useful (e.g. [5], and indirectly [3]). First, in order to have available an effective method of computing $G_F^3(R)$, a known result from [5, p. 73, 5.9] is rephrased (5.1), and then sharpened in (5.2). Then a class of rings is constructed by techniques similar to those used to construct algebraically compact abelian groups ([7, p. 174, Theorem 42.1], [9]), or certain characteristic subgroups of generalized Baer-Specker groups ([10]).

5.1. Definition and Theorem. For any ring $R (1 \in R)$, define $\mathscr{I}(R)$ to be the set of all complement right ideals $J \leqslant R$ such that $Z_2 R \subseteq J$, and such that $\hat{J} \leqslant \hat{R}$ is a fully invariant right R-submodule of $\hat{R}$. (Thus $J \triangleleft R$, and $J < R$ is fully invariant.)

Then $\mathscr{I}(R) \to G_F^3(R)$, $J \to [J/Z_2R]^3$ is a lattice isomorphism, where $J_1 \wedge J_2 = J_1 \cap J_2$ and $J_1 \vee J_2 = (J_1 + J_2)^-$ for $J_1, J_2 \in \mathscr{I}(R)$.

5.2. Lemma. *For any complement right ideal $J < R$ with $ZR \subset J$, let $C < R$ be any right ideal maximal with respect to $J \oplus C << R$.*

(1) *Then the following are all equivalent*
  (a) *$\hat{J} \leqslant \hat{R}$ is fully invariant;*
  (b) $\mathrm{Hom}_R\left(J, E(R/J)\right) = 0$;
  (c) $\forall b \in J$, $\mathrm{Hom}_R(bR, C) = 0$.

(2) *If in addition $ZR = 0$, then* (a), (b), *and* (c) *are equivalent to* (d):
  (d) *$\hat{C} \leqslant \hat{R}$ is fully invariant.*

Proof. (1) (a) $\Longleftrightarrow$ (b). Let $\pi\colon \widehat{R} = \widehat{J} \oplus \widehat{C} \to \widehat{C}$ and $\pi^*\colon \operatorname{Hom}_R(\widehat{J}, \widehat{R}) \to \operatorname{Hom}_R(\widehat{J}, \widehat{C})$. Since $(C + J)/J << R/J$, we have $\widehat{C} \cong E(R/J)$. Then

$$\pi^*[\operatorname*{Hom}_R(\widehat{J}, \widehat{R})] = \pi \circ \operatorname*{Hom}_R(\widehat{J}, \widehat{R}) = \operatorname*{Hom}_R(\widehat{J}, \widehat{C}).$$

Since $\widehat{J}/J$ is torsion and $\widehat{C}$ torsion free, $\operatorname{Hom}_R(\widehat{J}, \widehat{C}) \cong \operatorname{Hom}_R(J, \widehat{C})$ under the restriction map. Clearly (1) (a)$\Longleftrightarrow$ $\operatorname{Hom}_R(\widehat{J}, \widehat{C}) = 0$.

(1) (c) $\Longrightarrow$ (a) is clear. (1) (b) $\Longrightarrow$ (c). This follows from $(C + J)/J << R/J << E(R/J) \cong \widehat{C}$. □

5.3. Definition. For an infinite set X, let $\mathscr{B} \subset \mathscr{P}(X)$ be a subring ($C \bullet D = C \cap D$, $C + D = C \cup D \backslash S \cap D \in \mathscr{B}$; C, D, $\emptyset$, $X \in \mathscr{B}$). Let $\{R_x \mid x \in X\}$ be any indexed set of right Ore domains R_x with $1 \in R_x$. For $r \in \Pi\{R_x \mid x \in X\} = \Pi R_x$, write $r_x = r(x) \in R_x$, and $r = (r_x)_{x \in X} = (r_x) = (r(x))$. The *support* of r is $\operatorname{supp} r = \{x \in X \mid r(x) \neq 0\}$ and $r^{-1}0 = \{x \in X \mid r(x) \neq 0\}$. For any subset $H \subseteq X$, $\chi_H \in \Pi R_x$ is the characteristic function of H, i.e. $\chi_H(x_0) = 1 \in R_x$ if $x_0 \in H$, and $\chi_H(x_0) = 0$ when $x_0 \notin H$. Define R to be the subring $R \subseteq \Pi R_x$ consisting of all those r such that $\operatorname{supp} r \in \mathscr{B}$. Hence 1, $\chi_{\operatorname{supp} r} \in R$, and also $\chi_{r^{-1}0} = 1 - \chi_{\operatorname{supp} r} \in R$.

5.4. Thus $\mathscr{B} = \{\operatorname{supp} r \mid r \in R\}$. The (unique over $\mathscr{B}$) minimal completion of $\mathscr{B}$ is denoted by r.o.($\mathscr{B}$). (See Banaschewski [2, p. 123, Corollaries 3 and 4], Halmos [16, p. 13, p. 91, and p. 93], and Jech [17, p. 153]). Then $\mathscr{B} \subset$ r.o.($\mathscr{B}$) is dense and every element of $\mathscr{B}$ is the supremum of those elements which it dominates. By [16, p. 93], there is a complete lattice (ring) monomorphism $g\colon$ r.o.$(\mathscr{B}) \to \mathscr{P}(X)$ such that $\mathscr{B} \subset g(\text{r.o.}(\mathscr{B}))$, i.e. there is a commutative diagram.

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\text{inclusion}} & \text{r.o.}(\mathscr{B}) \\ & \searrow^{\text{inclusion}} & \swarrow g \\ & \mathscr{P}(X) & \end{array}$$

In $\mathscr{B} \subset \mathscr{P}(X)$ we use $\cup$, $\cap$, $\subseteq$; in $\mathscr{B} \subset$ r.o.($\mathscr{B}$), $\vee$, $\wedge$, and $\leqslant$. (For some restriction on $|\text{r.o.}(\mathscr{B})|$, see Pierce [19, p. 896]).

5.5. Lemma. *Let $\{a_\gamma\} \subset \mathscr{B}$ be any infinite subset, and $0 \neq b \in \mathscr{B}$. Set $\xi = \vee a_\gamma \in$ r.o.($\mathscr{B}$). Then*

(i) $\bigcup_\gamma a_\gamma \subseteq g(\xi)$;

(ii) $g(\bigwedge_\gamma a_\gamma) \subseteq \bigcap_\gamma a_\gamma$;

(iii) $b \leqslant \xi \Longrightarrow \exists\gamma,\ b \cap a_\gamma \neq \emptyset$.

(iv) $b \not\leqslant \xi \Longrightarrow \exists d \in \mathscr{B}, 0 \neq d \leqslant b, d \wedge \xi = 0 \Longrightarrow 0 \neq d \subseteq b$ and $d \cap (\bigcup_\gamma a_\gamma) = \emptyset$.

Proof. (i) and (ii) are trivial. (iii) This follows from the fact that any complete Boolean lattice satisfies a limited infinite distributive law ([16, p. 28]). Thus $b \wedge (\bigwedge_\gamma a_\gamma) = \bigwedge_\gamma (b \wedge a_\gamma) = \bigwedge_\gamma (b \cap a_\gamma)$, and $b \leqslant \xi$ implies that $b \cap a_\gamma \neq \emptyset$ for some γ.

(iv) Any Boolean lattice such as r.o.($\mathscr{B}$) is separative ([17, p. 153]), i.e. whenever $b \not\leqslant \xi$, there exists an $\eta \in$ r.o.($\mathscr{B}$) such that $0 \neq \eta \leqslant b$, but $\eta \wedge \xi = 0$. Since $\mathscr{B} \subset$ r.o.($\mathscr{B}$) is dense, there exists a $d \in \mathscr{B}$, $o \neq d \leqslant \eta$. Thus $d \wedge \xi = 0$.

If $d \cap (\bigcup a_\gamma) \neq 0$, then $d \cap a_\gamma \neq 0$ for some γ. Since $g(d \wedge a_\gamma) = d \cap a_\gamma$, also $d \wedge a_\gamma \neq 0$. Then $0 \neq d \wedge a_\gamma \leqslant d \wedge \xi$ is a contradiction. Thus $d \cap (\bigcup a_\gamma) = \emptyset$.

The next theorem determines the complement right ideals of R which in turn will determine $G_F^3(R)$. □

5.6. Theorem IV. *Let $L \leqslant R$. For any $\xi \in$ r.o.($\mathscr{B}$), define*

$$L_\xi = \{b \in R \mid \operatorname{supp} b \leqslant \xi\}.$$

Then

(i) *$L_\xi = \bar{L}_\xi$ is a right complement, and every right complement is of the above form for a unique $\xi \in$ r.o.($\mathscr{B}$).*

(ii) *If $\xi = \bigvee\{\operatorname{supp} a \mid a \in L\} \in$ r.o.($\mathscr{B}$), then $L << L_\xi$, and hence L_ξ is the complement closure of L.*

Proof. (i) It suffices to show that for any $b \in R \backslash L_\xi$, also $b \notin \bar{L}_\xi$. For this, it suffices to show that for some $d \in R$ with $bdR \neq 0$, $bdR \cap L_\xi = 0$. ([5, p. 53, Lemma 1.2]). Thus $b \not\leqslant \xi$. By 5.5 (iv), there is a $d \in R$ with $0 \neq \operatorname{supp} d \subseteq \operatorname{supp} b$, hence with $bd \neq 0$, but where

$$\operatorname{supp} d \cap \bigcup_{a \in L} \operatorname{supp} a = \emptyset.$$

Now suppose that $0 \neq c \in bdR \cap L_\xi$. Then $0 \neq \operatorname{supp} c \leqslant \xi$ By 5.5 (iii), there exists an $a \in L$ such that $\emptyset \neq \operatorname{supp} c \cap \operatorname{supp} a$. But then $\operatorname{supp} c \subseteq \operatorname{supp} bd \subseteq d$ is a contradiction. Hence $L_\xi = \bar{L}_\xi$.

If $\xi \neq \eta \in$ r.o.($\mathscr{B}$) and $\xi \not\leqslant \eta$, then there exists an $a \in R$ with $0 \neq \operatorname{supp} a \leqslant \xi$, but $\operatorname{supp} a \wedge \eta = 0$ by the separativeness property ([17, p. 153]). Since $\operatorname{supp} a \not\leqslant \eta$, $a \notin L_\eta$ whereas $o \neq a \in L_\xi$. Therefore $L_\xi \neq L_\eta$. It suffices to prove (ii).

(ii) It will be shown that for any $0 \neq b \in L_\xi$, also $bR \cap L \neq 0$. By 5.5 (iii), there exists an $a \in L$ such that $\operatorname{supp} b \cap \operatorname{supp} a \neq \emptyset$. Since all the R_x are right Ore, there exists an a_1, $b_1 \in R$ with $\operatorname{supp} a_1 = \operatorname{supp} b_1 = \operatorname{supp} b \cap \operatorname{supp} a$ such that $0 \neq ba_1 = ab_1 \in bR \cap aR \subseteq bR \cap L$. □

5.7. Corollary 1 to Theorem IV. *Let α, $\beta \in$ r.o.($\mathscr{B}$). Then*

(i) $L_{\alpha\wedge\beta} = L_\alpha \cap L_\beta$; *hence in particular if* $\alpha \wedge \beta = 0$, *then* $L_\alpha + L_\beta = L_\alpha \oplus L_\beta$.

(ii) $L_{\alpha\vee\beta} = (L_\alpha + L_\beta)^-$.

Proof. (i) Conclusion (i) is clear. (ii) If $\alpha = 0$ or $\beta = 0$, we are done. So let $\alpha \neq 0$ and $\beta \neq 0$. Clearly, $L_\alpha + L_\beta \subseteq L_{\alpha\vee\beta}$. We will show that $L_\alpha + L_\beta << L_{\alpha\vee\beta}$. Let $0 \neq c \in L_{\alpha\vee\beta}$ with $c \notin L_\beta$. The latter implies that $\operatorname{supp} c \nleqslant \beta$. Use of the separativeness of r.o.($\mathscr{B}$), and then the density of $\mathscr{B} \subset$ r.o.($\mathscr{B}$) shows that there exists an $d \in R$ such that $0 \neq \operatorname{supp} d \leqslant \operatorname{supp} c$ and $\operatorname{supp} d \wedge \beta = 0$. Then

$$\begin{aligned}\operatorname{supp} d = \operatorname{supp} d \wedge (\alpha \vee \beta) &= [\operatorname{supp} d \wedge \alpha] \vee [\operatorname{supp} d \wedge \beta] \\ &= \operatorname{supp} d \wedge \alpha, \text{ and } \operatorname{supp} d \leqslant \alpha.\end{aligned}$$

By right Ore condition, there exist r, $s \in R$ such that $0 \neq dr = cs$ with $\emptyset \neq \operatorname{supp} r = \operatorname{supp} s = \operatorname{supp} d$. Hence $0 \neq cs = dr \in L_\alpha$. □

5.8. Main Corollary 2 to Theorem IV. *For any infinite set X, let $\mathscr{B}$ be a Boolean subring $\mathscr{B} \subseteq \mathscr{P}(X)$ and $\mathscr{B} \subset$ r.o.($\mathscr{B}$) the minimal completion of $\mathscr{B}$. For any family of right Ore domains $\{R_x \mid x \in X\}$, let $R \subseteq \Pi R_x$ be the subring $R = \{r \in \Pi R_x \mid \operatorname{supp} r \in \mathscr{B}\}$. Let $\mathscr{I}(R)$ and $G^3_F(R)$ be as in* 5.1 *and* 1.3. *Then*

(i) $\mathscr{I}(R) = \{L_\xi \mid \xi \in \text{r.o.}(\mathscr{B})\}$ *where* $L_\xi = \{r \in R \mid \operatorname{supp} r \leqslant \xi\}$.

(ii) *There is a natural Boolean lattice isomorphism* $G^3_F(R) = \{[L_\xi]^3 \mid \xi \in \text{r.o.}(\mathscr{B})\} \to \text{r.o.}(\mathscr{B})$, $[L_\xi]^3 \to \xi$.

Proof. (i) and (ii). Let $\eta \in$ r.o.($\mathscr{B}$) with $\eta \wedge \xi = 0$ and $\xi \vee \eta = 1$. By 5.7 (i), $L_\xi \oplus L_\eta << L_1 = R$. In 5.2, take $J + L_\xi$ and $C = L_\eta$. If 5.2 (1)(c) fails, there exists a $0 \neq \Phi \colon bR \to L_\eta$ for $b \in L_\xi$. But then $0 \neq \Phi(b) = \Phi(b\chi_{\operatorname{supp} b}) = (\Phi b)\chi_{\operatorname{supp} b} \in L_\eta\chi_{\operatorname{supp} b}$. Let $c \in L_\eta$. Then $\operatorname{supp} c \leqslant \eta$, and $\operatorname{supp} b \cap \operatorname{supp} c \leqslant \xi \wedge \eta = 0$. Hence $L_\eta\chi_{\operatorname{supp} b} = 0$, a contradiction. The rest follows from 5.1 and 5.7. □

5.9 Corollary 3 to Theorem IV. *In addition to the hypotheses of the last theorem assume that $\mathscr{B} \subseteq \mathscr{P}(X)$ is closed under arbitrary unions and intersections. For any $L \leqslant R$ define $H = \bigcup\{\operatorname{supp} a \mid a \in L\}$. Then*

(i) $L << \bar{L} = \chi_H R$;

(ii $\mathscr{I}(R) = \{\chi_H R \mid H \in \mathscr{B}\}$ *and* $G^3_F(R) = \{[\chi_H R]^3 \mid H \in \mathscr{B}\} \cong \mathscr{B}$, *under* $[\chi_H R]^3 \to H$.

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Author's address: Department of Mathematics, Tulane University, New Orleans, Lousiana 70118, U.S.A.