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CHORDS AND ARCLENGTH OF A CLOSED SPACE CURVE

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The inequalities (1) and (2) established in [3] belong to the family of Wirtinger's inequalities. In the present paper some geometric inequalities derived from (1) and (2) are discussed. The results may be compared with those from [1], [2].

Let us recall two basic inequalities.

1. Let $\mathscr{A} = A_0, A_1, \ldots, A_{n-1}$ be a closed space n-gon in $\mathbf{R}^N$, where $A_{n+k} = A_k$ for $k = 0, 1, 2, \ldots$. Then for all $p = 0, 1, \ldots, n-1$,

$$\sum_{\nu=0}^{n-1} |A_{\nu+p} - A_\nu|^2 \leqslant \Big(\frac{\sin p\frac{\pi}{n}}{\sin\frac{\pi}{n}}\Big)^2 \sum_{\nu=0}^{n-1} |A_{\nu+1} - A_\nu|^2. \tag{1}$$

For $p = 2, 3, \ldots, n-2$, equality is attained if and only if $\mathscr{A}$ is a plane affine-regular n-gon or, for $N = 1$, its 1-dimensional projection. (For $p = 0, 1, n-1$ equality always occurs.)

2. Let $f(x)$ be a smooth function with period 2π. Then for all real t,

$$\int_0^{2\pi} \big(f(x) - f(x+t)\big)^2 \mathrm{d}x \leqslant 4\sin^2\frac{t}{2}\int_0^{2\pi} f'(x)^2 \mathrm{d}x. \tag{2}$$

Equality is attained if and only if $f(x) = A\cos x + B\sin x + C$, where A, B, C are real constants (for $t = 0$ equality always holds).

First we shall give a geometric interpretation of the inequality (2).

Theorem 1. *Let Γ be a closed rectifiable curve in $\mathbf{R}^N$ of length L. Assume Γ is parametrized by $x = 2\pi s/L$, where s is the arclength. Then for all $t \in \mathbf{R}$*

$$\int_0^{2\pi} |\Gamma(x) - \Gamma(x+t)|^2 \mathrm{d}x \leqslant \frac{2L^2}{\pi}\cdot\sin^2\frac{t}{2}, \tag{3}$$

with equality only for a circle.

P r o o f. In view of $|\Gamma'(x)| = \frac{L}{2\pi}$, (3) follows immediately from (2). As to equality in (3), we have $G(x) = \vec{A}\cos x + \vec{B}\sin x + \vec{C}$, where $\vec{A}$, $\vec{B}$, $\vec{C}$ are constant vectors in $\mathbf{R}^N$. Differentiating this relation we get

$$\vec{A}^2 \sin^2 x - 2\vec{A}\vec{B}\sin x \cos x + \vec{B}^2\cos^2 x = \frac{L^2}{4\pi^2}$$

for all $x \in \langle 0, 2\pi)$. Substituting $x = 0$ and $x = \frac{1}{2}\pi$ into this equation we have $|\vec{A}| = |\vec{B}| = \frac{L}{2\pi}$ and $\vec{A}\cdot\vec{B} = 0$. Then Γ is a circle. □

The next two theorems are related to the problem of Herda [2]. By means of the inequalities stated in them, we shall give a new characterization of a regular n-gon and a circle.

Theorem 2. *Let $\mathscr{A} = A_0, A_1, \ldots, A_{n-1}$ be a closed equilateral n-gon in $\mathbf{R}^N$ of length L, where $A_{n+k} = A_k$, $k = 0, 1, 2, \ldots$. Then for all $p = 0, 1, \ldots, n-1$*

$$\min_{\nu=0,1,\ldots,n-1} |A_\nu - A_{\nu+p}| \leqslant L\,\frac{\sin(p\frac{\pi}{n})}{n\sin\frac{\pi}{n}} \tag{4}$$

holds, with equality holding only if $\mathscr{A}$ is a regular n-gon.

P r o o f. From (1) and $\sum_{\nu=0}^{n-1} |A_\nu - A_{\nu+1}|^2 = L^2/n$ we have

$$\sum_{\nu=0}^{n-1} |A_{\nu+p} - A_\nu|^2 \leqslant L^2\,\frac{\sin^2(p\frac{\pi}{n})}{n\sin^2\frac{\pi}{n}}.$$

The Cauchy-Schwarz inequality implies

$$\sum_{\nu=0}^{n-1} |A_{\nu+p} - A_\nu| \leqslant L\,\frac{\sin(p\frac{\pi}{n})}{\sin\frac{\pi}{n}} \tag{5}$$

and

$$\min_{\nu=0,1,\ldots,n-1} |A_{\nu+p} - A_\nu| \leqslant \frac{1}{n}\sum_{\nu=0}^{n-1} |A_{\nu+p} - A_\nu| \leqslant L\,\frac{\sin(p\frac{\pi}{n})}{n\sin\frac{\pi}{n}}.$$

It is easily seen that equality in (4) holds only if $\mathscr{A}$ is a regular n-gon. □

R e m a r k . The requirement of equilaterality of an n-gon cannot be weakened as the example of a rectangle shows.

Now we will investigate the continuous case.

Theorem 3. *Let Γ be a closed rectifiable curve in $\mathbf{R}^N$ with length L, which is parametrized by $x = 2\pi s/L$, where s is the arclength. Then for all $t \in \mathbf{R}$*

$$\min_{x\in\langle 0,2\pi)} |\Gamma(x) - \Gamma(x+t)| \leqslant \frac{L}{\pi} \sin\frac{t}{2}, \tag{6}$$

where equality holds only if Γ is a circle.

Proof. The Cauchy-Schwarz inequality and (3) give

$$\int_0^{2\pi} |\Gamma(x) - \Gamma(x+t)| \, dx \leqslant 2L \sin\frac{t}{2}. \tag{7}$$

We get

$$\min_{x\in\langle 0,2\pi)} |\Gamma(x) - \Gamma(x+t)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |\Gamma(x) - \Gamma(x+t)| \, dx \leqslant \frac{L}{\pi} \sin\frac{t}{2}.$$

The case of equality follows immediately from (3). □

From the previous theorems we easily get further properties of a circle and a regular n-gon.

Theorem 4. *Let Γ be a closed rectifiable curve in $\mathbf{R}^N$ with length L. Let $x = y = 2\pi s/L$, where s is the arclength. Then*

$$\int_0^{2\pi}\int_0^{2\pi} |\Gamma(x) - \Gamma(y)| \, dx \, dy \leqslant 8L, \tag{8}$$

with equality attained only for a circle.

Proof. Integrating (7) we get (8). □

Theorem 5. *Let $\mathscr{A} = A_0, A_1, \ldots, A_{n-1}$ be a closed equilateral n-gon in $\mathbf{R}^N$ of length L. Then*

$$\sum_{k,j=0}^{n-1} |A_k - A_j| \leqslant \frac{1}{1-\cos\frac{\pi}{n}} L, \tag{9}$$

with equality attained only for a regular n-gon.

Proof. Summing the inequalities (5) for $p = 0, 1, \ldots, n-1$ and using the relation $\sum_{p=0}^{n-1} \sin(p\frac{\pi}{n}) = \operatorname{cotg}\frac{\pi}{2n}$ we get (9). □

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