Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 3, 549–576

Persistent URL: http://dml.cz/dmlcz/128342

Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

MEASURE-VALUED SOLUTIONS AND ASYMPTOTIC BEHAVIOR OF A MULTIPOLAR MODEL OF A BOUNDARY LAYER

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(Received May 17, 1991)

0. PREFACE

The physical theory of multipolar fluids was introduced in the paper [18] by Nečas and Šilhavý and follows general ideas by Green and Rivlin [5], [6]. The theory is compatible with the second law of thermodynamics as well as with the principle of material frame indifference.

As formulated in [18] and also in the survey paper [19], the theory takes into account both the linear and nonlinear dependences of the stress tensors on gradients of velocity. Nevertheless, the linear dependence was studied in more detail. We would like to mention a detailed discussion of bipolar compressible fluids in [18] or [19] as well as the series of papers by Nečas, Novotný, Šilhavý [15], [16], [17], or Nečas, Novotný [14] dealing with some qualitative properties of compressible multipolar fluids as global weak solvability, uniqueness and cavitation of density.

In the present paper, we focus our attention on the case of an incompressible fluid which has turned out to be very important in applications. It was shown in Bellout, Bloom, Nečas [3] that the multipolarity has only a minor perturbative effect to exhibit the flattening out phenomena predicted by the boundary layer theory and the imperfection in this particular case can be removed by considering a slightly nonlinear version of the constitutive relations, which is completely admissible within the context of the original formulation of the theory in [18] or [19].

Following [3], we choose from the very broad class of nonlinearities the simplest one, i.e. the case when the first viscosity coefficient depends on the invariants of the first spatial gradient of velocity. While in [3] some special cases as plane Poiseuille flow, Poiseuille flow in a pipe, plane Couette flow were studied analytically and discussed from the physical point of view, we concentrate our effort on the general formulation of the initial boundary value problem in two or three dimensions.

The paper is organized as follows. The first chapter is devoted to the formulation of the problem for bipolar fluids. In the following two chapters, we deal with the global existence of weak solutions, regularity and smoothing effect. In the fourth chapter, the measure-valued solutions are defined and the behavior of the weak solutions under the vanishing higher viscosity is studied. The limits of strong viscous solutions exhibit the loss of regularity and satisfy equations of motion for monopolar fluids in the sense of regular measures. These conclusions are proved in Chapter five. The sixth chapter concerns the uniqueness of weak solutions for bipolar fluids. In the last two chapters, we study the asymptotic behavior. The existence of the universal attractor with a finite Hausdorff dimension is proved.

1. Formulation of the problem

Let $\Omega \subset \mathbf{R}^N$, $N = 2$ or 3 be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. For $T > 0$, $t \in (0,T)$ we denote $Q_t = (0,t) \times \Omega$ and $I = (0,T)$. We will consider the unsteady flow of an incompressible (put $\varrho \equiv 1$), nonlinear, bipolar fluid. This model is described by the following system of equations:

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} + f_i \quad \text{in} \quad Q_T \quad \text{for} \quad i = 1, \dots, N, \tag{1.2}$$

where $\mathbf{v}$ is the velocity, $\mathbf{f}$—the specific external body force and (τ_{ij}) is the stress tensor.

Further, we will denote: p—the pressure, (τ_{ijk})—the bipolar stress tensor and $(e_{ij}(\mathbf{v}))$—the symmetric part of velocity tensor $(e_{ij}(\mathbf{v}) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}))$. All quantities are evaluated at (t, x).

The stress tensor τ_{ij}, provided it is symmetric, has the form

$$\tau_{ij} = -p\delta_{ij} + \beta e_{ij}(\mathbf{v}) - \mu \Delta e_{ij}(\mathbf{v}), \tag{1.3}$$

where β, μ can depend on the invariants of gradients of the velocity field with respect to the material frame. Especially, we will suppose in the sequel (cf. Bellout, Bloom, Nečas [3]) $\beta = \beta(\hat{v}^2, D(\mathbf{v}))$, $\mu = \text{const.} > 0$, where $\hat{v}^2 = e_{ij}(\mathbf{v})e_{ij}(\mathbf{v})$, $D(\mathbf{v}) = \det(e_{ij}(\mathbf{v}))_{i,j=1}^N$ are the invariants of the velocity tensor with respect to the principle of the material frame indifference (cf. Nečas, Šilhavý [18] or Novotný [19]). Let us mention that the dependence β on the third invariant does not appear in (1.3) due to the continuity equation (1.1). The constitutive equation for the bipolar stress tensor is considered in the form

$$\tau_{ijk} = \mu \frac{\partial e_{ij}(\mathbf{v})}{\partial x_k}. \tag{1.4}$$

We suppose in the sequel that

$$\beta \in \mathscr{C}^1(\mathbf{R}^2) \tag{1.5}$$

and

$$\mu_1 + \mu_1' \hat{v}^{2\bar{\gamma}} \leqslant \beta(\hat{v}^2, D(\mathbf{v})) \leqslant \mu_2 + \mu_2' \hat{v}^{2\gamma} \tag{1.6}$$

with $\mu_2 > \mu_1 > 0$, $\bar{\gamma}$, $\gamma \geqslant 0$, $\bar{\gamma} \leqslant \gamma \leqslant \bar{\gamma} + \frac{1}{2}$, $\mu_1' > 0$, $\mu_2' \geqslant 0$ being constants.

We will study the system of equations (1.1)–(1.3) with an initial condition

$$\mathbf{v}(0) = \mathbf{v}_0, \tag{1.7}$$

no-slip boundary condition

$$\mathbf{v} = 0 \quad \text{on} \quad I \times \partial\Omega \tag{1.8}$$

and unstable boundary conditions

$$\tau_{ijk}(\mathbf{v})\nu_j\nu_k = 0 \quad \text{on} \quad I \times \partial\Omega \tag{1.9}$$

expressing zero power of internal stresses on the boundary (cf. Novotný [19]). Here $\nu = (\nu_1, \dots, \nu_N)$ denotes the exterior normal of $\partial\Omega$.

Now we can give the weak formulation of our problem. First, put

$$V_2 = \left\{\mathbf{v};\ \mathbf{v} \in W_0^{1,2}(\Omega, \mathbf{R}^N) \cap W^{2,2}(\Omega, \mathbf{R}^N);\ \operatorname{div} \mathbf{v} = 0\right\}, \tag{1.10}$$

$$\mathscr{H} = \left\{\mathbf{v} \in \mathscr{C}_0^\infty(\Omega, \mathbf{R}^N);\ \operatorname{div} \mathbf{v} = 0\right\}, \tag{1.11}$$

$$H = \overline{\mathscr{H}}^{L^2(\Omega, \mathbf{R}^N)} \tag{1.12}$$

and (γ'' denotes $\frac{2\bar{\gamma}+2}{2\gamma+1}$)

$$\mathscr{V} = \begin{cases} V_2 \quad \text{if} \quad V_2 \subset W^{1,\gamma'}(\Omega, \mathbf{R}^N) & \left(\dfrac{1}{\gamma'} + \dfrac{1}{\gamma''} = 1\right) \quad \text{or} \\ V_2 \cap W^{1,\gamma'}(\Omega, \mathbf{R}^N) & \text{otherwise.} \end{cases} \tag{1.13}$$

Note that $\gamma' \geqslant 2$ because of (1.6).

Let us further denote

$$((\mathbf{v}, \omega)) = \mu\left(\frac{\partial e_{ij}(\mathbf{v})}{\partial x_k}, \frac{\partial e_{ij}(\omega)}{\partial x_k}\right) + \mu_1 \left(e_{ij}(\mathbf{v}), e_{ij}(\omega)\right), \tag{1.14}$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega, \mathbf{R}^N)$. Then $((\cdot,\cdot))$ is a scalar product on V_2 and $\|v\|_2 = ((v,v))^{\frac{1}{2}}$ is a V_2-norm. We denote $\tilde{\beta}(\hat{v}^2, D(\mathbf{v})) = \beta(\hat{v}^2, D(\mathbf{v})) - \mu_1$. The usual Bochner spaces are denoted by $L^p(I, V_2)$, $L^p(I, H)$ and $L^p(I, \mathscr{V})$, for $1 \leqslant p \leqslant \infty$. The norm $\|\cdot\|$ denotes the usual L_2-norm.

Definition 1.15. A function $\mathbf{v}$ is called a *weak solution* of (1.1)–(1.9) if and only if

$$\mathbf{v} \in L^2(I, V_2) \cap L^\infty(I, H), \tag{1.16}$$

$$\frac{\hat{\partial} \mathbf{v}}{\partial t} \in L^{\gamma''}(I, \mathcal{V}^*) \tag{1.17}$$

and (1.1)–(1.4), (1.7)–(1.9) are satisfied in the following sense:

$$\begin{aligned}\int_0^T \left\langle \frac{\hat{\partial} \mathbf{v}}{\partial t}, \varphi \right\rangle \mathrm{d}t + \iint_{Q_T} v_j \frac{\partial v_i}{\partial x_j} \varphi_i \,\mathrm{d}x\,\mathrm{d}t + \int_0^T ((\mathbf{v}, \varphi))\,\mathrm{d}t \\ + \iint_{Q_T} \tilde{\beta}(\hat{v}^2, D(\mathbf{v})) e_{ij}(\mathbf{v}) e_{ij}(\varphi)\,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T} f_i \varphi_i \,\mathrm{d}x\,\mathrm{d}t \\ \text{for every } \varphi \in L^{\gamma'}(I, \mathcal{V}),\end{aligned} \tag{1.18}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality in $\mathcal{V}$ and the functional $\frac{\hat{\partial} \mathbf{v}}{\partial t}$ is connected with $\mathbf{v}$ by the relation

$$\int_0^T \left\langle \frac{\hat{\partial} \mathbf{v}}{\partial t}, \varphi \right\rangle \mathrm{d}t = -\int_0^T \left\langle \mathbf{v}, \frac{\partial \varphi}{\partial t} \right\rangle \mathrm{d}t \quad \forall \varphi \in \mathscr{C}_0^\infty(I, \mathcal{V}).$$

We will need another weak formulation which does not involve the time derivative of $\mathbf{v}$:

$$\begin{aligned}-\iint_{Q_T} v_i \frac{\partial \omega_i}{\partial t} \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \mathbf{v}_0 \omega(0)\,\mathrm{d}x - \iint_{Q_T} v_i v_j \frac{\partial \omega_i}{\partial x_j} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T ((\mathbf{v}, \omega))\,\mathrm{d}t \\ + \iint_{Q_T} \tilde{\beta}(\hat{v}^2, D(\mathbf{v})) e_{ij}(\mathbf{v}) e_{ij}(\omega)\,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T} f_i \omega_i \,\mathrm{d}x\,\mathrm{d}t \\ \text{for every } \omega \in \mathscr{C}^\infty(\overline{Q}_T, \mathbf{R}^N), \omega(t, \cdot) \in W_0^{1,2}(\Omega, \mathbf{R}^N)\ \forall\, t \in \bar{I},\ \omega(T) = 0.\end{aligned} \tag{1.19}$$

If we consider a monopolar fluid, then (1.3) is replaced by

$$\tau_{ij} = -p\delta_{ij} + \beta(\hat{v}^2, D(\mathbf{v})) e_{ij}(\mathbf{v}) \tag{1.20}$$

and only the initial condition (1.7) and the stable boundary condition (1.8) are considered. Therefore, the appropriate weak formulation for the unsteady flow of a monopolar fluid reads

$$(1.21)\quad -\iint\limits_{Q_T} v_i \frac{\partial \omega_i}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \mathbf{v}_0\omega(0)\,\mathrm{d}x - \iint\limits_{Q_T} v_i v_j \frac{\partial \omega_i}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \iint\limits_{Q_T} \beta(\hat{v}^2, D(\mathbf{v}))e_{ij}(\mathbf{v})e_{ij}(\omega)\,\mathrm{d}x\,\mathrm{d}t = \iint\limits_{Q_T} f_i\omega_i\,\mathrm{d}x\,\mathrm{d}t$$

for every $\omega \in \mathscr{C}^\infty(\bar{Q}_T, \mathbf{R}^N), \omega(t,\cdot) \in W_0^{1,2}(\Omega, \mathbf{R}^N)\ \forall\, t \in \bar{I},\ \omega(T) = 0$.

2. Global weak solvability to the bipolar problem

2.1 Preliminaries. We define the operator A:

$$(2.1.1)\qquad (A\mathbf{v}, \omega) = ((\mathbf{v}, \omega)) \quad \forall \omega \in V_2$$

with the natural domain of definition
(2.1.2)

$$\mathscr{D}(A) = \{\mathbf{v} \in V_2;\ \text{there exists } \mathbf{y} \in H \text{ such that } \ (\mathbf{y}, \omega) = ((\mathbf{v}, \omega)) \quad \forall \omega \in V_2\}.$$

A is a self-adjoint positive operator on H, hence A^{-1} exists and due to the compact imbedding of V_2 into H it is compact. We can define powers A^s, $s \in \mathbf{R}$, being also self-adjoint operators with the domain of definition $\mathscr{D}(A^s)$. The linear space $V_{4s} := \mathscr{D}(A^s)$ is a Hilbert space equipped with the scalar product $(A^s\mathbf{v}, A^s\omega) + (\mathbf{v}, \omega)$. We denote $\|\mathbf{v}\|_{V_{4s}}^2 := (A^s\mathbf{v}, A^s\mathbf{v}) + (\mathbf{v}, \mathbf{v})$. It is clear that the norms in $\|\cdot\|_{V_{4s}}$ and $(A^s\cdot, A^s\cdot)^{\frac{1}{2}}$ are equivalent norms in V_{4s} provided $s > 0$.

The following properties will be useful in the sequel:

(2.1.3) $\quad A^{s_1}\mathbf{v} \in \mathscr{D}(A^{s-s_1})$ for every $\mathbf{v} \in \mathscr{D}(A^s)$ and $s > s_1 \geqslant 0$.

(2.1.4) $\quad$ There exists an orthonormal basis $\{\omega^j\} \subset V_2$ such that
- (i) $\omega^j \in \mathscr{C}^\infty(\bar{\Omega})\ \forall j = 1, 2, \ldots,$
- (ii) $((\omega^j, \omega)) = \lambda_j(\omega^j, \omega) \forall \omega \in V_2,$
- (iii) $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \lambda_j \to \infty$ if $j \to \infty$.

(2.1.5) $\quad$ Let us put $P_n\mathbf{v} = \sum\limits_{j=1}^{n} \lambda_j(\omega^j, \mathbf{v})\omega^j$ and $Q_n := I - P_n$.
If $(H)_n := \operatorname{span}\{\omega^1, \omega^2, \ldots, \omega^n\}$ in $L^2(\Omega, \mathbf{R}^N)$ and
$(V_2)_n := \operatorname{span}\{\omega^1, \omega^2, \ldots, \omega^n\}$ in V_2, then P_n is an ortho-

gonal projector of H onto $(H)_n$ and of V_2 onto V_{2n}.

(2.1.6) $P_n A\mathbf{v} = AP_n\mathbf{v}$ and also $Q_n A\mathbf{v} = AQ_n\mathbf{v}$.

(2.1.7) $(A^s\mathbf{v},\mathbf{v}) \geqslant \lambda_1^{s_1}(A^{s-s_1}\mathbf{v},\mathbf{v})$ for every $\mathbf{v} \in V_2$ and $s \geqslant s_1 > 0$.

(2.1.8) $(A^s Q_n\mathbf{v}, Q_n\mathbf{v}) \geqslant \lambda_{n+1}^{s_1}(A^{\frac{s-s_1}{2}}Q_n\mathbf{v}, A^{\frac{s-s_1}{2}}Q_n\mathbf{v})$
for every $\mathbf{v} \in V_2$ and $s \geqslant s_1 > 0$.

(2.1.9) For $\mathbf{v} \in H$ we define $A^s\mathbf{v} \in \mathscr{D}(A^{-s})$ by $\langle A^s\mathbf{v},\varphi\rangle = (\mathbf{v}, A^s\varphi)$ for every $\varphi \in \mathscr{D}(A^s)$, where $\langle\cdot,\cdot\rangle$ denotes duality in $\mathscr{D}(A^s)$.

(2.1.10) For $\mathbf{v} \in L^2(I,\mathscr{D}(A^s))$, $\frac{\partial\mathbf{v}}{\partial t} \in L^2(I,H)$ and $\varphi \in \mathscr{C}_0^\infty(Q_T,\mathbf{R}^N)$, we have $\langle\langle A^s\frac{\partial\mathbf{v}}{\partial t},\varphi\rangle\rangle = \int_0^T(\frac{\partial A^s\mathbf{v}}{\partial t},\varphi)\,\mathrm{d}t$, where $\langle\langle\cdot,\cdot\rangle\rangle$ denotes duality in $L^2(I,\mathscr{D}(A^s))$.

Let us note that for $0 \leqslant s \leqslant \frac{1}{2}$

(2.1.11) $$V_{4s} = \overline{V_2},$$

where the closure is taken in $W^{4s,2}$, hence the norms $\|\cdot\|_{W^{4s,2}}$ and $\|\cdot\|_{V_{4s}}$ are equivalent. Consequently, the norms $\|\mathbf{v}\|_{W^{s,2}}$ and $\|A^{s/4}\mathbf{v}\|$,$s \geqslant 0$ are equivalent on $\mathscr{D}(A^{s/4})$. The proofs and a more precise discussion of these facts can be found in Lions, Magenes [12].

Let $p > 1$. Put

(2.1.12) $$\|\mathbf{v}\|_{V_{1,p}} = \left(\int_\Omega |e_{ij}(\mathbf{v})|^p\right)^{\frac{1}{p}},$$

where $V_{1,p} = \left\{\mathbf{v};\,\mathbf{v} \in W_0^{1,p},\,\operatorname{div}\mathbf{v} = 0\right\}$. Then $\|\cdot\|_{V_{1,p}}$ and $\|\cdot\|_{W^{1,p}}$ are equivalent (cf. Nečas [13]).

2.2 Existence of Approximations. We choose an orthonormal basis $\{\omega^j\}_{j=1}^\infty$ in H (see (2.1.14)). In order to get appropriate approximations of (1.18) or (1.19) we use the Faedo-Galerkin method. Put

(2.2.1) $$\mathbf{v}^n(t,x) = \sum_{j=1}^n \gamma_j^n(t)\omega^j(x).$$

Then

(2.2.2) $$\Gamma^n = (\gamma_1^n,\gamma_2^n,\ldots,\gamma_n^n) \in \mathscr{C}^1(\bar{I},\mathbf{R}^n)$$

is a solution to the system of ordinary differential equations ($r = 1, 2, \dots, n$)

$$(2.2.3)\qquad \left(\frac{\mathrm{d}\mathbf{v}^n}{\mathrm{d}t}, \omega^r\right) + \int_\Omega v_j^n \frac{\partial v_i{}^n}{\partial x_j} \omega_i^r \,\mathrm{d}x + ((\mathbf{v}^n, \omega^r)) + \int_\Omega \tilde{\beta}((\hat{v}^n)^2, D(\mathbf{v}^n)) e_{ij}(\mathbf{v}^n) e_{ij}(\omega^r) \,\mathrm{d}x = \int_\Omega f_i \omega_i^r \,\mathrm{d}x$$

with the initial condition

$$(2.2.4)\qquad \mathbf{v}^n(0) = P_n \mathbf{v}_0.$$

As usual, this system has a solution (2.2.2).

Now, we prove appropriate apriori estimates.

Lemma 2.2.5. *Let the assumptions* (1.5), (1.6) *be satisfied and let*

$$(2.2.6)\qquad \mathbf{v}_0 \in L^2(\Omega) \quad \textit{and} \quad \mathbf{f} \in L^2(I, V_2^*).$$

Then

$$(2.2.7)\ \|\mathbf{v}^n\|^2_{L^\infty(I,H)} + \|\mathbf{v}^n\|^2_{L^2(I,V_2)} + \iint\limits_{Q_T} \tilde{\beta}((\hat{v}^n)^2, D(\mathbf{v}^n)) e_{ij}(\mathbf{v}^n) e_{ij}(\mathbf{v}^n) \,\mathrm{d}x \,\mathrm{d}t \leqslant c\left(\|\mathbf{f}\|^2_{L^2(I,\mathcal{V}^*)} + \|\mathbf{v}_0^n\|^2_{L^2(\Omega)}\right) \leqslant \text{const.},$$

$$(2.2.8)\qquad \|\mathbf{v}^n\|_{L^{2(\bar{\gamma}+1)}(I, W^{1,2(\bar{\gamma}+1)})} \leqslant \text{const.}$$

Proof. The estimates (2.2.7) can be derived by multiplying the equation (2.2.3) by γ_r^n and adding the resulting equalities. We get

$$(2.2.9)\quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathbf{v}^n\|^2 + ((\mathbf{v}^n, \mathbf{v}^n)) + \int_\Omega \tilde{\beta}((\hat{v}^n)^2, D(\mathbf{v}^n)) e_{ij}(\mathbf{v}^n) e_{ij}(\mathbf{v}^n) \,\mathrm{d}x = (\mathbf{f}, \mathbf{v}^n).$$

After integrating the last equation over $(0, T)$, one gets (2.2.7); (2.2.8) follows from (2.2.7) and (1.6). □

Lemma 2.2.10. *Under the assumptions of Lemma* 2.2.5

$$(2.2.11)\qquad \left\|\frac{\partial \mathbf{v}^n}{\partial t}\right\|_{L^{\gamma''}(I,\mathcal{V}^*)} \leqslant \text{const.},$$

where $\mathcal{V}^*$ is the dual space to $\mathcal{V}$ (see (1.13)*).*

Proof follows directly from (2.2.7) and (2.2.8). □

Lemma 2.2.12. *Let*

$$\mathbf{v}_0 \in \mathscr{D}(A^{1/4}) \quad \textit{and} \quad \mathbf{f} \in L^2(I, \mathscr{D}(A^{-1/4})) \tag{2.2.13}$$

and let

$$\kappa_1 \leqslant \tilde{\beta}(\hat{v}^2, D(\mathbf{v})) \leqslant \kappa_2 \quad (\textit{it means that } \gamma = \bar{\gamma} = 0 \textit{ in } (1.6)). \tag{2.2.14}$$

Then

$$\begin{aligned} &\|\mathbf{v}^n\|_{L^\infty(I,\mathscr{D}(A^{1/4}))} \leqslant \text{const.}, \\ &\|\mathbf{v}^n\|_{L^2(I,\mathscr{D}(A^{3/4}))} \leqslant \text{const.} \end{aligned} \tag{2.2.16}$$

and

$$\left\| \frac{\partial \mathbf{v}^n}{\partial t} \right\|_{L^2(I,\mathscr{D}(A^{-1/4}))} \leqslant \text{const.}\,. \tag{2.2.17}$$

Proof. We multiply (2.2.3) by $\lambda_r^{1/2}\gamma_r^n$, add these equations for $r = 1, 2, \dots, n$, which gives

$$\begin{aligned} (\frac{\partial \mathbf{v}^n}{\partial t}, A^{1/2}\mathbf{v}^n) + ((\mathbf{v}^n, A^{1/2}\mathbf{v}^n)) &= -\int_\Omega v_j^n \frac{\partial v_i^n}{\partial x_j}(A^{1/2}\mathbf{v}^n)_i \mathrm{d}x \\ &- \int_\Omega \tilde{\beta}((\hat{v}^n)^2, D(\mathbf{v}^n)) e_{ij}(\mathbf{v}^n) e_{ij}(A^{1/2}\mathbf{v}^n)\, \mathrm{d}x + \int_\Omega f_i (A^{1/2}\mathbf{v}^n)_i\, \mathrm{d}x. \end{aligned} \tag{2.2.18}$$

Clearly we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\frac{1}{2}\|A^{1/4}\mathbf{v}^n\|^2 + \|A^{3/4}\mathbf{v}^n\|^2 &\leqslant \left| \int_\Omega v_j^n v_i^n \frac{\partial (A^{1/2}\mathbf{v}^n)_i}{\partial x_j}\, \mathrm{d}x \right| \\ &+ \kappa_2 \int_\Omega |e_{ij}(\mathbf{v}^n) e_{ij}(A^{1/2}\mathbf{v}^n)|\, \mathrm{d}x + \int_\Omega |f_i (A^{1/2}\mathbf{v}^n)_i|\, \mathrm{d}x. \end{aligned} \tag{2.2.19}$$

Let us denote the integrals on the right-hand side of (2.2.19) by Y_1, Y_2, Y_3, respectively, and estimate them separately. We use the Hölder and Young inequalities. Then

$$\begin{aligned} Y_1 &\leqslant c\|A^{1/4}\mathbf{v}^n\|\,\|A^{1/2}\mathbf{v}^n\|\,\|A^{3/4}\mathbf{v}^n\| \\ &\leqslant \varepsilon\|A^{3/4}\mathbf{v}^n\|^2 + K_1(\varepsilon)\|A^{1/4}\mathbf{v}^n\|^2\|A^{1/2}\mathbf{v}^n\|^2, \end{aligned} \tag{2.2.20}$$

$$Y_2 \leqslant \kappa_2\|A^{3/4}\mathbf{v}^n\|\,\|A^{1/4}\mathbf{v}^n\| \leqslant \varepsilon\|A^{3/4}\mathbf{v}^n\|^2 + K_2(\varepsilon)\|A^{1/4}\mathbf{v}^n\|^2 \tag{2.2.21}$$

and

$$Y_3 \leqslant \|A^{3/4}\mathbf{v}^n\| \, \|A^{-1/4}\mathbf{f}\| \leqslant \varepsilon \|A^{3/4}\mathbf{v}^n\|^2 + K_3(\varepsilon)\|A^{-1/4}\mathbf{f}\|^2. \tag{2.2.22}$$

By integrating (2.2.19) over $(0,t)$ and using (2.2.7),(2.2.20)–(2.2.22) and the Gronwall lemma we obtain (2.2.16). The assertion (2.2.17) follows from (2.2.16) and from the definition of the norm $L^2(0,T,\mathscr{D}(A^{-1/4}))$. □

Let us denote $I_\delta = (\delta, T)$ for $\delta \in (0,T\rangle$ and $Q_{\delta,T} = I_\delta \times \Omega$. The next lemma deals with the smoothing property.

Lemma 2.2.23. *Let*

$$\mathbf{v}_0 \in \mathscr{D}(A^{1/4}), \quad \mathbf{f} \in L^2(Q_T). \tag{2.2.24}$$

Let $\tilde{\beta}$ depend only on $\hat{v}^2$, and let (2.2.14) *hold.*
Then for every $\delta > 0$

$$\|\mathbf{v}^n\|_{L^\infty(I_\delta,\mathscr{D}(A^{1/2}))} \leqslant \text{const.}, \tag{2.2.25}$$

$$\left\| \frac{\partial \mathbf{v}^n}{\partial t} \right\|_{L^2(Q_{\delta,T})} \leqslant \text{const.} \tag{2.2.26}$$

Moreover, let

$$|\tilde{\beta}'(\hat{v}^2)| \leqslant c_1, \tag{2.2.27}$$

where c_1 is a positive constant. Then

$$\|\mathbf{v}^n\|_{L^2(I_\delta,\mathscr{D}(A))} \leqslant \text{const.} \tag{2.2.28}$$

Proof. Let $\xi \in \mathscr{C}^\infty(\langle 0,T))$, $\xi(t) = 0$ for $t \in \langle 0, \frac{\delta}{2}\rangle$, $\xi(t) = 1$ for $t \in \langle \delta, T)$. Let us multiply the r-th equation in (2.2.3) by $\dot{\gamma}_r^n(t)\xi^2(t)$ and add the resulting equalities. We get

$$\begin{aligned} \frac{1}{2}\left\|\xi \frac{\partial \mathbf{v}^n}{\partial t}\right\|^2 &+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\xi A^{1/2}\mathbf{v}^n\right\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \xi^2 \tilde{B}(\hat{v}^n)^2)\,\mathrm{d}x \\ &= \xi\frac{\mathrm{d}\xi}{\mathrm{d}t}\|A^{1/2}\mathbf{v}^n\|^2 + \int_\Omega \xi^2 f_i \frac{\partial v_i^n}{\partial t}\,\mathrm{d}x \\ &\quad - \int_\Omega \xi^2 v_j^n \frac{\partial v_i^n}{\partial x_j}\frac{\partial v_i^n}{\partial t}\,\mathrm{d}x + \int_\Omega \xi \frac{\mathrm{d}\xi}{\mathrm{d}t}\tilde{B}(\hat{v}^n)^2)\,\mathrm{d}x \end{aligned} \tag{2.2.29}$$

where $\tilde{B}(s) = \int\limits_0^s \tilde{\beta}(\sigma)\,\mathrm{d}\sigma$.

The terms denoted by $Y_1, \ldots, Y_4$ on the right-hand side of (2.2.29) can be estimated by

$$|Y_1| \leqslant K_1 \|A^{3/4}\mathbf{v}^n\|^2, \tag{2.2.30}$$

$$|Y_2| \leqslant \left\|\xi \frac{\partial \mathbf{v}^n}{\partial t}\right\| \|\xi \mathbf{f}\| \leqslant \varepsilon \left\|\xi \frac{\partial \mathbf{v}^n}{\partial t}\right\|^2 + K_2(\varepsilon)\|\mathbf{f}\|^2, \tag{2.2.31}$$

$$\begin{aligned} |Y_3| &\leqslant \left\|\xi \frac{\partial \mathbf{v}^n}{\partial t}\right\| \|A^{1/4}\mathbf{v}^n\| \, \|A^{3/4}\mathbf{v}^n\| \\ &\leqslant \varepsilon \left\|\xi \frac{\partial \mathbf{v}^n}{\partial t}\right\|^2 + K_3(\varepsilon)\|A^{1/4}\mathbf{v}^n\|^2 \, \|A^{3/4}\mathbf{v}^n\|^2, \end{aligned} \tag{2.2.32}$$

$$|Y_4| \leqslant K_4 \int_\Omega |e_{ij}(\mathbf{v}^n)e_{ij}(\mathbf{v}^n)|\,\mathrm{d}x \leqslant K_5\|A^{1/4}\mathbf{v}^n\|^2. \tag{2.2.33}$$

Thus the estimates (2.2.25) and (2.2.26) follow immediately from Lemma 2.2.12. Now, we multiply (2.2.3) by $\lambda_r \xi^2(t)\gamma_r^n(t)$ and again add these relations. We get

$$\begin{aligned} &\left(\frac{\partial \mathbf{v}^n}{\partial t}, \xi^2 A\mathbf{v}^n\right) + ((\mathbf{v}^n, \xi^2 A\mathbf{v}^n)) + \int_\Omega \xi^2 \tilde{\beta}((\hat{v}^n)^2)e_{ij}(\mathbf{v}^n)e_{ij}(A\mathbf{v}^n)\,\mathrm{d}x \\ &\qquad = -\int_\Omega \xi^2 v_j^n \frac{\partial v_i^n}{\partial x_j}(A\mathbf{v}^n)_i\,\mathrm{d}x + \int_\Omega f_i(A\mathbf{v}^n)_i\,\mathrm{d}x. \end{aligned} \tag{2.2.34}$$

The last equation can be rewritten as

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi A^{1/2}\mathbf{v}^n\|^2 + \|\xi A\mathbf{v}^n\|^2 \\ &\quad = \int_\Omega \xi \frac{\mathrm{d}\xi}{\mathrm{d}t}(A^{1/2}\mathbf{v}^n)_i(A^{1/2}\mathbf{v}^n)_i\,\mathrm{d}x - \int_\Omega \xi^2 v_j^n \frac{\partial v_i^n}{\partial x_j}(A\mathbf{v}^n)_i\,\mathrm{d}x \\ &\qquad + \int_\Omega f_i(A\mathbf{v}^n)_i\,\mathrm{d}x - \xi^2 \int_\Omega \frac{\partial}{\partial x_j}(\tilde{\beta}((\hat{v}^n)^2)e_{ij}(\mathbf{v}^n))(A\mathbf{v}^n)_i\,\mathrm{d}x. \end{aligned} \tag{2.2.35}$$

The right-hand side of (2.2.35) can be estimated by

$$\begin{aligned} \int_\Omega \xi^2 v_j^n \frac{\partial v_i^n}{\partial x_j}(A\mathbf{v}^n)_i\,\mathrm{d}x &\leqslant \|\xi A\mathbf{v}^n\| \, \|A^{1/4}\mathbf{v}^n\| \, \|A^{3/4}\mathbf{v}^n\| \\ &\leqslant \varepsilon\|\xi A\mathbf{v}^n\|^2 + K_1(\varepsilon)\|A^{3/4}\mathbf{v}^n\|^2 \, \|A^{1/4}\mathbf{v}^n\|^2 \end{aligned} \tag{2.2.36}$$

and

$$\begin{aligned} &\left|\xi^2 \int_\Omega \frac{\partial}{\partial x_j}(\tilde{\beta}((\hat{v}^n)^2)e_{ij}(\mathbf{v}^n))(A\mathbf{v}^n)_i\,\mathrm{d}x\right| \\ &\qquad \leqslant 2\left|\int_\Omega \xi^2 \tilde{\beta}'((\hat{v}^n)^2)\frac{e_{rs}(\mathbf{v}^n)}{\hat{v}^{2n}}\frac{\partial e_{rs}(\mathbf{v}^n)}{\partial x_j}e_{ij}(\mathbf{v}^n)(A\mathbf{v}^n)_i\,\mathrm{d}x\right| \\ &\qquad\quad + \left|\xi^2 \int_\Omega \tilde{\beta}((\hat{v}^n)^2)\frac{\partial e_{ij}(\mathbf{v}^n)}{\partial x_j}(A\mathbf{v}^n)_i\,\mathrm{d}x\right| =: Y_5 + Y_6 \end{aligned} \tag{2.2.37}$$

where
(2.2.38)
$$Y_5 \leqslant c\|\xi A\mathbf{v}^n\|\,\|A^{3/4}\mathbf{v}^n\|\,\|A^{1/2}\mathbf{v}^n\|^2 \leqslant \varepsilon\|\xi A\mathbf{v}^n\|^2 + K_2(\varepsilon)\|A^{1/2}\mathbf{v}^n\|^4\,\|A^{3/4}\mathbf{v}^n\|^2,$$

$$Y_6 \leqslant K\|\xi A\mathbf{v}^n\|\,\|A^{1/2}\mathbf{v}^n\| \leqslant \varepsilon\|\xi A\mathbf{v}^n\|^2 + K_3(\varepsilon)\|A^{1/2}\mathbf{v}^n\|^2. \tag{2.2.39}$$

The proof of the estimate (2.2.28) is complete. □

3. Existence theorems and smoothing effect

Theorem 3.1. *Let the assumptions of Lemma 2.2.5 be satisfied. Then there exists a weak solution* $\mathbf{v}$ *of* (1.1)–(1.9) *(see* (1.18)*) such that*

$$\mathbf{v} \in L^2(I, V_2) \cap L^{2(\bar{\gamma}+1)}(I, W^{1,2(\bar{\gamma}+1)}) \cap L^\infty(I, H), \tag{3.2}$$

$$\frac{\partial \mathbf{v}}{\partial t} \in L^{\gamma''}(I, \mathscr{V}^*). \tag{3.3}$$

Moreover, if $\gamma = \bar{\gamma} = 0$ *(in condition* (1.6)*) then*

$$\mathbf{v} \in \mathscr{C}_0(I, H). \tag{3.4}$$

Theorem 3.5. *Let the assumptions of Lemma 2.2.12 be satisfied. Then there exists a weak solution* $\mathbf{v}$,

$$\mathbf{v} \in \mathscr{C}_0(I, \mathscr{D}(A^{1/4})) \cap L^2(I, \mathscr{D}(A^{3/4})), \tag{3.6}$$

$$\frac{\partial \mathbf{v}}{\partial t} \in L^2(I, \mathscr{D}(A^{-1/4})) \tag{3.7}$$

such that (1.18) *holds.*

Theorem 3.8 (smoothing effect). *Let the assumptions of Lemma 2.2.8 be satisfied. Then the weak solution* $\mathbf{v}$ *(see Theorem 3.5) is such that for every* $\delta > 0$

$$\frac{\partial \mathbf{v}}{\partial t} \in L^2(Q_{\delta,T}), \tag{3.9}$$

$$\mathbf{v} \in L^2(I_\delta, \mathscr{D}(A)), \tag{3.10}$$

$$\mathbf{v} \in \mathscr{C}_0(I_\delta, \mathscr{D}(\mathscr{A}^{1/2})). \tag{3.11}$$

Proof (of Theorems 3.1, 3.5, 3.8) . From (2.2.7), (2.2.8) and the well-known Aubin lemma (cf. Lions [11] or Simon [20]) about three Banach spaces concerning the compact imbedding of the space

$$\left\{\mathbf{v} \in L^p(I, B_0)\,;\ \frac{\partial \mathbf{v}}{\partial t} \in L^q(I, B_1)\right\} \quad \text{into} \quad L^p(I, B),$$

we have

$$\mathbf{v}^n \to \mathbf{v} \quad *\text{-weakly in} \quad L^\infty(I, H), \tag{3.12}$$

$$\mathbf{v}^n \to \mathbf{v} \quad \text{weakly in} \quad L^2(I, V_2), \tag{3.13}$$

$$\frac{\partial \mathbf{v}^n}{\partial t} \to \frac{\partial \mathbf{v}}{\partial t} \quad *\text{-weakly in} \quad L^{\gamma''}(I, \mathcal{V}^*), \tag{3.14}$$

$$\mathbf{v}^n \to \mathbf{v} \quad \text{strongly in} \quad L^2(I, W^{s,2}),\ s < 2, \tag{3.15}$$

where (3.12)–(3.15) hold at least for a suitable subsequence.

By virtue of (3.15)

$$e_{ij}(\mathbf{v}^n) \to e_{ij}(\mathbf{v}) \quad \text{a.e. in} \quad Q_T, \tag{3.16}$$

hence

$$\tilde{\beta}((\hat{v}^n)^2, D(\mathbf{v}^n))e_{ij}(\mathbf{v}^n) \to \tilde{\beta}(\hat{v}^2, D(\mathbf{v}))e_{ij}(\mathbf{v}) \quad \text{a.e. in} \quad Q_T \tag{3.17}$$

and because of

$$\|\tilde{\beta}((\hat{v}^n)^2, D(\mathbf{v}^n))e_{ij}(\mathbf{v}^n)\|_{L^{\frac{2\tilde{\gamma}+2}{2\gamma+1}}(Q_T)} \leqslant \text{const.} \tag{3.18}$$

we have

$$\tilde{\beta}((\hat{v}^n)^2, D(\mathbf{v}^n))e_{ij}(\mathbf{v}^n) \to \tilde{\beta}(\hat{v}^2, D(\mathbf{v}))e_{ij}(\mathbf{v}) \quad \text{weakly in} \quad L^{\frac{2\tilde{\gamma}+2}{2\gamma+1}}(Q_T), \tag{3.19}$$

which implies convergence of the last term on the right side in the weak formulation (1.18).

Similarly, we prove 3.5 to be a consequence of 2.2.12 and 3.8 to be a consequence of 2.2.23. □

In Chapter 3 we have proved the existence of weak solutions to the problem (1.1)–(1.9). Nevertheless, for monopolar fluids described by (1.21) in many situations, such a question becomes an outstanding open problem.[2] Therefore an appropriate generalization of weak solutions is useful.

We will consider the space $L^2(Q_T, \mathscr{C}_0(\mathbf{R}^{N^2}))$, where $\mathscr{C}_0(\mathbf{R}^{N^2})$ is the Banach space of continuous functions $f\colon \mathbf{R}^{N^2} \longmapsto \mathbf{R}$ satisfying $\lim\limits_{|\lambda|\to\infty} f(\lambda) = 0$. A well known form of the Riesz representation theorem (cf. Hewitt, Stromberg [7], Edwards [4]) states that

$$L^2(Q_T, \mathscr{C}_0(\mathbf{R}^{N^2}))^* = L^2_{(w)}(Q_T, M(\mathbf{R}^{N^2})), \tag{4.1}$$

where $M(\mathbf{R}^{N^2})$ is a Banach space of bounded Radon measures. Let $\langle\cdot,\cdot\rangle$ denote duality in $\mathscr{C}_0(\mathbf{R}^{N^2})$. Then $\nu \in L^2_{(w)}(Q_T, M(\mathbf{R}^{N^2}))$ if and only if ν is a *-weak measurable mapping[3] $\nu\colon Q_T \longmapsto M(\mathbf{R}^{N^2})$ such that

$$\|\nu\|_{L^2_{(w)}(Q_T, M(\mathbf{R}^{N^2}))} \equiv \operatorname*{ess\,sup}_{(t,x)\in Q_T} \|\nu_{(t,x)}\|_{M(\mathbf{R}^{N^2})}\,\mathrm{d}x\,\mathrm{d}t < \infty \tag{4.2}$$

with

$$\begin{aligned} \|\nu_{(t,x)}\|_{M(\mathbf{R}^{N^2})} &= \sup_{\substack{\varphi\in\mathscr{C}_0(\mathbf{R}^{N^2})\\ \|\varphi\|_{\mathscr{C}_0(\mathbf{R}^{N^2})}\leqslant 1}} \int_{\mathbf{R}^{N^2}} \varphi\,\mathrm{d}\nu_{(t,x)} \\ &= \sup_{\substack{\varphi\in\mathscr{C}_0(\mathbf{R}^{N^2})\\ \|\varphi\|_{\mathscr{C}_0(\mathbf{R}^{N^2})}\leqslant 1}} \langle \nu(t,x), \varphi\rangle . \end{aligned} \tag{4.3}$$

We denote by $b\colon \mathbf{R}^{N^2} \longmapsto R^{N^2}$, $\xi\colon R^{N^2} \longmapsto R^{N^2}$ mappings which are defined as follows:

$$b(\sigma) = (b_{ij}(\sigma))_{i,j=1}^N = (\beta(\sigma^2, D(\sigma))e_{ij}(\sigma))_{i,j=1}^N \quad\text{and}\quad \xi(\sigma) = \sigma .$$

Definition 4.4 (measure-valued solution). A couple $(\mathbf{v}, \nu)$ is said to be a *measure-valued solution to the initial boundary value problem* (1.1), (1.2), (1.7), (1.8), (1.20) if and only if

$$\mathbf{v} \in L^\infty(I, H) \cap L^2(I, V_1) \cap L^{2(\bar\gamma+1)}(I, W^{1,2(\bar\gamma+1)}), \tag{4.5}$$

[2] For some special cases this problem was solved in the framework of weak solutions by Ladyženskaya [9] (cf. Remark 5.32), other results can be found in Lions [11], §2, Sect. 5.

[3] ν is a *-weak measurable mapping if and only if $\langle \nu_{(t,x)}, \varphi\rangle$ is Lebesgue measurable for every $\varphi \in \mathscr{C}_0(\mathbf{R}^{N^2})$.

where $V_1 = \left\{\mathbf{v};\ \mathbf{v}(t) \in W_0^{1,2};\ \operatorname{div}\mathbf{v} = 0\right\}$;

$$\nu \in L^2_{(w)}(Q_T, M(\mathbf{R}^{N^2})), \tag{4.6}$$

$$\|\nu_{(t,x)}\|_{M(\mathbf{R}^{N^2})} = 1 \quad \text{a.e. in } Q_T, \tag{4.7}$$

(4.8) $\quad b_{ij}(\sigma), \xi_{ij}(\sigma)$ are ν-integrable functions on $\mathbf{R}^{N^2}$ for $i,j = 1,2,\ldots,N$

and

$$\int\limits_{\mathbf{R}^{N^2}} \sigma \mathrm{d}\nu_{(t,x)}(\sigma) = \nabla\mathbf{v}(t,x) \text{ a.e. in } Q_T, \tag{4.9}$$

$$\begin{aligned} -\iint\limits_{Q_T} v_i \frac{\partial\varphi_i}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \mathbf{v}_0\varphi(0)\,\mathrm{d}x - \iint\limits_{Q_T} v_i v_j \frac{\partial\varphi_i}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t \\ + \iint\limits_{Q_T} \frac{\partial\varphi_i}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t \int\limits_{\mathbf{R}^{N^2}} \beta(\hat{\sigma}^2, D(\sigma))e_{ij}(\sigma)\mathrm{d}\nu_{(t,x)} = \iint\limits_{Q_T} f_i\varphi_i\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.10}$$

for every $\varphi \in L^2(I, W^{1,\gamma'}(\Omega, \mathbf{R}^N))$, $\frac{\partial\varphi}{\partial t} \in L^{\gamma'}(Q_T, \mathbf{R}^N)$ for every $t \in \bar{I}$, $\varphi(T) = 0$ (for γ' see (1.13)).

Remark 4.11. Let $\mathbf{v}$ be a weak solution to a monopolar fluid, i.e. $\mathbf{v} \in L^\infty(I,H) \cap L^2(I,V_1) \cap L^{2(\bar{\gamma}+1)}(I, W^{1,2(\bar{\gamma}+1)})$ and (1.21) is satisfied. Then $(\mathbf{v},\nu)$, where $\nu_{(t,x)}$ is defined by $\nu_{(t,x)} = \delta(\sigma_{ij} - \frac{\partial v_i}{\partial x_j}(t,x))$ a.e. in Q_T, is a measure-valued solution.

On the other hand, if a couple $(\mathbf{v},\nu)$, where $\mathbf{v} \in L^\infty(I,H) \cap L^2(I,V_1) \cap L^{2(\bar{\gamma}+1)}(I, W^{1,2(\bar{\gamma}+1)})$ and $\nu_{(t,x)} = \delta(\sigma - \nabla\mathbf{v}(t,x))$, satisfies (4.8)–(4.10), then $\mathbf{v}$ satisfies (1.21).

Now, we are ready to prove the following theorems.

Theorem 4.12. *Let the assumptions of Lemma* 2.2.5 *be satisfied. Then in the class* (4.5)–(4.7) *there exists at least one measure-valued solution* $(\mathbf{v},\nu)$ *satisfying* (4.8)–(4.10).

Theorem 4.13. *Let the assumptions of Theorem* 4.12 *be satisfied and let* $\mathbf{v}^\mu$ *be a weak solution of* (1.18) *in the class of solutions* (3.2), (3.3) *(see Theorem* 3.1*) corresponding to* μ. *Put*

$$\nu^\mu_{(t,x)} = \delta(\sigma - \nabla\mathbf{v}^\mu(t,x)). \tag{4.14}$$

Then there exists a subsequence of $(\mathbf{v}^\mu, \nu^\mu)$ *such that*

$$(4.15) \qquad \mathbf{v}^\mu \to \mathbf{v} \text{ weakly in } L^2(I, V_1),$$

$$(4.16) \qquad \mathbf{v}^\mu \to \mathbf{v} \text{ *-weakly in } L^\infty(I, H),$$

$$(4.17) \qquad \nu^\mu \to \nu \text{ *-weakly in } L^2_{(w)}(Q_T, M(\mathbf{R}^{N^2})),$$

$$(4.18) \qquad \mathbf{v}^\mu \to \mathbf{v} \text{ strongly in } L^2(Q_T),$$

$$(4.19) \qquad \mu\left(\frac{\partial}{\partial x_k} e_{ij}(\mathbf{v}^\mu), \frac{\partial}{\partial x_k} e_{ij}(\varphi)\right) \to 0 \text{ for every } \varphi \in \mathscr{C}^\infty(\bar{Q}_T),$$

and $(\mathbf{v}, \nu)$ *is a measure-valued solution to the monopolar fluid.*

5. Proof of the theorems

First, recall one fundamental theorem about Young measures due to Ball [2].

Theorem 5.1. *Let* $Q \subset \mathbf{R}^m$ *be a Lebesgue measurable set and let* $z^j : Q \to \mathbf{R}^M$, $j = 1, 2, \ldots$, *be a sequence of Lebesgue measurable functions. Then there exist a measure* ν *and a subsequence* $\{z^s\} \subset \{z^j\}$ *such that*

$$(5.2) \qquad \nu \in L^\infty_{(w)}(Q_T, M(\mathbf{R}^M)),$$

$$(5.3) \qquad \|\nu_y\|_{M(\mathbf{R}^M)} \leqslant 1 \text{ for a.e. } y \in Q,$$

$$(5.4) \qquad \varphi(z^s) \to \langle \nu_y, \varphi \rangle = \int_{\mathbf{R}^M} \varphi(\sigma)\, \mathrm{d}\nu_y(\sigma) \text{ *-weakly in } L^\infty(Q)$$

for every $\varphi \in \mathscr{C}_0(\mathbf{R}^M)$ *and* $s \to \infty$. *Moreover, if*

$$(5.5) \qquad \lim_{k\to\infty} \sup_{s=1,2,\ldots} \operatorname{meas}\{x \in Q \cap B_R;\ |z^s(x)| \geqslant k\} = 0$$

for every $R > 0$, *where* $B_R = \{y \in Q;\ |y| \leqslant R\}$, *then*

$$(5.6) \qquad \|\nu_y\|_{M(\mathbf{R}^M)} = 1 \text{ for a.e. } y \in Q.$$

Further, let the condition (5.5) hold and let

$$(5.7) \qquad \sup_{s=1,2,\ldots} \int_Q \Psi(|\varphi(z^s)|)\, \mathrm{d}y < \infty$$

be satisfied for any continuous function φ. Here $\Psi: \langle 0,\infty) \longmapsto \mathbf{R}$ is some Young function [4].

Then (for $s \to \infty$)

$$\varphi(z^s) \to \langle \nu_y, \varphi\rangle \text{ *-weakly in } L_\Psi(Q). \tag{5.9}$$

P r o o f. We associate with z^j the mapping $\nu^j : Q \longmapsto M(\mathbf{R}^M)$ defined by

$$\nu^j_y = \delta_{z^j(y)}, \tag{5.10}$$

hence

$$\|\nu^j_y\|_{M(\mathbf{R}^M)} = 1 \text{ for a.e. } y \in Q. \tag{5.11}$$

Due to separability of $\mathscr{C}_0(\mathbf{R}^M)$, and consequently of $L^1(Q, \mathscr{C}_0(\mathbf{R}^M))$, there exist a subsequence $\{\nu^s\}$ of $\{\nu^j\}$ and an element $\nu \in L^\infty_{(w)}(Q_T, M(\mathbf{R}^M))$ such that

$$\nu^s \to \nu \text{ *-weakly in } L^\infty_{(w)}(Q_T, M(\mathbf{R}^M)). \tag{5.12}$$

Therefore

$$\langle \nu^s_y, \varphi\rangle \to \langle \nu_y, \varphi\rangle \text{ *-weakly in } L^\infty(Q) \tag{5.13}$$

for every $\varphi \in \mathscr{C}_0(\mathbf{R}^M)$ (we use the *-weak compactness of bounded sets in separable Banach spaces). So (5.4) is proved, and (5.3) follows from the *-weak lower semicontinuity of the norm $\|\cdot\|_{L^\infty_{(w)}(Q_T, M(\mathbf{R}^M))}$.

We define $\vartheta^k \in \mathscr{C}_0(\mathbf{R}^M)$ as follows:

$$\vartheta^k(\tau) = \begin{cases} 1 & \text{for} \quad |\tau| \leqslant k \\ 1 + k - |\tau| & \text{for} \quad k \leqslant |\tau| \leqslant k+1 \\ 0 & \text{for} \quad |\tau| \geqslant k+1. \end{cases}$$

[4] Definitions and properties of Young functions as well as Orlicz spaces are to be found in Krasnoselski, Ruticki [8]. We only note that a Young function satisfies

$$\lim_{\xi\to\infty} \frac{\Psi(\xi)}{\xi} = \infty. \tag{5.8}$$

Let (5.5) be satisfied and let $E \subset Q$ be a bounded measurable set. Then

$$(5.15) \qquad \begin{aligned} \lim_{s\to\infty} \frac{1}{\operatorname{meas}(E)} \int_E \vartheta^k(z^s(y))\, \mathrm{d}y &= \frac{1}{\operatorname{meas}(E)} \int_E \langle \nu_y, \vartheta^k \rangle\, \mathrm{d}y \\ &\leqslant \frac{1}{\operatorname{meas}(E)} \int_E \|\nu_y\|_{M(\mathbf{R}^M)}\, \mathrm{d}y \leqslant 1. \end{aligned}$$

However,

$$(5.16) \qquad \begin{aligned} 0 &\leqslant \frac{1}{\operatorname{meas}(E)} \int_E \left(1 - \vartheta^k(z^s(y))\right) \mathrm{d}y \\ &\leqslant \frac{\operatorname{meas}\{y \in E;\, |z^s(y)| \geqslant k\}}{\operatorname{meas}(E)} \\ &\leqslant \sup_{s=1,2,\ldots} \frac{\operatorname{meas}\{y \in E;\, |z^s(y)| \geqslant k\}}{\operatorname{meas}(E)} \leqslant \varepsilon_k, \end{aligned}$$

where $\varepsilon_k \to 0$ if $k \to \infty$. Letting $s \to \infty$ in (5.16) we get

$$(5.17) \qquad 1 - \varepsilon_k \leqslant \frac{1}{\operatorname{meas}(E)} \int_E \langle \nu_y, \vartheta^k \rangle\, \mathrm{d}y \leqslant \frac{1}{\operatorname{meas}(E)} \int_E \|\nu_y\|_{M(\mathbf{R}^M)}\, \mathrm{d}y.$$

From this and (5.15) we obtain

$$(5.19) \qquad 1 = \frac{1}{\operatorname{meas}(E)} \int_E \|\nu_y\|_{M(\mathbf{R}^M)}\, \mathrm{d}y,$$

which implies that $\|\nu_y\|_{M(\mathbf{R}^M)} = 1$ a.e. in Q. So (5.6) is proved.

Let $\varphi \colon \mathbf{R}^M \to \mathbf{R}^1$ be any continuous function satisfying (5.7). We can suppose without loss of generality that $\varphi \geqslant 0$. Put

$$(5.19) \qquad \varphi^k(\tau) = \varphi(\tau)\vartheta^k(\tau).$$

If Φ is the complementary Young function to Ψ, then $C_\Phi(Q)$ denotes the closure of the set of all measurable functions defined on $\overline{Q}$ with respect to the Orlicz norm $\|\cdot\|_{L_\Phi}$. To prove (5.9) it is sufficient to show that

$$(5.20) \qquad \langle \nu_y, \varphi \rangle \in L_\Psi(Q)$$

and to verify, for an arbitrary $g \in C_\Phi(Q)$, the validity of the following limiting processes:

$$\iint_Q g(y)\,\varphi^k(z^s(y))\,\mathrm{d}y \underset{k\to\infty}{\rightrightarrows} \iint_Q g(y)\,\varphi(z^s(y))\,\mathrm{d}y, \tag{5.21}$$

$$\iint_Q g(y)\varphi^k(z^s(y))\,\mathrm{d}y \underset{s\to\infty}{\longrightarrow} \iint_Q g(y)\,\langle \nu_y, \varphi^k\rangle\,\mathrm{d}y, \tag{5.22}$$

$$\iint_Q g(y)\,\langle \nu_y, \varphi^k\rangle\,\mathrm{d}y \underset{k\to\infty}{\longrightarrow} \iint_Q g(y)\,\langle \nu_y, \varphi\rangle\,\mathrm{d}y. \tag{5.23}$$

The convergence in (5.21) is the uniform one with respect to s. This can be proved directly. Indeed,

$$\begin{aligned}(5.24)\quad &\iint_Q |g(y)|\,|\varphi^k(z^s(y)) - \varphi(z^s(y))|\,\mathrm{d}y\\ &\leqslant \int_{A_k^s} |g(y)|\,(1-\vartheta^k(|z^s|))\varphi(z^s(y))\,\mathrm{d}y + \int_{B_k^s} |g(y)|\varphi(z^s(y))\,\mathrm{d}y\\ &\leqslant \|g\|_{L_\Phi(A_k^s)}\|\varphi(z^s)\|_{L_\Psi(A_k^s)} + \|g\|_{L_\Phi(B_k^s)}\|\varphi(z^s)\|_{L_\Psi(B_k^s)} \equiv J,\end{aligned}$$

where $A_k^s = \{y \in Q\,;\, k \leqslant |z^s(y)| \leqslant k+1\}$, $B_k^s = \{y \in Q\,;\, |z^s(y)| \geqslant k+1\}$. Using the fact that $\|h\|_{L_\Psi(Q)} \leqslant \big(1 + \iint_Q \Psi(h(y))\,\mathrm{d}y\big)$ together with (5.5) we obtain

$$\begin{aligned}(5.25)\quad J &\leqslant (\|g\|_{L_\Phi(A_k^s)} + \|g\|_{L_\Phi(B_k^s)})(1 + \iint_Q \Psi(\varphi(z^s))\,\mathrm{d}y\\ &\leqslant \text{const.}(\|g\|_{L_\Phi(A_k^s)} + \|g\|_{L_\Phi(B_k^s)}).\end{aligned}$$

By virtue of the uniform continuity [5] of Orlicz norm $\|g\|_{L_\Phi(Q')}$ with respect to Q', one gets that for every $\varepsilon > 0$ there exists k_0 such that for $k > k_0$ the right-hand side of (5.25) is less than ε. Thus (5.21) is proved.

To verify (5.23), note that $\varphi^{k+1} \geqslant \varphi^k \geqslant 0$. Due to the monotone convergence theorem we have

$$\int_{\mathbf{R}^M} \varphi^k \mathrm{d}\nu_y \longrightarrow \int_{\mathbf{R}^M} \varphi\,\mathrm{d}\nu_y \text{ for a.e. } y \in Q. \tag{5.26}$$

[5] This means: let $g \in L_\Phi(Q)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|g\|_{L_\Phi}(Q') \leqslant \varepsilon$ for every Q', meas $Q' < \delta$—cf. Krasnoselski, Ruticki [8].

For $g \geqslant 0$ we have

$$g(y)\,\langle \nu_y, \varphi^{k+1}\rangle \geqslant g(y)\,\langle \nu_y, \varphi^k\rangle \geqslant 0 \tag{5.27}$$

and

$$\lim_{k\to\infty} g(y)\,\langle \nu_y, \varphi^k\rangle = g(y)\,\langle \nu_y, \varphi\rangle\,. \tag{5.28}$$

Similarly, owing to the monotone convergence theorem, we obtain (5.23). (5.22) folows immmediately from (5.4) and the imbedding $C_\Phi(Q) \circlearrowleft L_1(Q)$. Now it suffices to prove

$$\|\langle \nu, \varphi^k\rangle\|_{L_\Psi(Q)} \leqslant \text{ const.} \tag{5.29}$$

Then there exists $\chi \in L_\Psi(Q)$ such that

$$\iint_Q g(y)\,\langle \nu_y, \varphi^k\rangle\,\mathrm{d}y \longrightarrow \iint_Q g(y)\chi\,\mathrm{d}y \tag{5.30}$$

for every $g \in C_\Phi(Q)$. Comparing (5.30) with (5.23), where we take smooth functions as the test functions, we get

$$\chi(y) = \langle \nu_y, \varphi\rangle \ \text{ for a.e. } y \in Q \tag{5.31}$$

and (5.20) is proved.
Clearly, $\varphi^k \leqslant \varphi^{k+1} \leqslant \varphi$. Thus

$$\sup_{s=1,2,\dots} \int_Q \Psi(|\varphi^k(z^s)|)\,\mathrm{d}y \leqslant \sup_{s=1,2,\dots} \int_Q \Psi(|\varphi(z^s)|)\,\mathrm{d}y \leqslant c \tag{5.32}$$

or

$$\|\langle \nu^s, \varphi^k\rangle\|_{L_\Psi(Q)} \leqslant c, \tag{5.33}$$

where c depends neither on k nor on s. Thus there exists $a \in L_\Psi(Q)$ (comparing (5.34) with (5.22) it can be seen that $a(y) = \langle \nu_y, \varphi^k\rangle$ for a.e. $y \in Q$) such that, for $s \to \infty$,

$$\langle \nu^s, \varphi^k\rangle \to a \text{ *-weakly in } L_\Psi(Q). \tag{5.34}$$

Taking $G = \Psi$ and $u_s = \langle \nu^s, \varphi^k\rangle$ in Lemma 5.35 below we finally obtain (5.29).

The proof of Theorem 5.1 is complete. □

Lemma 5.35. *Let $G\colon \mathbf{R}^m \longmapsto \mathbf{R}^+ \cup \{\infty\}$ be a lower semicontinous convex function. Let $u_s, u \in L^1(\Omega; \mathbf{R}^m)$ and*

$$\int f\,\mathrm{d}u_s \to \int f\,\mathrm{d}u \text{ for every } f \in \mathscr{C}_0(\mathbf{R}^m). \tag{5.36}$$

Then

$$\int_Q G(u) \leqslant \liminf_{j\to\infty} \int_Q G(u_j). \tag{5.37}$$

Proof (of Theorems 4.12,4.13) .

Let $\{\mathbf{v}^\mu\}_{\mu>0}$ be a sequence of solutions to (1.18) corresponding to $\mu > 0$. We have uniformly with respect to μ

$$\|\mathbf{v}^\mu\|_{L^\infty(I,L^2(\Omega))} \leqslant \text{const.}, \tag{5.38}$$

$$\|\mathbf{v}^\mu\|_{L^2(I,\mathscr{D}(A^{1/4}))} \leqslant \text{const.}, \tag{5.39}$$

$$\iint_{Q_T} \beta(\hat{\mathbf{v}}^\mu) e_{ij}(\mathbf{v}^\mu) e_{ij}(\mathbf{v}^\mu) \leqslant \text{const.}, \tag{5.40}$$

$$\mu \left(\frac{\partial e_{ij}(\mathbf{v}^\mu)}{\partial x_k}, \frac{\partial e_{ij}(\mathbf{v}^\mu)}{\partial x_k} \right) \leqslant \text{const.}, \tag{5.41}$$

hence due to (1.18)

$$\left\| \frac{\partial \mathbf{v}^\mu}{\partial t} \right\|_{L^{\gamma''}(I,\mathscr{V}^*)} \leqslant \text{const.} \tag{5.42}$$

It is clear that

$$\mu \left(\frac{\partial e_{ij}(\mathbf{v}^\mu)}{\partial x_k}, \frac{\partial e_{ij}(\omega)}{\partial x_k} \right) \to 0 \tag{5.43}$$

for every $\omega \in \mathscr{C}_0^\infty(Q_T, \mathbf{R}^N)$. Further, due to (5.38)–(5.41),

$$\mathbf{v}^\mu \to \mathbf{v} \text{ weakly in } L^2(I, \mathscr{D}(A^{1/4})), \tag{5.44}$$

$$\mathbf{v}^\mu \to \mathbf{v} \text{ *-weakly in } L^\infty(I, H), \tag{5.45}$$

$$\mathbf{v}^\mu \to \mathbf{v} \text{ strongly in } L^2(Q_T). \tag{5.46}$$

Due to (5.39),

$$\|\nu^\mu(t,x)\|_{M(\mathbf{R}^{N^2})} \leqslant 1 \text{ a.e in } Q_T, \tag{5.47}$$

hence (5.3), (5.4) hold. In Theorem 5.1 we put $Q = Q_T$, $z^\mu = \nabla \mathbf{v}^\mu$, $\nu^\mu = \delta(\sigma - \nabla \mathbf{v}^\mu)$, $\varphi = \xi_{ij}$ or b_{ij}, $i, j = 1, 2, \dots, N$, $\Psi(\tau) = \frac{1}{2}\tau^{\frac{2\bar{\gamma}+2}{2\gamma+1}}$ provided $\gamma \neq 0$ or $\Psi(\tau) = \frac{1}{2}\tau^2$ provided $\gamma = 0$ (note that the condition (5.8) is satisfied). The validity of (5.5) follows, for instance, from the L^1-estimate of $\nabla \mathbf{v}^\mu$ in Q_T, which is clearly guaranteed in this case.

The assumptions of Theorem 5.1 hold. Thus we get

$$\iint\limits_{Q_T} \varphi(t,x)\ \langle \nu^\mu(t,x), \beta_{ij}(\sigma_{rs})\rangle\ \mathrm{d}x\,\mathrm{d}t \longrightarrow \iint\limits_{Q_T} \varphi(t,x)\ \left\langle \nu_{(t,x)}, \beta_{ij}(\sigma_{rs})\right\rangle\ \mathrm{d}x\,\mathrm{d}t \tag{5.48}$$

for every $\varphi \in L^{\gamma'}(Q_T)$ and

$$\iint\limits_{Q_T} \varphi(t,x)\,\langle \nu^\mu(t,x), \sigma_{rs}\rangle\ \mathrm{d}x\,\mathrm{d}t \longrightarrow \iint\limits_{Q_T} \varphi(t,x)\ \left\langle \nu_{(t,x)}, \sigma_{rs}\right\rangle\ \mathrm{d}x\,\mathrm{d}t \tag{5.49}$$

for every $\varphi \in L^2(Q_T)$. The rest is obvious. □

Remark 5.50 (Sufficient condition for ν to be the Dirac measure).
Put $\tau_{ij}^{(v)} = \beta(\hat{v}^2)e_{ij}(\mathbf{v})$, *hence* $\tau_{ij} = -p\delta_{ij} + \tau_{ij}^{(v)}$ *(cf. 1.19), and suppose*

$$\left|\tau_{ij}^{(v)}\right| \leqslant c_1(1 + |\hat{v}|^{2\bar{\gamma}})|\hat{v}| \ \text{for}\ \bar{\gamma} \geqslant \frac{N-2}{4}, \tag{5.51}$$

$$\tau_{ij}^{(v)} \frac{\partial v_i}{\partial x_j} \geqslant c_2\,|\hat{v}|^2(1 + c_3|\hat{v}|^{2\gamma}), \tag{5.52}$$

$$\begin{aligned}&\int_\Omega (\tau_{ij}^{(v)}(\mathbf{v}) - \tau_{ij}^{(v)}(\tilde{\mathbf{v}}))(e_{ij}(\mathbf{v}) - e_{ij}(\tilde{\mathbf{v}}))\,\mathrm{d}x\\ &\geqslant c_4 \sum_{i,j=1}^{N} (e_{ij}(\mathbf{v}) - e_{ij}(\tilde{\mathbf{v}}))\,(e_{ij}(\mathbf{v}) - e_{ij}(\tilde{\mathbf{v}})),\end{aligned} \tag{5.53}$$

where c_1, c_2, c_3, c_4 are positive constants.

Then we have according to Ladyženskaya [9]:

Theorem 5.54. *Let* $\mathbf{v}_0 \in V_1$, $\mathbf{f} \in L^2(I, W^{1,2}(\Omega, \mathbf{R}^N)$ *and let* (5.51)–(5.53) *be satisfied. Then there exists a unique* $\mathbf{v}$

$$\mathbf{v} \in L^\infty(I,H) \cap L^2(I,V_1) \cap L^{2(\bar{\gamma}+1)}(I, W^{1,2(\bar{\gamma}+1)}), \tag{5.55}$$

such that (1.21) *is fulfilled.*

Thus, under the assumptions of Remark 5.50 *(cf. Remark* 4.11*) among measure-valued solutions of* (1.21) *(see Theorem* 4.12*) there exists a unique exceptional solution which is the Dirac measure, i.e.* $\nu = \delta(\sigma - \nabla \mathbf{v})$.

6. Uniqueness of weak solution

Let us denote $\beta_{ij}(\alpha) = \beta(\hat{\alpha}^2, D(\alpha))\alpha_{ij}$ with $\hat{\alpha}^2 = \frac{1}{4}(\alpha_{rs} + \alpha_{sr})(\alpha_{rs} + \alpha_{sr})$, $D(\alpha) = \det(\alpha_{rs})_{r,s=1}^{N}$. We prove the following theorem.

Theorem 6.1. *Let the conditions* (2.2.13), (2.2.14) *be satisfied and let*

$$\left| \frac{\partial \beta_{ij}}{\partial \alpha_{rs}} \right| \leqslant \text{const.} \qquad \textit{for every } i,j,r,s = 1,\dots,N. \tag{6.2}$$

Then in the class of functions $\mathbf{v}$ *satisfying* (3.6)–(3.7) *there exists only one solution to* (1.18).

P r o o f. Let $\mathbf{u} = \mathbf{v} - \tilde{\mathbf{v}}$ be the difference of two solutions of (1.18) corresponding to the same initial conditions. Then $\mathbf{u}$ has to comply with

$$\begin{aligned} \left(\frac{\partial \mathbf{u}}{\partial t}, \varphi\right) + ((\mathbf{u}, \varphi)) &= -\int_\Omega u_j \frac{\partial v_i}{\partial x_j} \varphi_i \, \mathrm{d}x \\ &- \int_\Omega \tilde{v}_j \frac{\partial u_i}{\partial x_j} \varphi_i \, \mathrm{d}x - \int_\Omega \left[\beta_{ij}\left(\frac{\partial v_r}{\partial x_s}\right) - \beta_{ij}\left(\frac{\partial \tilde{v}_r}{\partial x_s}\right)\right] \frac{\partial \varphi_i}{\partial x_j} \, \mathrm{d}x \\ &\qquad \text{for every } \varphi \in \mathcal{V}. \end{aligned} \tag{6.3}$$

We take $\varphi = A^{1/2}\mathbf{u}$ as a test function in (6.3). The terms on the right-hand side are bounded as follows (cf. (2.2.19)):

$$\begin{aligned} \left| \int_\Omega u_j \frac{\partial v_i}{\partial x_j} (A^{1/2}\mathbf{u})_i \, \mathrm{d}x \right| &\leqslant c_1 \|A^{3/4}\mathbf{u}\| \, \|A^{3/4}\mathbf{v}\| \, \|A^{1/4}\mathbf{u}\| \\ &\leqslant \varepsilon \|A^{3/4}\mathbf{u}\|^2 + K(\varepsilon) \, \|A^{3/4}\mathbf{v}\|^2 \|A^{1/4}\mathbf{u}\|^2, \end{aligned} \tag{6.4}$$

$$\begin{aligned} \left| \int_\Omega \tilde{v}_j \frac{\partial u_i}{\partial x_j} (A^{1/2}\mathbf{u})_i \, \mathrm{d}x \right| &\leqslant c_1 \|A^{3/4}\mathbf{u}\| \, \|A^{3/4}\tilde{\mathbf{v}}\| \, \|A^{1/4}\mathbf{u}\| \\ &\leqslant \varepsilon \|A^{3/4}\mathbf{u}\|^2 + K(\varepsilon) \, \|A^{3/4}\tilde{\mathbf{v}}\|^2 \|A^{1/4}\mathbf{u}\|^2, \end{aligned} \tag{6.5}$$

$$\begin{aligned} &\left| \int_\Omega \left[\beta_{ij}\left(\frac{\partial v_r}{\partial x_s}\right) - \beta_{ij}\left(\frac{\partial \tilde{v}_r}{\partial x_s}\right)\right] \frac{\partial (A^{1/2}\mathbf{u})_i}{\partial x_j} \, \mathrm{d}x \right| \\ &\leqslant \int_\Omega \left| \frac{\partial \beta_{ij}}{\partial \alpha_{rs}} \left(\frac{\partial v_r}{\partial x_s} + \vartheta\left(\frac{\partial \tilde{v}_r}{\partial x_s} - \frac{\partial v_r}{\partial x_s}\right)\right)\right| \left| \frac{\partial u_r}{\partial x_s} \right| \left| \frac{\partial (A^{1/2}\mathbf{u})_i}{\partial x_j} \right| \, \mathrm{d}x \\ &\leqslant c_1 \|A^{3/4}\mathbf{u}\| \, \|A^{1/4}\mathbf{u}\| \leqslant \varepsilon \|A^{3/4}\mathbf{u}\|^2 + c_1^2 K(\varepsilon) \, \|A^{1/4}\mathbf{u}\|^2, \end{aligned} \tag{6.6}$$

where $0 \leqslant \vartheta \leqslant 1$. From (6.3)–(6.6) and the Gronwall lemma one gets

$$\|A^{1/4}\mathbf{u}(t)\|^2 \leqslant \|A^{1/4}\mathbf{u}(0)\|^2 \exp \mu(t), \tag{6.7}$$

where $\mu(t) = c_2 \int_0^t \left(1 + \|A^{3/4}\mathbf{v}\|^2 + \|A^{3/4}\tilde{\mathbf{v}}\|^2\right) \mathrm{d}\tau$ with $c_2 > 0$. The uniqueness is proved. □

Now, let $\mathbf{v}_0 \in \mathscr{D}(A^{1/4})$ and $\mathbf{f} \in L^2(\Omega)$. Under the assumptions (2.2.14), (6.2) we can define a family of operators $S_t \colon \mathscr{D}(A^{1/4}) \longmapsto \mathscr{D}(A^{1/4})$, $t > 0$ by

$$S_t \mathbf{v}_0 = \mathbf{v}(t, \mathbf{v}_0), \tag{6.8}$$

where $\mathbf{v}(t, \mathbf{v}_0)$ is a solution to (1.18) with initial condition $\mathbf{v}_0$.

Due to the existence (Theorem 3.5) and uniqueness (Theorem 6.1) we can state

Lemma 6.9. *The operators $\{S_t\}_{t \geqslant 0}$ defined above form a semigroup on $\mathscr{D}(A^{1/4})$, i.e.*

$$\begin{aligned} &S_0 \mathbf{v}_0 = \mathbf{v}_0, \\ &S_{t+t'} \mathbf{v}_0 = S_t\, S_{t'} \mathbf{v}_0 \quad \forall\, \mathbf{v}_0 \in \mathscr{D}(A^{1/4}). \end{aligned} \tag{6.10}$$

7. Existence of the universal attractor

Definition 7.1. A closed compact set $\mathscr{A}$ of a Banach space X is a universal attractor of the semigroup $\{S_t\}_{t \geqslant 0}$, $S_t \colon X \longmapsto X$ if and only if

(i) $\mathscr{A}$ is an invariant set, i.e. $S_t \mathscr{A} = \mathscr{A}$ for every $t \geqslant 0$;

(ii) for every bounded set $B \subset X$ we have $\lim\limits_{t \to \infty} \operatorname{dist}(S_t B, \mathscr{A}) = 0$, where $\operatorname{dist}(M, N) = \inf\limits_{x \in M} \sup\limits_{y \in N} \|x - y\|_X$.

The following theorem gives sufficient conditions for the semigroup $\{S_t\}_{t \geqslant 0}$ to have a universal attractor. For the proof see Babin, Višik [1].

Theorem 7.2. *Let $S_t \colon X \longmapsto X$ be a semigroup on X. We suppose that*

(i) $\{S_t\}_{t \geqslant 0}$ is uniformly bounded, i.e. for every bounded $B \subset X$ there exists a constant $R(B)$ such that $\|S_t B\|_X \leqslant R(B)$ for every $t \geqslant 0$;

(ii) there exists a bounded closed set $B_0 \subset X$ which is attracting, i.e. for every bounded set $B \subset X$ there exists $T(B) > 0$ such that $S_t B \subseteq B_0$ for each $t \geqslant T(B)$;

(iii) the operator S_t is continuous for $t \geqslant 0$ and compact for $t > 0$.

Then the semigroup $\{S_t\}_{t\geqslant 0}$ has a universal attractor.

Lemma 7.3. *Let B be an arbitrary bounded set in $\mathcal{D}(A^{1/4})$. For the semigroup $\{S_t\}_{t\geqslant 0}$ defined by (6.8) there exist numbers $R_0 > 0$ and $\alpha > 0$ such that*

$$\|S_t B\| \leqslant R_0 \quad \text{for every} \quad t > t(B) > 0 \ , \tag{7.4}$$

$$\|S_t \mathbf{v}_0\| \leqslant R_0 \quad \text{for every} \quad t > 0 \qquad \text{provided } \mathbf{v}_0 \in B_0 = \{\mathbf{v};\, \|\mathbf{v}\| \leqslant R_0\}, \tag{7.5}$$

$$\int_t^{t+1} \|A^{1/2} S_\tau B\|^2 \mathrm{d}\tau \leqslant \alpha \quad \text{for every} \quad t \geqslant T(B). \tag{7.6}$$

P r o o f. We take $\varphi = \mathbf{v}$ as the test function in the weak formulation (1.18). We have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathbf{v}\|^2 + \|A^{1/2}\mathbf{v}\|^2 + \int_\Omega \beta(\hat{v}^2, D(\mathbf{v}))e_{ij}(\mathbf{v})e_{ij}(\mathbf{v})\,\mathrm{d}x \\ &\qquad \leqslant \|\mathbf{f}\|\,\|\mathbf{v}\| \leqslant \frac{1}{2\lambda_1}\|\mathbf{f}\|^2 + \frac{\lambda_1}{2}\|\mathbf{v}\|^2. \end{aligned} \tag{7.7}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2 + \lambda_1\|\mathbf{v}\|^2 \leqslant \frac{1}{\lambda_1}\|\mathbf{f}\|^2. \tag{7.8}$$

After solving the ordinary diferential equation, we obtain

$$\|\mathbf{v}\|^2 \leqslant \|\mathbf{v}_0\|^2 e^{-\lambda_1 t} + \frac{1}{\lambda_1^2}\left(1 - e^{-\lambda_1 t}\right)\|\mathbf{f}\|^2. \tag{7.9}$$

The proof is completed by putting $R_0 \geqslant \frac{\|\mathbf{f}\|}{\lambda_1}$ and $\alpha = (\lambda_1 + 1)R_0^2$. □

Let us recall the so called generalized Gronwall lemma (Cf. Foias, Sell, Temam [5] or Temam [21, p. 89]).

Lemma 7.10. *Let g, h, y be three positive locally integrable functions for $t_0 \leqslant t < \infty$ which satisfy*

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leqslant gy + h \qquad \text{for all} \qquad t \geqslant t_0 \tag{7.11}$$

and

$$\int_t^{t+r} g\,\mathrm{d}s \leqslant \alpha_1, \qquad \int_t^{t+r} h\,\mathrm{d}s \leqslant \alpha_2, \qquad \int_t^{t+r} y\,\mathrm{d}s \leqslant \alpha_3 \quad \text{for all} \quad t \geqslant t_0, \tag{7.12}$$

where α_1, α_2, α_3 are positive constants. Then

$$y(t+1) \leqslant (\alpha_3 + \alpha_2)\exp\alpha_1 \qquad \text{for all} \quad t \geqslant t_0. \tag{7.13}$$

Now, we are in a position to prove the existence of a universal attractor. We will verify that the semigroup (6.8) satisfies the conditions (i), (ii), (iii) from Theorem 7.2.

Theorem 7.14. *Let $\{S_t\}_{t\geqslant 0}$ be the semigroup of operators $S_t\colon \mathscr{D}(A^{1/4}) \longmapsto \mathscr{D}(A^{1/4})$ defined by (6.8). Then there exists a universal attractor $\mathscr{A} \subset \mathscr{D}(A^{1/4})$.*

Proof. Let us consider the weak formulation (1.18) with the test function $\varphi = A^{1/2}\mathbf{v}$. We get (cf. Lemma 2.2.12)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|A^{1/4}\mathbf{v}\|^2 + \|A^{3/4}\mathbf{v}\|^2 &\leqslant c\Big(\|A^{1/4}\mathbf{v}\|\,\|A^{3/4}\mathbf{v}\|\,\|A^{1/2}\mathbf{v}\| \\ &\quad + \|A^{3/4}\mathbf{v}\|\,\|A^{1/2}\mathbf{v}\| + \|\mathbf{f}\|\,\|A^{1/2}\mathbf{v}\|\Big) \\ &\leqslant \frac{1}{2}\|A^{3/4}\mathbf{v}\|^2 + K\Big(\|A^{1/4}\mathbf{v}\|^2\,\|A^{1/2}\mathbf{v}\|^2 + \|A^{1/2}\mathbf{v}\|^2 + \|\mathbf{f}\|^2\Big), \end{aligned} \tag{7.15}$$

so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/4}\mathbf{v}\|^2 \leqslant K_1\|A^{1/2}\mathbf{v}\|^2\,\|A^{1/4}\mathbf{v}\|^2 + K_1\Big(\|A^{1/2}\mathbf{v}\|^2 + \|\mathbf{f}\|^2\Big). \tag{7.16}$$

Let us denote $\alpha_1 = K_1\alpha$, $\alpha_2 = K_1(\alpha + \|\mathbf{f}\|^2)$, $\alpha_3 = \alpha($ for α see Lemma 7.3). Due to Lemma 7.3 and the generalized Gronwall lemma with $y = \|A^{1/4}\mathbf{v}\|^2$, $g = K_1\|A^{1/2}\mathbf{v}\|^2$, $h = K_1(\|A^{1/2}\mathbf{v}\|^2 + \|\mathbf{f}\|^2)$ we have

$$\|A^{1/4}\mathbf{v}(t)\|^2 \leqslant R' \quad \text{for all sufficiently large } t, \tag{7.17}$$

where $R' = (\alpha_2 + \alpha_3)(\exp\alpha_1)$. Hence $\|S_tB\|_{\mathscr{D}(A^{1/4})} \leqslant R'$ for $t \geqslant T(B)$. Thus the condition (ii) in 7.2 is fulfilled.

Moreover, we also have (see 7.15)

$$\begin{aligned}(7.18)\qquad &\frac{1}{2}\sup_{(0,T(B))}\|A^{1/4}S_tB\|^2 \leqslant \sup_{\mathbf{v}\in B}\Big\{\frac{1}{2}\|A^{1/4}\mathbf{v}_0\|^2\\ &+\int_0^{T(B)}\|A^{3/4}S_t\mathbf{v}_0\|^2\,\mathrm{d}\tau + K\int_0^{T(B)}\|A^{1/2}S_t\mathbf{v}_0\|^2\|A^{1/4}\mathbf{v}_0\|^2\,\mathrm{d}\tau\\ &+K\int_0^{T(B)}\|A^{1/2}S_t\mathbf{v}_0\|^2\,\mathrm{d}\tau + T(B)\|\mathbf{f}\|^2\Big\} \leqslant R''.\end{aligned}$$

Thus also (i) in 7.2 holds.

Continuity of $\{S_t\}_{t\geqslant 0}$ follows from the proof of uniqueness, see (6.7), and the compactness is a consequence of Theorem 3.8., exactly (3.11), and of the compact imbedding of $\mathscr{D}(A^{1/2})$ into $\mathscr{D}(A^{1/4})$. □

8. Hausdorff dimension of the attractor

In order to prove the finite Hausdorff dimension of the universal attractor, we use the well-known lemma (cf. Ladyženskaya [10]).

Lemma 8.1. *Let M be a bounded set in X (Hilbert space) and let $T\colon X \longmapsto X$ be a mapping such that*

(i) $\quad TM \supseteq M,$

(ii) $\quad \|T(v_1) - T(v_2)\|_X \leqslant L\,\|v_1 - v_2\|_X \qquad L > 0 \quad \forall v_1, v_2 \in M$

(iii) $\quad \|QT(v_1) - QT(v_2)\|_X \leqslant \delta\,\|v_1 - v_2\|_X \qquad 0 < \delta < 1$

where $Q \equiv I - P$; P and Q are orthogonal projectors of X onto X_n and $X_n^{\perp}$, respectively ($\dim X_n = n$).

Then M has a finite Hausdorff dimension d_0 which can be estimated by

$$(8.2)\qquad d_0(M) \leqslant n\log\left(\frac{2k^2L^2}{1-\delta^2} - \frac{2}{1+\delta^2}\right)$$

with an absolute constant k.

We apply this lemma for the universal attractor $\mathscr{A}$, whose existence is guaranteed by Theorem 7.2, and put $T = S_t$. Then (i) holds automatically. For the definition of $P := P_n$ and $Q := Q_n$ see (2.1.5).

We denote $\mathbf{u}(t) = \mathbf{v}(t, \mathbf{v}^1) - \tilde{\mathbf{v}}(t, \mathbf{v}^2)$, where $\mathbf{v}^1$, $\mathbf{v}^2 \in \mathscr{A}$. Clearly, $\mathbf{u}(0) = \mathbf{v}^1 - \mathbf{v}^2$. Thus we need to verify

$$\|A^{1/4}\mathbf{u}(t)\| \leqslant L\|A^{1/4}\mathbf{u}(0)\|, \tag{8.3}$$

$$\|A^{1/4}Q\mathbf{u}(t)\| \leqslant \delta\|A^{1/4}\mathbf{u}(0)\| \tag{8.4}$$

at least for some $t > 0$.

First, we have already proved the estimate (8.3) in the proof of uniqueness (see Theorem 6.1). Here $L = \exp \int_0^t (1 + \|A^{3/4}\mathbf{v}^1\|^2 + \|A^{3/4}\mathbf{v}^2\|^2)\, \mathrm{d}\tau$.

Second, we proceed similarly as in the proof of uniqueness (Theorem 6.1) but using $A^{1/2}Q\mathbf{u}$ as the test function.

We obtain (cf. (2.1.11), (2.1.12)):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|A^{1/4}Q\mathbf{u}\|^2 + \|A^{3/4}Q\mathbf{u}\|^2 &\leqslant \|A^{3/4}Q\mathbf{u}\|\ \|A^{1/4}\mathbf{u}\|\ \|A^{1/4}\mathbf{v}\| \\ &\quad + \|A^{1/4}\tilde{\mathbf{v}}\|\ \|A^{1/4}\mathbf{u}\|\ \|A^{3/4}Q\mathbf{u}\| + c_1\|A^{1/4}\mathbf{u}\|\ \|A^{3/4}Q\mathbf{u}\| \\ &\leqslant R(\mathscr{A})\left(\frac{1}{3R(\mathscr{A})}\|A^{3/4}Q\mathbf{u}\|^2 + 3R(\mathscr{A})\|A^{1/4}\mathbf{u}\|^2\right) \\ &\quad + c_1\left(\frac{1}{6c_1}\|A^{3/4}Q\mathbf{u}\|^2 + \frac{3}{2}c_1\|A^{1/4}\mathbf{u}\|^2\right). \end{aligned} \tag{8.5}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/4}Q\mathbf{u}\|^2 + \|A^{3/4}Q\mathbf{u}\|^2 \leqslant \left(3R^2(\mathscr{A}) + \frac{3}{2}c_1^2\right)\|A^{1/4}\mathbf{u}\|^2. \tag{8.6}$$

Denoting $L_1 = (3R^2(\mathscr{A}) + \frac{3}{2}c_1^2)$ and using (2.1.8), we rewrite (8.6) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/4}Q\mathbf{u}\|^2 + \lambda_{n+1}\|A^{1/4}Q\mathbf{u}\|^2 \leqslant L_1\|A^{1/4}\mathbf{u}\|^2. \tag{8.7}$$

Thus, we have

$$\|A^{1/4}Q\mathbf{u}\|^2 \leqslant \|A^{1/4}\mathbf{u}(0)\|e^{-\lambda_{n+1}t} + \frac{L_1}{\lambda_{n+1}}\|A^{1/4}\mathbf{u}\|^2\left(e^{-\lambda_{n+1}t} - 1\right); \tag{8.8}$$

using (8.3), (2.1.4), we verify (8.4). The proof is complete.

Acknowledgements. We thank Michael Růžička for his careful reading of the preprint version of this paper and for many valuable discussions.

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