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LOCALLY HOMOGENEOUS GRAPHS WITH DENSE LINKS
AT VERTICES

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Let G be a locally finite graph and u its vertex. Denote by *link* (u, G) the subgraph of G induced on the set of vertices adjacent to u. Let G_0 be a finite graph. The graph G is called *locally G_0*, or *locally homogeneous*, if *link* $(u, G) \cong G_0$ for each vertex u of G.

The original motivation for the study of locally homogeneous graphs probably came from the theory of finite automata. At one of the first conferences on graph theory, held in Smolenice in 1963, Zykov [9] formulated the following two problems:

(P1) Which are the finite graphs G_0 that there is a locally G_0 graph G?

(P1F) Which are the finite graphs G_0 that there is a *finite* locally G_0 graph G?

Another problem related to locally homogeneous graphs reads as follows:

(P2) For a given finite graph G_0 characterize all graphs G that are locally G_0.

A survey paper [3] dealing with (P1) and (P1F) was published in 1976. New information on the research related to (P1), (P1F) and (P2) can be found in [4].

It is quite a simple observation that the complete graph K_{n+1} is the only locally K_n graph. The complete graph K_n has the maximum number of edges among all n-vertex graphs. We say that a locally G_0 graph has a dense link if the number of edges of G_0 is great in comparison with the number of its vertices. The example of a complete graph suggests that the structure of the graph G is strongly determined by its link provided G_0 is dense. In this note we shall present two ways how to make this sentence more precise. The first way comes from the following result of Šoltés [7]. We shall call a graph G an (r, t)-graph if there are integers $r > t \geqslant 0$ such that G is r-regular and the link of each vertex is a t-regular graph. It is easy to see that G is an (r, t)-graph if and only if G is r-regular and each edge of G lies in t triangles.

Nice examples of (r, t)-graphs are strongly regular graphs. It should be noted that an (r, t)-graph is not necessarily locally homogeneous. In [7] Šoltés proved the following

Theorem 1 (Šoltés). *Let t and r be integers such that $t < r < t + \sqrt{\frac{8}{9}(t - 1)} + \frac{4}{3}$ holds. Then a connected finite graph G is an (r, t)-graph if and only if $r - t$ divides r and G is the complete $\left(1 + \frac{r}{r-t}\right)$-partite graph whose each part has $r - t$ vertices.*

As a consequence we obtain

Corollary 2. *Let G_0 be a t-regular graph with r vertices such that $t < r < t + \sqrt{\frac{8}{9}(t - 1)} + \frac{4}{3}$ holds. Then a connected finite graph G is locally G_0 if and only if $r - t$ divides r and G is the complete $\left(1 + \frac{r}{r-t}\right)$-partite graph whose each part has $r - t$ vertices.*

Let $t \geqslant 1$ be a square. Then the tensor product $K_{2+\sqrt{t}} \otimes K_{2+\sqrt{t}} \cong G$ is locally $K_{1+\sqrt{t}} \otimes K_{1+\sqrt{t}}$ and G is $t + 2\sqrt{t} + 1$ regular. Thus the coefficient at $\sqrt{t - 1}$ in Theorem 1 and Corollary 2 cannot exceed 2.

The second way of making the notion of a dense link in a graph precise was motivated by the following result of Bugata, Horňák and Jendroľ [2]. If H is a graph then $\overline{H}$ denotes its complement.

Theorem 3 (Bugata, Horňák, Jendroľ). *Let T be a tree. Then a locally $\overline{T}$ graph exists if and only if T is a path or a star.*

The main ideas of the proof of the following theorem were extracted from the proof of Theorem 3 in [2].

Theorem 4. *Let H be a graph satisfying the following two conditions:*
(i) there are two adjacent vertices in H whose degrees differ by at least 2,
(ii) each 4-vertex induced subgraph of H contains at least two adjacent edges.
Then there is no locally H graph.

P r o o f . Suppose there is a locally H graph G. Let v be a vertex in G and let x, y be vertices in *link* (v, G) such that $\deg_H x + 2 \leqslant \deg_H y$. Since G is $v(H)$-regular there are two vertices w, z in $V(G)$-(*link* $(v, G) \cap \{v\}$) adjacent to x but not to y. Any of the vertices y, and v is adjacent neither to w nor to z. Thus there are no two adjacent edges in the subgraph of G induced on the vertices v, y, w, z. However, all these vertices belong to *link* (x, G), contradicting the assumption. $\square$

It is easy to see that conditions (i) and (ii) can be replaced by the following two conditions for the complementary graph:

(j) there are two non-adjacent vertices in $\overline{H}$ whose degrees differ by at least 2,

(jj) $\overline{H}$ contains no quadrangle.

Since each tree which is not a path or a star contains a pendant vertex v and a vertex of a degree $\geqslant 3$ nonadjacent to v, one implication in Theorem 3 follows.

Theorem 4 is in a sense complementary to Corollary 2 because it covers irregular cases (but not all) of locally homogeneous graphs with a dense link.

Since the cartesian product $K_p \times K_2$ is locally the disjoint union $K_{p-1} \cup K_1$, by [8] we have for every finite group Γ a locally $K_{p-1} \cup K_1$ graph G such that $\Gamma \cong \mathrm{Aut}(G)$. This means that we must be careful in an attempt to formulate more precisely the above mentioned idea that the structure of a locally homogeneous graph with a dense link is more strongly determined by its link than in the case of a thin link.

In this note we were interested in locally homogeneous graphs with dense links at vertices. One can believe that if, on the other hand, the number of edges at a link G_0 of a vertex in a locally homogeneous graph G is small, then the structure of G is more free that it is if G_0 is dense. The results of the papers [5, 6] support this idea.

Concluding remark. It paid on my attention that Brouwer, Cohen and Neumaier in [1; Theorem 1.4.3] proved that an (r, t)-graph with $\sqrt{r} + \frac{1}{2} \geqslant r - t$ is complete multipartite. Since $\sqrt{r} + \frac{1}{2} > \sqrt{t} + \frac{1}{2} > \sqrt{\frac{8}{9}(t-1)} + \frac{4}{3}$, this result implies Theorem 1.

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