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SPANNING SUBGRAPHS OF EMBEDDED GRAPHS

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INTRODUCTION

A well-known folklore result states that the set of edges of an arbitrary cotree of a plane graph G corresponds to the edge-set of a spanning tree of the dual graph G^*. Thus there exists a natural 1–1 correspondence between the spanning trees of G and those of G^*.

The aim of the present note is to extend this correspondence to surfaces other than the plane (or sphere). Let G be a connected graph 2-cell embedded in a closed surface S with Euler characteristic χ, and let $G^* \to S$ be the dual embedding. Suppose that G dissects S into $r \geqslant 2$ faces. Since G is connected, there exists an edge e_1 of G which lies on the boundary of two distinct faces. Thus $G - e_1$ is embedded in S with $r - 1$ faces. If $r - 1 \geqslant 2$ the process can be repeated. Continuing in this way one obtains a set $\{e_1, \ldots, e_{r-1}\} = A$ of edges of G such that $G - A$ is a connected spanning subgraph of G 2-cell embedded in S with a single face. It turns out that the subset $A^* \subseteq E(G^*)$ of the edges dual to those in A induces a spanning tree of G^*. Moreover, each spanning tree of G^* arises in this way. More generally, we show that the assignment $X \mapsto E(G^*) - X$, X a subset of $E(G)$, defines a 1–1 correspondence between certain spanning subgraphs of G with Betti number $k \in [0, 2 - \chi]$ and those of G^* with Betti number $2 - \chi - k$.

The relationship between the surface duality and spanning trees has already been examined. It is known (see Biggs [1] for the orientable case and Richter and Shank [4] for the general case) that, for any spanning tree T of a graph G with a cellular embedding in a surface S, the complement of the edges dual to T contains a spanning tree of G^*. The methods include rotation systems ([1]) and the cycle-cocycle duality in embedded graphs ([4]). Our approach is based on the result of Edmonds [2] and Richter and Shank [4] that, in dual graphs, bounding cycles and cocycles correspond to each other.

We use the standard graph theoretic terminology with minor deviations. We also assume that the reader is familiar with the basic notions of topological graph theory (see, e.g., [3]).

Let G be a connected graph with edge-set E. The term *subgraph* will always refer to a *spanning subgraph*. Thus every subgraph of G may be identified with its edge-set; conversely, every subset of E can be viewed as a subgraph of G. Accordingly, if H is a subgraph of G, we denote by $G-H$ the subgraph obtained from G by deleting the edges of H.

A *cycle* in G is a subgraph $z \subseteq G$ such that for every vertex v of G $\deg_z(v)$ is even. For a set W of vertices of G let δW denote the set of edges with one end in W and the other end not in W. A *cocycle* in G is a set q of edges of the form $q = \delta W$, for some W.

Now assume that G is 2-cell embedded in a closed surface S, possibly non-orientable. That is, there exists an embedding $i\colon G \to S$ such that each connected component of the space $S - i(G)$ is homeomorphic to the Euclidean plane $\mathbf{R}^2$. For a subset D of faces of G, let ∂D denote the set of edges e of G such that there exists a face f in D and a face f' not in D, both containing the edge e on its boundary. It is a straightforward matter to see that, for every subset D of faces, ∂D is a cycle in G. We say that z is a *bounding cycle* of G if $z = \partial D$ for some D.

It is well-known that the set of cycles (cocycles, bounding cycles) of G forms a vector space over $\mathbf{GF}(2)$ under the operation of the symmetric difference of sets.

Let G^* be the dual graph of G with respect to the given embedding of G in S. For any edge e of G the corresponding dual edge of G^* will be denoted by e^*. If P is a subset of E, the symbol P^* will stand for $\{e^*;\ e \in P\}$. In general, for any pair of mutually corresponding dual sets of edges P and P^*, the presence of $*$ will indicate the context in G^* while its absence the context in G.

We shall use $\beta(G)$ for the Betti number (i.e., the cycle rank) of a graph G and $\chi(S)$ for the Euler characteristic of a surface S. We shall also employ the quantity $\bar{\gamma}(S) = 2 - \chi(S)$, the *Euler genus* of S. Recall that if G is 2-cell embedded in S and V, E and F are the vertex-set, the edge-set and the set of faces of G, respectively, then $\beta(G) = |E| - |V| + 1$ and $\chi(S) = |V| - |E| + |F|$. The latter is usually known as the Euler-Poincaré formula.

Our point of departure is the following elementary but important observation due to Edmonds [2] and Richter and Shank [4].

Theorem 0. *Let G be a connected graph 2-cell embedded in a surface S, and let G^* be the corresponding dual graph. Then z is a bounding cycle in G if and only if z^* is a cocycle in G^*.*

Proof ([4]). Assume that D is a set of faces of G with $\partial D = z$. Let $e \in \partial D$. Then there is a face f in D and a face f' not in D such that $e \in \partial f \cap \partial f'$. Let W be the set of vertices of G^* which correspond to the faces in D, and let v and v' be the vertices of G^* corresponding to f and f', respectively. Then e^* joins v to v'. Since v belongs to W and v' does not, we have $e^* \in \delta W$. Thus $z^* \subseteq \delta W$. By similar arguments, $\delta W \subseteq z^*$, and hence z^* is a cocycle. The converse statement is proved similarly. □

We shall continue with the above notation. Throughout the section we shall assume that G is a connected graph with a cellular embedding in a closed surface S of Euler characteristic χ and that G^* is the corresponding dual. It will be convenient to denote the Euler genus of S, $\bar{\gamma}(S) = 2 - \chi$, by n.

To proceed further we shall need one more notion which, in a sense, can be regarded as a higher-surface analogue of a spanning tree.

Definition. Let k be an integer with $0 \leqslant k \leqslant n$. A connected spanning subgraph $Q \subseteq G$ will be called a *k-frame* of G provided that:

(1) $\beta(Q) = k$; and

(2) Q does not contain a non-zero bounding cycle.

We shall see that the duality between spanning trees in the plane extends to the duality between "complementary" k-frames in closed surfaces. (Note that a 0-frame is simply a spanning tree.)

Theorem 1. *Let G be a connected graph 2-cell embedded in a closed surface of the Euler genus n, and let Q be a k-frame of G, $0 \leqslant k \leqslant n$. Then $(G - Q)^*$ is an $(n - k)$-frame of G^*.*

Proof. We first show that $R^* = (G - Q)^*$ is a connected spanning subgraph of G^*. Assume, on the contrary, that this is not the case. Then there exists a cocycle $q^* \neq \emptyset$ in G^* such that $q^* \subseteq Q^*$. So $q \subseteq Q$ is a bounding cycle. Since Q is a k-frame, we have $q = \emptyset$ and hence $q^* = \emptyset$, a contradiction. Thus R^* is a connected subgraph of G^*. We proceed to prove that R^* is an $(n-k)$-frame. Routine calculation involving the Euler-Poincaré formula and the surface duality yields $\beta(R^*) = n - k$. Thus it remains to show that no cycle of R^*, except zero, is bounding. To derive a contradiction, assume that R^* contains a non-zero bounding cycle z^*. It follows that z is a cocycle in G with $z \subseteq R = G - Q$. On the other hand, Q is a connected spanning subgraph of G, so it must contain an edge of z. Hence $Q \cap (G - Q) \neq \emptyset$, which is absurd. Thus we may conclude that $R^* = (G - Q)^*$ contains no bounding cycle $z \neq \emptyset$, that is, R^* is an $(n-k)$-frame. □

An immediate corollary of the above theorem is that the dual of a cotree of G is an n-frame of G^*. Since an n-frame is a connected spanning subgraph of the graph in question, this strengthens Theorem 1 of [4] (as well as similar results in [1, 2]): The dual of a cotree of G contains a spanning tree of G^*.

As far as n-frames are concerned, a little more can be said:

Theorem 2. *The following statements are equivalent:*

(i) *Q is an n-frame of G;*

(ii) *Q is a k-frame of G, $0 \leqslant k \leqslant n$, such that the induced embedding of Q in the surface S is cellular;*

(iii) *Q is a connected spanning subgraph of G such that the induced embedding of Q in S is cellular with one face.*

Proof. (i) $\Rightarrow$ (ii): We have to show that the induced embedding of an arbitrary n-frame is cellular. Assume that G divides S into r faces, and that Q is an n-frame of G. By Theorem 1, $R^* = (G - Q)^*$ is a 0-frame, that is, a spanning tree of G^*. Thus there are $r - 1$ edges of G not in Q, each of them lying on the boundary of two distinct faces. Now glue together the pairs of faces of G sharing a common boundary edge in $R = G - Q$. Note that the way of pasting the faces together is determined by the tree R^*. Since R^* is connected, without cycles, and each face of G is a 2-cell, the resulting space f is again a 2-cell. With some abuse of notation, $S - Q = f$, which means that the embedding of Q is cellular.

(ii) $\Rightarrow$ (iii): Assume that (ii) holds. Then $\beta(Q) \leqslant n$. Since the embedding of Q is cellular, the Euler-Poincaré formula implies the reverse inequality. Hence $\beta(Q) = n$ and the number of faces of Q is one.

(iii) $\Rightarrow$ (i): Let Q be a connected spanning subgraph of G for which the induced embedding is cellular with one face, denoted by f. From the Euler-Poincaré formula we have $\beta(Q) = n$. Furthermore, $\partial f = \emptyset$ and there are no other bounding cycles in Q. Therefore Q is an n-frame. This completes the proof. □

Note that if B is the set of face boundaries of G, less any one, and if U is a basis for the cycle space of an n-frame Q of G, then $B \cup U$ is a basis for the cycle of G. Indeed, B is independent since $B^* = \{C^* : C \in B\}$ is a basis for the cocycle space of

G^* by Theorem 0. Moreover, no cycle in U is bounding, so $B \cup U$ is independent as well, and $|B \cup U| = |B| + n = |B| + 2 - \chi(S) = \beta(G)$. This observation generalizes the well-known fact that if G is a graph embedded in the plane then B is the basis for the cycle space of G. A similar but more complicated generalization (leading to the same type of basis) can be found in [4, Theorem 2].

We now turn back to the general case. Our final result characterizes those edges of a k-frame that do not lie on a cycle. In fact, this is a higher surface version of the obvious fact that, for a spanning tree T of G and for any edge e of the corresponding cotree, $T + e$ contains a cycle.

Theorem 3. *Let Q be a k-frame of G, $0 \leqslant k \leqslant n$, and let e be an edge of $G - Q$. Then e^* is a cut-edge of the $(n-k)$-frame $(G-Q)^*$ if and only if $Q + e$ contains a non-zero bounding cycle.*

P r o o f. Let $R^* = (G - Q)^*$ be the $(n-k)$-frame corresponding to Q. First assume that $Q + e$ contains a non-zero bounding cycle z. Then e necessarily belongs to z. Theorem 0 implies that z^* is a cocycle of G^*, that is, $G^* - z^*$ is disconnected. Consequently, $R^* - e^* = (G^* - z^*) \cap R^*$ is disconnected, too, and hence e^* is a cut-edge of R^*.

For the converse, assume that e^* is a cut-edge of R^*. Let W be the vertex-set of one of the components of $R^* - e^*$. Consider the cocycle $q^* = \delta W$ in G^*. Then q is a bounding cycle of G. We show that $q \subseteq Q + e$. Since e^* is a cut-edge of R^*, we have $q^* \cap R^* = e^*$. Therefore $q^* - e^* \subseteq G^* - R^* = Q^*$, whence $q \subseteq Q + e$. Thus q is a non-zero bounding cycle contained in $Q + e$. □

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