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BIFURCATION OF PERIODIC SOLUTIONS TO DIFFERENTIAL INEQUALITIES IN $\mathbb{R}^3$

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1. Introduction

Consider the inequality

$$
\begin{aligned}
&U(t) \in K \text{ for } t \in [0,T),\\
&(\dot U(t) - A_\lambda U(t) - G(\lambda, U(t)), v - U(t)) \geqslant 0 \\
&\qquad\qquad\qquad\qquad \text{for all } v \in K, \text{ a.a. } t \in [0,T),
\end{aligned} \tag{1}
$$

where K is a closed convex cone with its vertex at the origin in $\mathbb{R}^3$, A_λ is a real 3×3 matrix depending continuously on a real parameter λ, $G\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous mapping locally lipschitzian in the variable u and satisfying the usual condition

$$
\lim_{u \to 0} \frac{G(\lambda, u)}{|u|} = 0 \quad \text{uniformly on compact } \lambda\text{-intervals.} \tag{2}
$$

Under certain assumptions concerning the eigenvalues of A_λ and a relation of the cone K to the eigenvectors of A_λ, we prove the existence of a bifurcation point λ_I at which periodic solutions to the inequality (1) bifurcate from the branch of trivial solutions. Main results of the paper are contained in Theorems 1, 2. While Theorem 1 contains the basic idea of our approach, Theorem 2 is in fact its consequence and can serve as a tool for verifying periodic bifurcation in examples (see Section 5). Both theorems are proved by elementary means. We investigate the solutions of (1) and those of the linearized inequality

$$
\begin{aligned}
&U(t) \in K \text{ for } t \in [0,+\infty),\\
&(\dot U(t) - A_\lambda U(t), v - U(t)) \geqslant 0 \text{ for all } v \in K, \text{ a.a. } t \in [0,+\infty).
\end{aligned} \tag{3}
$$

Note that a different approach to the investigation of bifurcations of periodic solutions to inequalities in $\mathbb{R}^n$ based on degree theory is described in [3], [4]. Further, recall that a bifurcation of stationary solutions to variational inequalities has been studied by several authors during the last 15 years (see e.g. [2], [5], [6], [8] and the references therein).

2. Main Results

Our assumptions concerning the matrix A_λ and the convex cone K will be the following: A_λ has eigenvalues $\alpha(\lambda)\pm i\beta(\lambda)$, $-\nu(\lambda)$ which depend continuously on $\lambda \in \mathbb{R}$ and eigenvectors $\bar{u}\pm i\bar{v}$, $\bar{w}$ independent of λ. Let $f_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, \ldots, N$ be convex functions continuously differentiable on $\mathbb{R}^2 \setminus \{[0,0]\}$ and satisfying $f_i(rx_1, rx_2) = rf_i(x_1, x_2)$, $i = 1, \ldots, N$ for all $r > 0$. We shall assume that the cone K is of the form

$$K = \{u \in \mathbb{R}^3;\, x_3 \geqslant f_i(x_1, x_2), i = 1, 2, \ldots, N\}, \tag{4}$$

where $x = [x_1, x_2, x_3]$ is the vector of the coordinates of u with respect to the basis $\{\bar{u}, \bar{v}, \bar{w}\}$, i.e. $u = x_1\bar{u} + x_2\bar{v} + x_3\bar{w}$. Moreover, we assume that

$$K \neq \{u \in \mathbb{R}^3;\, x_3 \geqslant 0\}, \tag{5}$$

i.e. not all the functions f_i are zero, and also that near any point $v \in K, v \neq 0$ the cone K can be locally described in terms of at most two of the functions $f_1, \ldots, f_N$. More precisely, we impose the following condition on K:

$$\begin{gathered}\text{for any } v \in K,\ v \neq 0 \text{ there exist a pair of indices } 1 \leqslant i, j \leqslant N\\ \text{and an open neighbourhood } W \text{ of the point } v \text{ such that}\\ W \cap K = \{u \in W;\, x_3 \geqslant f_i(x_1, x_2), x_3 \geqslant f_j(x_1, x_2)\}.\end{gathered} \tag{6}$$

Remark 1. By a solution of inequality (1) on $[0, T)$ we mean an absolutely continuous function satisfying (1). The following assertions are obtained by standard considerations from the existence results for general differential inclusions [1]. For any $u \in K$, $\lambda \in \mathbb{R}$ the solution of (1) satisfying $U(0) = u$ exists and is unique at least on some interval $[0, T)$, $T > 0$. This solution will be denoted by $U_\lambda(t, u)$. If $T_0 > 0$ and $U_\lambda(t, u)$ is bounded on any subinterval $[0, T)$ of $[0, T_0)$ on which it exists then $U_\lambda(t, u)$ exists on $[0, T_0)$. This together with simple a priori estimates (see Lemma 2.1 in [4]) imply that for any $T > 0$, $\Lambda > 0$ there is $R > 0$ such that $U_\lambda(t, u)$ exists on $[0, T)$ for any $u \in K$, $|u| \leqslant R$, $|\lambda| \leqslant \Lambda$. Particularly, for any $u \in K$, $\lambda \in \mathbb{R}$ there exists a unique solution of (3) satisfying $U(0) = u$ on the whole interval $[0, +\infty)$. It will be denoted by $U_{\lambda,0}(t, u)$.

The symbol $(\cdot,\cdot)$ will stand for the usual inner product in $\mathbf{R}^3$ with the corresponding norm denoted by $|\cdot|$. We denote by $\langle\cdot,\cdot\rangle$ the inner product $\langle u,v\rangle=(x,y)$, where x,y are the vectors of the coordinates of u,v with respect to the basis $\{\bar u,\bar v,\bar w\}$.

We set

$$S=\{r\bar w;\ r\in\mathbf{R}\}.$$

Any continuous function $U\colon[0,T]\to\mathbf{R}^3\setminus S$ can be uniquely written as

$$U(t)=\varrho(t)[\cos(\varphi_0-\varphi(t))\bar u+\sin(\varphi_0-\varphi(t))\bar v]+X_3(t)\bar w,$$

where $\varphi_0\in[0,2\pi)$, $\varrho(t)>0$, $\varphi(t)$, $X_3(t)$ are continuous functions defined on $[0,T]$ and φ satisfies $\varphi(0)=0$. Hence, for any $u\in K\setminus S$, $\lambda\in\mathbf{R}$ we can define $\varphi_\lambda(t,u)$ as the function $\varphi(t)$ corresponding to $U(t)=U_\lambda(t,u)$ on an interval $[0,T)$ on which $U_\lambda(t,u)\notin S$. Similarly, we define $\varphi_{\lambda,0}(t,u)$ as the function $\varphi(t)$ corresponding to $U_{\lambda,0}(t,u)$ on $[0,+\infty)$ (see also Lemma 2,(I)).

Remark 2. Let $U(t)=U_\lambda(t,u)\notin S$ for all $t\in[0,T]$ and let $X(t)$ be the vector of the coordinates of $U(t)$ with respect to the basis $\{\bar u,\bar v,\bar w\}$, i.e. $U(t)=X_1(t)\bar u+X_2(t)\bar v+X_3(t)\bar w$. It follows easily from the definition of $\varphi_\lambda(t,u)$ that

$$\dot\varphi_\lambda(t,u)=\frac{\langle\dot U(t),X_2(t)\bar u-X_1(t)\bar v\rangle}{X_1^2(t)+X_2^2(t)},\quad t\in[0,T).$$

For $u\in K\setminus S$, $\lambda\in\mathbf{R}$ we define

$$T(\lambda,u)=\inf\{t>0;\ \varphi_\lambda(t,u)=2\pi\}$$

and use the symbol $T_0(\lambda,u)$ in the linearized case (3). We note that $T(\lambda,u)=+\infty$ if one of the following three cases occurs:

$\varphi_\lambda(t,u)<2\pi$ for all $t>0$;

there exists $T>0$ such that $\varphi_\lambda(t,u)<2\pi$ for all $t\in[0,T)$ and $U_\lambda(T,u)\in S$;

$U_\lambda(t,u)$ is defined only on $[0,T)$ and $\varphi_\lambda(t,u)<2\pi$ for all $t\in[0,T)$.

Consider the inequality

$$\begin{aligned}&u\in K,\\&(\mu u-A_\lambda u,v-u)\geqslant 0\text{ for all }v\in K.\end{aligned}\tag{7}$$

A real number μ is called an eigenvalue of the inequality (7) (for a given $\lambda\in\mathbf{R}$) if there exists a nontrivial u satisfying (7). Any such u is called an eigenvector of (7) corresponding to μ.

We define

$$g(u)=\frac{x_3}{\sqrt{x_1^2+x_2^2}} \text{ for } u\notin S, u=x_1\bar{u}+x_2\bar{v}+x_3\bar{w},$$

$$\tau=\max\{g(u);\, 0\neq u\in\partial K\}.$$

Remark 3. In any cone K of the form (4) there exists at least one vector v satisfying

$$0\neq v\in\partial K,\ g(v)=\tau. \tag{8}$$

(This v represents the ray which is the closest one to S with respect to $\langle\cdot,\cdot\rangle$ among those lying on ∂K.)

We denote by $T_K(u)$ the contingent cone to K at a point $u\in K$, i.e.

$$T_K(u)=\operatorname{cl}\Big(\bigcup_{h>0}\bigcup_{v\in K}h(v-u)\Big).$$

Theorem 1. *Let $[\lambda_1,\lambda_2]\subset\mathbf{R}$ be an interval and v an arbitrary fixed element satisfying* (8). *Assume*

$$T_0(\lambda,v)<+\infty \quad \text{for } \lambda_1\leqslant\lambda\leqslant\lambda_2, \tag{9}$$

$$\alpha(\lambda)+\nu(\lambda)>0 \quad \text{for } \lambda_1\leqslant\lambda\leqslant\lambda_2, \tag{10}$$

$$\beta(\lambda)>0 \quad \text{for } \lambda_1\leqslant\lambda\leqslant\lambda_2, \tag{11}$$

$$|U_{\lambda,0}(T_0(\lambda,v),v)|<|v| \quad \text{for } \lambda=\lambda_1, \tag{12}$$

$$|U_{\lambda,0}(T_0(\lambda,v),v)|>|v| \quad \text{for } \lambda=\lambda_2. \tag{13}$$

Then for any sufficiently small $r>0$ there exists $\lambda\in(\lambda_1,\lambda_2)$ such that $U_\lambda(\cdot,rv)$ is a periodic solution of the inequality (1). *There is at least one bifurcation point $\lambda_I\in(\lambda_1,\lambda_2)$ at which periodic solutions of* (1) *bifurcate from the branch of trivial solutions.*

Idea of the proof of Theorem 1 (see Section 4 for details). The conditions (9), (10), (11) and Lemmas 2, 3 enable us to prove that the solution of the linearized inequality (3) starting from the particular initial condition v satisfies $\dot{\varphi}_{\lambda,0}(T_0(\lambda,v),v)>0$ when $\lambda\in[\lambda_1,\lambda_2]$. As a result, Lemma 1,(vi) implies $T(\lambda,rv)<+\infty$ for all $\lambda\in[\lambda_1,\lambda_2]$ and $r>0$ small. Combining Lemma 3 and Remark 5 we conclude that $U_\lambda(T(\lambda,rv),rv)=k(\lambda,r)v$ where $k(\lambda,r)$ is a positive function defined on $[\lambda_1,\lambda_2]\times(0,R)$. The conditions (12), (13) ensure $k(\lambda_1,r)<r<k(\lambda_2,r)$. Since k is continuous in the variable λ we obtain for any sufficiently small $r>0$ a value $\lambda\in[\lambda_1,\lambda_2]$ such that $k(\lambda,r)=r$. Thus we get $U_\lambda(T,rv)=rv$ where $T=T(\lambda,rv)$ and rv is the initial condition of a periodic solution. □

Theorem 2. *Let $[\Lambda_1, \Lambda_2] \subset \mathbf{R}$ be an arbitrary interval. Assume*

$$\alpha(\lambda) + \nu(\lambda) = 0, \alpha(\lambda) < 0 \quad \text{for } \lambda = \Lambda_1, \tag{14}$$

$$\alpha(\lambda) + \nu(\lambda) > 0 \quad \text{for } \Lambda_1 < \lambda \leqslant \Lambda_2, \tag{15}$$

$$\beta(\lambda) > 0 \quad \text{for } \Lambda_1 \leqslant \lambda \leqslant \Lambda_2, \tag{16}$$

$$0 \neq u \in \partial K \Longrightarrow A_\lambda u \notin T_K(u) \quad \text{for } \lambda = \Lambda_2, \tag{17}$$

$$0 \neq u \in \partial K \Longrightarrow (A_\lambda u, u) > 0 \quad \text{for } \lambda = \Lambda_2. \tag{18}$$

In addition, assume $\mu > 0$ whenever μ is an eigenvalue of (7) corresponding to an eigenvector $u \in \partial K$ for some $\lambda \in [\Lambda_1, \Lambda_2]$.

Then to any sufficiently small $r > 0$ there exist $\lambda \in (\Lambda_1, \Lambda_2)$ and $u \in K$, $|u| = r$ such that $U_\lambda(\cdot, u)$ is a periodic solution of (1).

Idea of the proof of Theorem 2 (see Section 4 for details). We shall find an interval $[\lambda_1, \lambda_2] \subset [\Lambda_1, \Lambda_2]$ for which the assumptions (9)–(13) are fulfilled. As in Theorem 1 the solutions of the inequality (3) starting at v are investigated. First we prove by using (14) that the solution $U_{\lambda,0}(t, v)$ of the inequality (3) with $\lambda = \Lambda_1$ is simultaneously a solution of the linear differential equation $\dot{U}(t) = A_\lambda U(t)$. Making use of the explicit form of this solution (see Remark 4) and of Lemma 1 we find $T_0(\lambda, v) < +\infty$ and $|U_{\lambda,0}(T_0(\lambda, v), v)| < |v|$ for all λ close to Λ_1. Hence λ_1 satisfying (12) is obtained. To find λ_2 we consider two cases: either $T_0(\lambda, v) < +\infty$ for all $\lambda \in [\Lambda_1, \Lambda_2]$ or there is a $\delta \in (\Lambda_1, \Lambda_2]$ such that $T_0(\delta, v) = +\infty$ and $T_0(\lambda, v) < +\infty$ for all $\lambda \in [\Lambda_1, \delta)$. In the first case we use the assumptions (17), (18) and Lemma 4 to get the inequality $|U_{\lambda,0}(T_0(\lambda, v), v)| > |v|$ for $\lambda = \Lambda_2$ and we can put $\lambda_2 = \Lambda_2$. In the case of $T_0(\delta, v) = +\infty$ we use Lemma 2 to prove

$$\frac{U_{\delta,0}(t, v)}{|U_{\delta,0}(t, v)|} \to u \text{ for } t \to +\infty$$

where $u \in \partial K$ is an eigenvector of (7). By our assumption, the corresponding eigenvalue μ is positive, which permits us to show $|U_{\delta,0}(t, v)| \to +\infty$ as $t \to +\infty$. This in turn leads to the inequality (13) with some $\lambda_2 < \delta$, λ_2 close to δ. □

3. Some General Remarks

Let $C \subset \mathbf{R}^3$ be a nonempty closed convex set and $w \in \mathbf{R}^3$ an arbitrary vector. The nearest point (with respect to the norm $|\cdot|$) to w in the set C will be hereafter referred to as the projection of w onto C.

We introduce some additional notation:

$K_i = \{u \in \mathbb{R}^3;\ x_3 \geqslant f_i(x_1, x_2)\}$, $1 \leqslant i \leqslant N$,

$T_i(u)$ for $u \in K_i$ is the contingent cone to K_i at a point u,

$n_i(u)$ is the unit inner normal to ∂K_i at a point $u \in \partial K_i$,

$P_u w$ for $u \in K$, $w \in \mathbb{R}^3$ is the projection of w onto $T_K(u)$,

$P_u^i w$ for $u \in K_i$, $w \in \mathbb{R}^3$ is the projection of w onto $T_i(u)$,

L is the 3×3 matrix with columns $\bar{u}, \bar{v}, \bar{w}$ and $B_\lambda = L^{-1} A_\lambda L$ is the canonical form of A_λ, i.e.

$$B_\lambda = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) & 0 \\ -\beta(\lambda) & \alpha(\lambda) & 0 \\ 0 & 0 & -\nu(\lambda) \end{pmatrix}.$$

While points in $\mathbb{R}^3$ are usually denoted by $u = [u_1, u_2, u_3]$, vector functions with values in $\mathbb{R}^3$ are denoted for instance by $U(t) = [U_1(t), U_2(t), U_3(t)]$. Throughout the paper the symbols $\dot{U}(t)$, $\dot{U}_\lambda(t, u)$, $\dot{\varphi}_{\lambda,0}(t, u)$ etc. denote the right derivatives of the corresponding functions.

Remark 4. Let $U(t) = X_1(t)\bar{u} + X_2(t)\bar{v} + X_3(t)\bar{w}$, $X(t) = [X_1(t), X_2(t), X_3(t)]$ be the solution of the equation $\dot{U}(t) = A_\lambda U(t)$ with the initial condition $U(0) = v$.

Then $\dot{X}(t) = B_\lambda X(t)$, $t \geqslant 0$ and

$$\begin{aligned} X_1(t) &= \mathrm{e}^{\alpha(\lambda)t}(X_1(0)\cos\beta(\lambda)t + X_2(0)\sin\beta(\lambda)t), \\ X_2(t) &= \mathrm{e}^{\alpha(\lambda)t}(X_2(0)\cos\beta(\lambda)t - X_1(0)\sin\beta(\lambda)t), \\ X_3(t) &= \mathrm{e}^{-\nu(\lambda)t}X_3(0). \end{aligned} \tag{19}$$

Remark 5. Let $v \in K$ satisfy (8) and let $T(\lambda, v) < +\infty$ for some $\lambda \in \mathbb{R}$. Then

(i) $g(U_\lambda(T(\lambda, v), v)) \leqslant \tau$ implies $U_\lambda(T(\lambda, v), v) = kv$ with some $k > 0$.

For any $u \in \mathbb{R}^3 \setminus S$

(ii) $g(u) \geqslant \tau$ implies $u \in K$ and $g(u) > \tau$ implies $u \in \operatorname{int} K$.

The proof of these assertions follows directly from the definitions of the function g and of the number τ.

Remark 6. Let $u \in K$, $w \in T_K(u)$, $z \in \mathbb{R}^3$. Then it is easy to see that

$$w = P_u z \Longleftrightarrow (w - z, x - w) \geqslant 0 \quad \text{for all } x \in T_K(u). \tag{20}$$

Thus it follows from the definition of the cone $T_K(u)$ that $P_u z$ is the unique point in $T_K(u)$ with the property

$$\begin{aligned} &(P_u z - z, P_u z) = 0, \\ &(P_u z - z, v - u) \geqslant 0 \text{ for all } v \in K. \end{aligned} \tag{21}$$

R e m a r k 7. An absolutely continuous function $U\colon [0,T) \to K$ is a solution of the inequality (1) if and only if

$$\dot{U}(t) = P_{U(t)}(A_\lambda U(t) + G(\lambda, U(t))) \text{ for a. a. } t \in [0,T) \tag{22}$$

(see [1]).

R e m a r k 8. Any solution $U\colon [0,T) \to K$ of (1) is right differentiable, its right derivative is right continuous in the interval $[0,T)$ and the equation (22) holds for all $t \in [0,T)$. For the proof see [7].

R e m a r k 9. Any eigenvalue μ of the inequality (7) with the corresponding eigenvector u satisfies

$$\mu|u|^2 = (A_\lambda u, u).$$

Further, it follows from Remark 6 that for any $u \in K$ and $\mu \in \mathbf{R}$ the inequality (7) is equivalent to

$$\mu u = P_u A_\lambda u.$$

R e m a r k 10. Suppose that at a point $t = t_0$ the solution $U(t) = U_{\lambda,0}(t,u)$ of the inequality (3) satisfies the equation $\dot{U}(t) = A_\lambda U(t)$. Then $\dot{\varphi}_{\lambda,0}(t_0, u) = \beta(\lambda)$ (see Remark 4). This occurs for instance when $U_{\lambda,0}(t_0,u) \in \operatorname{int} K$. More generally, it follows from Remark 8 that if $U(t)$ is a solution of (1) such that $U(t) \in \operatorname{int} K$ for all $t \in [t_1, t_2]$ then the equation $\dot{U}(t) = A_\lambda U(t) + G(\lambda, U(t))$ holds on this interval.

R e m a r k 11. For any solution $U\colon [0,T) \to K$ of the inequality (3) we have

$$(\dot{U}(t) - A_\lambda U(t), U(t)) = 0, \ t \in [0,T).$$

Lemma 1. *To any $T > 0$, $\Lambda > 0$ there exists $R > 0$ such that for any sequences $\lambda_n \to \lambda$, $|\lambda| < \Lambda$, $u_n \in K$, $u_n \to u$, $|u| < R$ we have*

(i) *$U_{\lambda_n}(\cdot, u_n) \to U_\lambda(\cdot, u)$ uniformly on $[0,T]$,*

(ii) *if $U_\lambda(t,u) \notin S$ for $t \in [0,T]$ then $\varphi_{\lambda_n}(\cdot, u_n) \to \varphi_\lambda(\cdot, u)$ uniformly on $[0,T]$,*

(iii) *if $T(\lambda, u) < T, \dot{\varphi}_\lambda(T(\lambda,u),u) > 0$ then $T(\lambda_n, u_n) \to T(\lambda, u)$.*

Let $\lambda_n \to \lambda \in \mathbf{R}$, $0 \neq u_n \in K$, $u_n \to 0$, $\frac{u_n}{|u_n|} \to w \in \mathbf{R}^3$, let $T > 0$ be arbitrary. Then

(iv) *$\dfrac{U_{\lambda_n}(\cdot, u_n)}{|u_n|} \to U_{\lambda,0}(\cdot, w)$ uniformly on $[0,T]$,*

(v) *if $w \notin S$ then $\varphi_{\lambda_n}(\cdot, u_n) \to \varphi_{\lambda,0}(\cdot, w)$ uniformly on $[0,T]$,*

(vi) *if $w \notin S$, $T_0(\lambda, w) < +\infty$ and $\dot{\varphi}_{\lambda,0}(T_0(\lambda,w),w) > 0$ then $T(\lambda_n, u_n) \to T_0(\lambda, w)$.*

For the proof see Theorems 2.1, 2.2 and Consequence 2.2 in [4].

Observation 1. *Let* $u = x_1\bar{u} + x_2\bar{v} + x_3\bar{w} \in \partial K \setminus \{0\}$ *and* $w \in \partial T_i(u)$ *for some* i, $1 \leqslant i \leqslant N$. *Then* $\langle w, x_2\bar{u} - x_1\bar{v}\rangle = 0$ *implies* $w = \mu u$, $\mu \in \mathbf{R}$.

Observation 2. *Let* $u_n \in K$, $u_n \to u$. *Then for any vector* $v \in T_K(u)$ *there exists a sequence* $v_n \to v$ *satisfying* $v_n \in T_K(u_n)$, $n = 1, 2, \ldots$.

The proof of this observation follows from results proved in [1].

Observation 3. *Let* $u_n \in K$, $z_n \in \mathbf{R}^3$, $u_n \to u$, $z_n \to z$. *Then the following implications hold:*

(i) *If* $P_{u_n}z_n \to w$, $P_{u_n}z_n \in \partial T_K(u_n)$, $n = 1, 2, \ldots$ *then* $w \in \partial T_i(u)$ *for some* i, $1 \leqslant i \leqslant N$.

(ii) *If* $P_{u_n}z_n \to w$, $w \in T_K(u)$ *then* $w = P_u z$.

(iii) *If there exists* j, $0 \leqslant j \leqslant N$ *such that* $u, u_n \in \partial K_1 \cap \partial K_2 \cap \ldots \cap \partial K_j \cap \operatorname{int} K_{j+1} \cap \ldots \cap \operatorname{int} K_N$, $n = 1, 2, \ldots$ *then* $P_{u_n}z_n \to P_u z$.

P r o o f. (i) Since $P_{u_n}z_n \in \partial T_K(u_n)$ we have $(P_{u_n}z_n, n_{i_n}(u_n)) = 0$ with some $1 \leqslant i_n \leqslant N$, $n = 1, 2, \ldots$. We may suppose that the sequence i_n is constant and therefore

$$(P_{u_n}z_n, n_i(u_n)) = 0, \ n = 1, 2, \ldots.$$

From the continuity of the normal $n_i(\cdot)$ we conclude $(w, n_i(u)) = 0$ and therefore $w \in \partial T_i(u)$.

(ii) Take an arbitrary $v \in T_K(u)$. Observation 2 implies $v_n \to v$ for some sequence $v_n \in T_K(u_n), n = 1, 2, \ldots$. We have

$$|v_n - z_n| \geqslant |P_{u_n}z_n - z_n|$$

and consequently $|v - z| \geqslant |w - z|$. This inequality, holding for all $v \in T_K(u)$, together with $w \in T_K(u)$ implies $w = P_u z$.

(iii) The case $j = 0$ is trivial. Let $j \geqslant 1$. As $|P_{u_n}z_n| \leqslant |z_n|$ and z_n is convergent, the sequence $P_{u_n}z_n$ is bounded. Therefore it is sufficient to prove the implication

$$P_{u_n}z_n \to w \Longrightarrow w = P_u z.$$

However, for $n = 1, 2, \ldots$ we have

$$(P_{u_n}z_n, n_i(u_n)) \geqslant 0, \ i = 1, 2, \ldots, j.$$

Consequently, $(w, n_i(u)) \geqslant 0$, $i = 1, 2, \ldots, j$, and w belongs to $T_K(u) = T_1(u) \cap \ldots \cap T_j(u)$. Now we use (ii) to prove $w = P_u z$. $\square$

Observation 4. *Let $u \in K$, $w \in \mathbf{R}^3$ be arbitrary vectors.*

If $P_u w \in \operatorname{int} T_{j+1}(u) \cap \ldots \cap \operatorname{int} T_N(u)$ where $1 \leqslant j \leqslant N-1$ then $P_u w$ coincides with the projection of w onto $T_1(u) \cap T_2(u) \cap \ldots \cap T_j(u)$.

Further, $P_u w = w$ whenever $P_u w \in \operatorname{int} T_K(u)$.

Proof. Denote by Π the set $T_1(u) \cap \ldots \cap T_j(u)$. We have $P_u w \in \Pi$ and therefore it is sufficient to prove $(P_u w - w, x - P_u w) \geqslant 0$ for all $x \in \Pi$ (see Remark 6). Choose $x \in \Pi$. Then $(1-t)P_u w + tx \in \Pi$, $0 \leqslant t \leqslant 1$. Moreover,

$$(1-t)P_u w + tx \in T_{j+1}(u) \cap \ldots \cap T_N(u) \quad \text{for } t > 0,\ t \text{ small.}$$

Hence $P_u w + t(x - P_u w) \in T_K(u)$ for some $t > 0$. Since $P_u w$ is the projection of w onto $T_K(u)$ we have

$$(P_u w - w, x - P_u w) = \frac{1}{t}(P_u w - w, P_u w + t(x - P_u w) - P_u w) \geqslant 0.$$

□

4. Proof of Main Results

Lemma 2. *Let $\lambda \in \mathbf{R}$, $\beta(\lambda) > 0$, and let $v \in K \setminus S$. Then*

(I) *$U_{\lambda,0}(t, v) \notin S$ for all $t > 0$,*

(II) *if $\dot{\varphi}_{\lambda,0}(t_0, v) = 0$ then $U_{\lambda,0}(t_0, v)$ is an eigenvector of (7) and $\dot{\varphi}_{\lambda,0}(t, v) = 0$ for all $t > t_0$,*

(III) *if*

$$\lim_{t \to +\infty} \varphi_{\lambda,0}(t, v) = \varphi \tag{23}$$

then

$$\lim_{t \to +\infty} \frac{U_{\lambda,0}(t, v)}{|U_{\lambda,0}(t, v)|} = u \in \partial K \tag{24}$$

where u is an eigenvector of (7),

(IV) *if $\dot{\varphi}_{\lambda,0}(t_0, v) \leqslant 0$ for some $t_0 \geqslant 0$ then $\dot{\varphi}_{\lambda,0}(t, v) \leqslant 0$ for all $t \geqslant t_0$,*

(V) *if $T_0(\lambda, v) < +\infty$ then $\dot{\varphi}_{\lambda,0}(t, v) > 0$ for all $t \in [0, T_0(\lambda, v))$.*

Proof. Throughout the proof we shall write $U(t) = U_{\lambda,0}(t, v)$, $\varphi(t) = \varphi_{\lambda,0}(t, v)$.

(I) If the statement were false there would exist $t_0 > 0$ such that $U(t_0) \in S$, $U(t) \notin S$ for all $t \in [0, t_0)$. Remark 11 implies

$$\frac{\mathrm{d}}{\mathrm{d}t}(|U(t)|^2) = 2(\dot{U}(t), U(t)) = 2(A_\lambda U(t), U(t)) \geqslant -C|U(t)|^2$$

with some $C > 0$. Thus $|U(t)|^2 \geqslant \mathrm{e}^{-Ct}|v|^2$ and therefore $U(t) \neq 0$ for all $t > 0$. Now it follows from the assumption (4) that $U(t_0) \in \operatorname{int} K$. Therefore $U(t)$ is also a solution of the equation $\dot{U}(t) = A_\lambda U(t)$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$, $\varepsilon > 0$ small. However, one can see from Remark 4 that no solution to this equation starting from a point $u \notin S$ can reach S in a finite time.

(II) Let $u = U(t_0)$, $w = \dot{U}(t_0)$ and let $u = x_1\bar{u} + x_2\bar{v} + x_3\bar{w}$. It follows from (I) that $u \notin S$, and Remark 2 yields

$$\langle w, x_2\bar{u} - x_1\bar{v}\rangle = 0. \tag{25}$$

We have $w \in T_K(u)$ by Remarks 7, 8. We shall prove $w \in \partial T_K(u)$. Indeed, if $w \in \operatorname{int} T_K(u)$ we would obtain from Remark 8

$$P_u A_\lambda u = \dot{U}(t_0) \in \operatorname{int} T_K(u),$$

and Observation 4 would imply $P_u A_\lambda u = A_\lambda u$. Hence $\dot{U}(t_0) = A_\lambda U(t_0)$ and Remark 10 would yield $\dot{\varphi}(t_0) = \beta(\lambda) > 0$.

Now $w \in \partial T_K(u)$ implies $w \in \partial T_i(u)$ for some i, $1 \leqslant i \leqslant N$ and thus Observation 1 together with (25) yields $P_u A_\lambda u = w = \mu u$ with some $\mu \in \mathbf{R}$. By Remark 9 we conclude that u is an eigenvector of (7).

Let us set $V(t) = \mathrm{e}^{\mu t}u$ and prove that $V(t) = U_{\lambda,0}(t,u)$. Indeed, using (7) we get

$$\begin{aligned}(\dot{V}(t) - A_\lambda V(t), z - V(t)) &= (\mu\mathrm{e}^{\mu t}u - \mathrm{e}^{\mu t}A_\lambda u, z - \mathrm{e}^{\mu t}u)\\ &= \mathrm{e}^{2\mu t}(\mu u - A_\lambda u, \mathrm{e}^{-\mu t}z - u) \geqslant 0\\ &\qquad \text{for all } z \in K,\ t \geqslant 0.\end{aligned}$$

Consequently, since $V(0) = U(t_0)$, we have $\dot{U}(t) = \dot{V}(t-t_0) = \mu\mathrm{e}^{\mu(t-t_0)}u = \mathrm{e}^{\mu(t-t_0)}w$ for $t \geqslant t_0$ and so the statement follows from (25) by Remark 2.

(III) To prove that the limit in (24) exists we shall verify that there is exactly one $u \in \mathbf{R}^3$ that satisfies

$$\frac{U(t_n)}{|U(t_n)|} \to u \text{ for some } t_n \to +\infty. \tag{26}$$

Let us prove that (26) implies $u \in \partial K$. Suppose there is $u \in \operatorname{int} K$ satisfying (26). Then $U_{\lambda,0}(t,u) \in \operatorname{int} K$ for all t in a small interval $[0,T]$ and Lemma 1,(i) yields

$$U_{\lambda,0}\left(t, \frac{U(t_n)}{|U(t_n)|}\right) \in \operatorname{int} K,\ t \in [0,T]$$

for n sufficiently large. Hence $U(t+t_n) = U_{\lambda,0}(t, U(t_n)) \in \operatorname{int} K$, $t \in [0,T]$ and therefore $\dot{U}(t) = A_\lambda U(t), t \in [t_n, t_n+T)$. By Remark 10

$$\varphi(t_n+T) - \varphi(t_n) = \int_0^T \dot{\varphi}(t_n+t)\mathrm{d}t = \int_0^T \beta(\lambda)\mathrm{d}s = T\beta(\lambda) > 0,$$

which is a contradiction as (23) yields $\varphi(t_n+T) - \varphi(t_n) \to 0$ for $n \to +\infty$. We have proved that (26) implies $u \in \partial K$. Finally, it follows from (4) that there is exactly one vector $u \in \partial K$ with a given argument (determined by (23)) and a given norm $|u| = 1$.

To show that u is an eigenvector of (7) we shall prove $\dot{\varphi}_{\lambda,0}(0,u) = 0$ and then use (II). Suppose for a moment that $\dot{\varphi}_{\lambda,0}(0,u) > 0$. Then $\varphi_{\lambda,0}(T,u) > 0$ for some $T > 0$ and Lemma 1 together with (24) yields

$$0 < \varepsilon < \varphi_{\lambda,0}\left(T, \frac{U(t)}{|U(t)|}\right) = \varphi_{\lambda,0}(T, U(t))$$

for t large and some $\varepsilon > 0$. Since $\varphi_{\lambda,0}(0,w) = 0$ for all $w \in K \setminus S$ we have $\varphi_{\lambda,0}(T,U(t)) = \varphi(t+T) - \varphi(t)$ and so the last inequality contradicts (23). By excluding in a similar way the inequality $\dot{\varphi}_{\lambda,0}(0,u) < 0$ we complete the proof of (III).

(IV) It follows from (I) (and Remark 8) that $\varphi(t), \dot{\varphi}(t)$ are defined for all $t \geqslant 0$. We set $t_1 = \inf\{t > t_0 : \dot{\varphi}(t) > 0\}$ and suppose $t_0 \leqslant t_1 < +\infty$. It follows from Remark 8 that $\lim_{t \to t_1+} \dot{\varphi}(t) = \dot{\varphi}(t_1)$ and so $\dot{\varphi}(t_1) \geqslant 0$. On the other hand, if $\dot{\varphi}(\bar{t}) = 0$ for some $\bar{t} \in [t_0, t_1]$ we would obtain from (II) that $\dot{\varphi}(t) = 0$ for all $t \geqslant \bar{t}$ which would contradict the assumption $t_1 < +\infty$.

Thus we are left with the situation

$$\dot{\varphi}(t_1) > 0, \tag{27}$$

$$\dot{\varphi}(t) < 0, \quad t \in [t_0, t_1). \tag{28}$$

To show that (27) and (28) contradict each other we shall prove

$$\lim_{t \to t_1-} \dot{U}(t) = \dot{U}(t_1) \tag{29}$$

and therefore

$$\lim_{t \to t_1-} \dot{\varphi}(t) = \dot{\varphi}(t_1). \tag{30}$$

First, note that because of (6) we may suppose

$$K = \{u \in \mathbb{R}^3;\, x_3 \geqslant f_i(x_1,x_2), x_3 \geqslant f_j(x_1,x_2)\} \tag{31}$$

where i, j are not necessarily distinct indices. Indeed, all our considerations will be confined to a suitable neighborhood of the point $u = U(t_1)$. Now Remark 10 and (28) imply $U(t) \in \partial K$ for all $t \in [t_0, t_1)$ and therefore $U(t_1) \in \partial K$. Moreover, we may suppose $x_3 = f_i(x_1, x_2) = f_j(x_1, x_2)$. Indeed, if $f_j(x_1, x_2) < x_3$ then $u \in \partial K$ would imply $f_i(x_1, x_2) = x_3$ and we could take $i = j$ in (31). Thus the normals $n_i(u)$, $n_j(u)$ are defined and we first consider the case where $n_i(u) = n_j(u)$. We have $T_K(u) = T_i(u) = T_j(u)$ and therefore $P_u A_\lambda u = P_u^i A_\lambda u = P_u^j A_\lambda u$. To prove (29) it is sufficient to show $P_{u_n} A_\lambda u_n \to P_u A_\lambda u$ whenever $u_n \to u$, $u_n \in \partial K, n = 1, 2, \ldots$ (see Remark 8). The continuity of the normals n_i, n_j implies

$$\begin{array}{lcl} u_n \in \partial K_i \cap \operatorname{int} K_j & \Longrightarrow & P_{u_n} A_\lambda u_n = P_{u_n}^i A_\lambda u_n \quad \to \quad P_u^i A_\lambda u, \\ u_n \in \operatorname{int} K_i \cap \partial K_j & \Longrightarrow & P_{u_n} A_\lambda u_n = P_{u_n}^j A_\lambda u_n \quad \to \quad P_u^j A_\lambda u. \end{array}$$

Recalling Observation 3, (iii) we find

$$u_n \in \partial K_i \cap \partial K_j \Longrightarrow P_{u_n} A_\lambda u_n \to P_u A_\lambda u$$

and (29) is proved.

Finally, let us deal with the case $n_i(u) \neq n_j(u)$. We set

$$\begin{aligned} a &= \left[-\frac{\partial f_i}{\partial x_1}(x), -\frac{\partial f_i}{\partial x_2}(x), 1\right], \\ b &= \left[-\frac{\partial f_j}{\partial x_1}(x), -\frac{\partial f_j}{\partial x_2}(x), 1\right], \\ c &= [x_2, -x_1, 0], \end{aligned}$$

where $u = x_1 \bar{u} + x_2 \bar{v} + x_3 \bar{w}$. (Note that a, b are normals to ∂K_i, ∂K_j with respect to $\langle \cdot, \cdot \rangle$.) Assume for a moment that $(a, c) = (b, c)$. Then $(a - b, c) = 0$ and it follows from the properties of the functions f_i, f_j that $(a - b, x) = 0$, $(a - b, [0, 0, 1]) = 0$. Thus the vector $a - b$ would be orthogonal to three independent vectors and therefore would equal zero. However, the assumption $n_i(u) \neq n_j(u)$ implies $a \neq b$. Hence $(a, c) \neq (b, c)$. We can assume $(a, c) < (b, c)$ and write this inequality as

$$\begin{aligned} -\frac{\partial f_i}{\partial x_1} \sin(\varphi_0 - \varphi(t_1)) &+ \frac{\partial f_i}{\partial x_2} \cos(\varphi_0 - \varphi(t_1)) \\ &< -\frac{\partial f_j}{\partial x_1} \sin(\varphi_0 - \varphi(t_1)) + \frac{\partial f_j}{\partial x_2} \cos(\varphi_0 - \varphi(t_1)), \end{aligned}$$

where $x_1 = \varrho \cos(\varphi_0 - \varphi(t_1)), x_2 = \varrho \sin(\varphi_0 - \varphi(t_1))$. Hence we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} f_i(\cos(\varphi_0 - \varphi), \sin(\varphi_0 - \varphi)) > \frac{\mathrm{d}}{\mathrm{d}\varphi} f_j(\cos(\varphi_0 - \varphi), \sin(\varphi_0 - \varphi))$$

at the point $\varphi = \varphi(t_1)$. Consequently,

$$f_i(\cos(\varphi_0 - \varphi), \sin(\varphi_0 - \varphi)) > f_j(\cos(\varphi_0 - \varphi), \sin(\varphi_0 - \varphi)) \tag{32}$$

whenever $\varphi > \varphi(t_1)$ and φ is sufficiently close to $\varphi(t_1)$. It follows from (27), (28) that the function $\varphi(t)$ attains its strict local minimum at the point $t = t_1$. Taking (4) into account we obtain from (32) an $\varepsilon > 0$ satisfying

$$U(t) \in \operatorname{int} K_j,\ t \in (t_1 - \varepsilon, t_1) \cup (t_1, t_1 + \varepsilon). \tag{33}$$

Consequently, $T_K(U(t)) = T_i(U(t))$ and

$$P_{U(t)} A_\lambda U(t) = P^i_{U(t)} A_\lambda U(t) \text{ a.e. on } (t_1 - \varepsilon, t_1 + \varepsilon). \tag{34}$$

By Remark 7 we conclude that the function $U(t)$ on $(t_1 - \varepsilon, t_1 + \varepsilon)$ is a solution of the inequality (3) with K replaced by K_i. Remark 8 implies that formula (34) is valid everywhere on $(t_1 - \varepsilon, t_1 + \varepsilon)$. In particular, $P_u A_\lambda u = P^i_u A_\lambda u$. Moreover, as we have noted above, $U(t)$ belongs to ∂K for $t \in (t_1 - \varepsilon, t_1]$. Thus it follows from (33) that $U(t) \in \partial K_i$ for $t \in (t_1 - \varepsilon, t_1]$ and therefore

$$\begin{aligned} \lim_{t \to t_1-} P_{U(t)} A_\lambda U(t) &= \lim_{t \to t_1-} P^i_{U(t)} A_\lambda U(t) = P^i_{U(t_1)} A_\lambda U(t_1) \\ &= P^i_u A_\lambda u = P_u A_\lambda u = P_{U(t_1)} A_\lambda U(t_1). \end{aligned}$$

Thus (29) follows from from Remark 8 and the proof of (IV) is complete.

(V) The assertion follows immediately from the definition of $T_0(\lambda, v)$ and from (IV). □

Lemma 3. *Let $\alpha(\lambda) + \nu(\lambda) > 0$ for all $\lambda \in [\lambda_1, \lambda_2]$. Then for any $T > 0$ there exists $R > 0$ such that the following implications hold for any $u \in K \setminus S$:*

$$\begin{gathered} |u| \leqslant R,\ g(u) \leqslant \tau \Longrightarrow g(U_\lambda(t, u)) \leqslant \tau \text{ for all } \lambda \in [\lambda_1, \lambda_2],\ t \in [0, T], \\ g(u) \leqslant \tau \Longrightarrow g(U_{\lambda,0}(t, u)) \leqslant \tau \text{ for } \lambda \in [\lambda_1, \lambda_2],\ t \in [0, +\infty). \end{gathered}$$

Proof. First of all we realize (see Remark 1) that if $|u|$ is small enough the solution $U_\lambda(t, u)$ exists on $[0, T)$ for all $[\lambda_1, \lambda_2]$. We shall prove

$$|u| \leqslant R,\ g(u) \leqslant \tau \Longrightarrow U_\lambda(t, u) \notin S \text{ for all } \lambda \in [\lambda_1, \lambda_2],\ t \in [0, T].$$

Indeed, suppose $U_{\lambda_n}(t_n, u_n) \in S$, $g(u_n) \leqslant \tau$ for some $u_n \to 0$, $t_n \in [0, T]$, $\lambda_n \in [\lambda_1, \lambda_2]$. We may suppose $\lambda_n \to \lambda$, $t_n \to t$ and $\frac{u_n}{|u_n|} \to w$. Then $w \in K \setminus S$ and by Lemma 1, (iv)

$$\frac{U_{\lambda_n}(t_n, u_n)}{|u_n|} \to U_{\lambda,0}(t, w).$$

Hence $U_{\lambda,0}(t,w) \in S$, which contradicts Lemma 2,(I).

Now if the first implication of the lemma were false there would necessarily exist sequences $u_n \in K$, $u_n \to 0$, $t_n \to t$, $\lambda_n \to \lambda \in [\lambda_1, \lambda_2]$, $\varepsilon_n > 0$ such that

$$g(U_{\lambda_n}(t_n, u_n)) = \tau, \tag{35}$$

$$g(U_{\lambda_n}(t, u_n)) > \tau \text{ for } t \in (t_n, t_n + \varepsilon_n),\ n = 1, 2, \dots \tag{36}$$

Recalling Remark 5,(ii) we can see from (36) that $U_{\lambda_n}(t, u_n) \in \operatorname{int} K$ for $t \in (t_n, t_n + \varepsilon_n)$, $n = 1, 2, \dots$. Therefore the equation

$$\dot{U}_{\lambda_n}(t, u_n) = A_{\lambda_n} U_{\lambda_n}(t, u_n) + G(\lambda_n, U_{\lambda_n}(t, u_n))$$

is valid on $(t_n, t_n + \varepsilon_n)$. Particularly, Remark 8 gives

$$\dot{U}_{\lambda_n}(t_n, u_n) = A_{\lambda_n} U_{\lambda_n}(t_n, u_n) + G(\lambda_n, U_{\lambda_n}(t_n, u_n)).$$

As a result of (35), (36) the right derivative of the function $g(U_{\lambda_n}(\cdot, u_n))$ is nonnegative at the point t_n. Setting $v_n = U_{\lambda_n}(t_n, u_n)$ we get

$$0 \leqslant (\operatorname{grad} g(U_{\lambda_n}(t_n, u_n)), \dot{U}_{\lambda_n}(t_n, u_n)) = (\operatorname{grad} g(v_n), A_{\lambda_n} v_n + G(\lambda_n, v_n)).$$

Lemma 1,(i) implies $v_n \to 0$ and, since $v_n \neq 0$, we may suppose $\frac{v_n}{|v_n|} \to w$. By passing to the limit in the inequality

$$0 \leqslant \left(\operatorname{grad} g\left(\frac{v_n}{|v_n|}\right), A_{\lambda_n} \frac{v_n}{|v_n|} + \frac{G(\lambda_n, v_n)}{|v_n|}\right)$$

we obtain from (2)

$$0 \leqslant (\operatorname{grad} g(w), A_\lambda w). \tag{37}$$

We set

$$\tilde{g}(x) = \frac{x_3}{\sqrt{x_1^2 + x_2^2}}, \quad x \in \mathbb{R}^3 \setminus S$$

and obtain

$$\operatorname{grad} g(w) L = \operatorname{grad} \tilde{g}(x),$$

where $Lx = w$. We have $\tilde{g}(x) = \tau$ and simple calculation yields

$$\operatorname{grad} \tilde{g}(x) = \left[-\frac{\tau x_1}{x_1^2 + x_2^2}, -\frac{\tau x_2}{x_1^2 + x_2^2}, \frac{1}{\sqrt{x_1^2 + x_2^2}}\right].$$

Consequently,

$$\begin{aligned}(\operatorname{grad} g(w), A_\lambda w) &= (\operatorname{grad} g(w), L B_\lambda x) = (\operatorname{grad} g(w) L, B_\lambda x) \\ &= (\operatorname{grad} \tilde{g}(x), B_\lambda x) = -\tau(\alpha(\lambda) + \nu(\lambda)).\end{aligned}$$

By virtue of (5) we have $\tau > 0$ and therefore by our assumption $(\operatorname{grad} g(w), A_\lambda w) < 0$, which contradicts (37).

The second implication of the lemma is an easy consequence of the first one. □

Proof of Theorem 1. We shall successively prove that the following assertions (I)–(VII) hold with some $R > 0$ sufficiently small.

(I) $\dot\varphi_\lambda(0, rv) > 0$ for all $\lambda \in [\lambda_1, \lambda_2]$, $r \in (0, R)$. In particular, $\dot\varphi_{\lambda,0}(0, v) > 0$ for all $\lambda \in [\lambda_1, \lambda_2]$.

The second inequality (the linearized case) follows directly from the assumption (9) and Lemma 2,(V). Suppose that there exist sequences $\lambda_n \to \lambda$, $r_n \to 0$ such that

$$\dot\varphi_{\lambda_n}(0, r_n v) \leqslant 0, \ n = 1, 2, \dots \tag{38}$$

Remark 8 together with (2) and the fact that the cones $T_K(v), T_K(r_n v)$ coincide, imply

$$\begin{aligned}\frac{1}{r_n}\dot U_{\lambda_n}(0, r_n v) &= \frac{1}{r_n} P_{r_n v}(A_{\lambda_n} r_n v + G(\lambda_n, r_n v)) \\ &= P_v\Big(A_{\lambda_n} v + \frac{1}{r_n} G(\lambda_n, r_n v)\Big) \to P_v A_\lambda v = \dot U_{\lambda,0}(0, v).\end{aligned} \tag{39}$$

Now Remark 2 implies $\dot\varphi_{\lambda_n}(0, r_n v) \to \dot\varphi_{\lambda,0}(0, v)$ and therefore (38) yields $\dot\varphi_{\lambda,0}(0, v) \leqslant 0$, which is impossible by the second inequality.

(II) $\dot\varphi_{\lambda,0}(T_0(\lambda, v), v) > 0$ for all $\lambda \in [\lambda_1, \lambda_2]$.

Since $g(v) = \tau$ it follows from the assumption (10) and from Lemma 3 that $g(U_{\lambda,0}(t, v)) \leqslant \tau$ for all $t > 0$. Therefore Remark 5,(i) yields

$$U_{\lambda,0}(T_0(\lambda, v), v) = k(\lambda)v, \ \lambda \in [\lambda_1, \lambda_2],$$

where $k(\lambda) > 0$. By (I) we get

$$\dot\varphi_{\lambda,0}(T_0(\lambda, v), v) = \dot\varphi_{\lambda,0}(0, k(\lambda)v) = \dot\varphi_{\lambda,0}(0, v) > 0.$$

(III) There exists $T > 0$ such that $T(\lambda, rv) < T$ for all $r \in (0, R)$, $\lambda \in [\lambda_1, \lambda_2]$.

We use (II) and Lemma 1,(vi) to find that $T(\lambda_n, r_n v) \to T_0(\lambda, v)$ whenever $\lambda_n \to \lambda$, $r_n \to 0$. As a result, any such sequence $T(\lambda_n, r_n v)$ is bounded.

(IV) For any $r \in (0, R)$ and $\lambda \in [\lambda_1, \lambda_2]$ there exists a unique $k(\lambda, r) > 0$ such that $U_\lambda(T(\lambda, rv), rv) = k(\lambda, r)v$.

We use Lemma 3 together with (III) to obtain

$$g(U_\lambda(T(\lambda, rv), rv)) \leqslant \tau, \ \lambda \in [\lambda_1, \lambda_2], \ r \in (0, R).$$

Since $rv \in \partial K$ and $g(rv) = \tau$, the statement is a direct consequence of Remark 5,(i).

(V) $\dot\varphi_\lambda(T(\lambda, rv), rv)) > 0$ for all $r \in (0, R)$, $\lambda \in [\lambda_1, \lambda_2]$.

Suppose

$$\dot\varphi_{\lambda_n}(T(\lambda_n, r_n v), r_n v) \leqslant 0, \ n = 1, 2, \ldots \tag{40}$$

where $\lambda_n \to \lambda$, $r_n \to 0$. It follows from (2) that $G(\lambda, 0) = 0$ and therefore $U_\lambda(t, 0) = 0$, $t \geqslant 0$. Lemma 1,(i) together with (III) implies $U_{\lambda_n}(T(\lambda_n, r_n v), r_n v) \to 0$. We use (IV) to write $U_{\lambda_n}(T(\lambda_n, r_n v), r_n v) = k(\lambda_n, r_n)v$, $n = 1, 2, \ldots$ and so (40) yields

$$0 \geqslant \dot\varphi_{\lambda_n}(T(\lambda_n, r_n v), r_n v) = \dot\varphi_{\lambda_n}(0, k(\lambda_n, r_n)v).$$

Since $k(\lambda_n, r_n) \to 0$, this contradicts (I).

(VI) The function $\lambda \to k(\lambda, r)$ is continuous on $[\lambda_1, \lambda_2]$ for each $r \in (0, R)$.

It follows from (III), (V) by Lemma 1,(iii) that $T(\lambda_n, rv) \to T(\lambda, rv)$ whenever $\lambda_n \to \lambda$, $\lambda_n \in [\lambda_1, \lambda_2]$ and $r \in (0, R)$ is fixed. Recalling (IV), we obtain from Lemma 1,(i) that

$$k(\lambda_n, r)v = U_{\lambda_n}(T(\lambda_n, rv), rv) \to U_\lambda(T(\lambda, rv), rv) = k(\lambda, r)v$$

and consequently, $k(\lambda_n, r) \to k(\lambda, r)$.

(VII) We have $k(\lambda_1, r) < r < k(\lambda_2, r)$ for all $r \in (0, R)$.

Suppose $k(\lambda_1, r_n) \geqslant r_n > 0$, $r_n \to 0$. As in (III) we find $T(\lambda_1, r_n v) \to T_0(\lambda_1, v)$ and therefore by Lemma 1,(iv)

$$\frac{k(\lambda_1, r_n)v}{r_n} = \frac{U_{\lambda_1}(T(\lambda_1, r_n v), r_n v)}{r_n} \to U_{\lambda_1,0}(T_0(\lambda_1, v), v).$$

Finally,

$$|v| \leqslant \frac{k(\lambda_1, r_n)|v|}{r_n} \to |U_{\lambda_1,0}(T_0(\lambda_1, v), v)|,$$

which contradicts (12).

Analogously, the assumption $k(\lambda_2, r_n) \leqslant r_n$, $r_n \to 0$ leads to a contradiction with (13).

It follows from (IV), (VI) and (VII) that for any v satisfying (8) and for each $r \in (0, R)$ there exists a value $\lambda \in [\lambda_1, \lambda_2]$ satisfying $U_\lambda(T(\lambda, rv), rv) = rv$, which completes the proof. □

Lemma 4. *Let $0 \neq v \in \partial K$ and let $\lambda \in \mathbb{R}$ be such that*

$$0 \neq u \in \partial K \Longrightarrow A_\lambda u \notin T_K(u).$$

Then $0 \neq U_{\lambda,0}(t, v) \in \partial K$ for $t \geqslant 0$.

Proof. Set $U(t) = U_{\lambda,0}(t,v)$. Since $v \notin S$ we obtain from Lemma 2,(I) that $U(t) \neq 0$ for all $t \geqslant 0$. Now if the statement were false there would exist $t_0 \geqslant 0$ and a sequence $t_n \to t_0+$ satisfying

$$0 \neq U(t_0) \in \partial K,$$
$$U(t_n) \in \operatorname{int} K, n = 1, 2, \ldots$$

We get $T_K(U(t_n)) = \mathbb{R}^3$ and by Remark 8 we obtain $\dot{U}(t_n) = A_\lambda U(t_n)$. By the same remark we get $P_{U(t_0)} A_\lambda U(t_0) = \dot{U}(t_0) = \lim_{n \to +\infty} A_\lambda U(t_n) = A_\lambda U(t_0)$ and therefore $A_\lambda U(t_0) \in T_K(U(t_0))$. This contradicts our assumption. □

Proof of Theorem 2 is based on Theorem 1. We take an arbitrary fixed element v satisfying (8) (see Remark 3) and verify the assumptions of Theorem 1 for an interval $[\lambda_1, \lambda_2] \subset [\Lambda_1, \Lambda_2]$.

We set

$$\delta = \sup\{\bar{\lambda} \in [\Lambda_1, \Lambda_2];\, T_0(\lambda, v) < +\infty \text{ for all } \lambda \in [\Lambda_1, \bar{\lambda}]\} \tag{41}$$

and prove successively the following assertions (i)–(vii).

(i) We have $\Lambda_1 < \delta$.

Let $U(t) = X_1(t)\bar{u} + X_2(t)\bar{v} + X_3(t)\bar{w}$ be the solution of the equation $\dot{U}(t) = A_\lambda U(t)$ with the initial condition $U(0) = v$ for $\lambda = \Lambda_1$. Using the formulas (19) we get

$$g(U(t)) = \frac{X_3(0)}{\sqrt{X_1^2(0) + X_2^2(0)}} \mathrm{e}^{-(\alpha(\lambda)+\nu(\lambda))t} = g(v)\mathrm{e}^{-(\alpha(\lambda)+\nu(\lambda))t}, \ t \geqslant 0,$$

where $\lambda = \Lambda_1$. By virtue of (8) and (14) the last relation becomes $g(U(t)) = \tau$, $t \geqslant 0$ and from Remark 5,(ii) we conclude that $U(t) \in K$ for all $t \geqslant 0$. Therefore $U(t) = U_{\lambda,0}(t,v)$, $t \geqslant 0$ and we have

$$\dot{U}_{\lambda,0}(t,v) = A_\lambda U_{\lambda,0}(t,v) \text{ for } \lambda = \Lambda_1, t \geqslant 0. \tag{42}$$

Remark 10 implies

$$\dot{\varphi}_{\lambda,0}(t,v) = \beta(\lambda) \text{ for } \lambda = \Lambda_1,\ t \geqslant 0. \tag{43}$$

Consequently,

$$T_0(\Lambda_1, v) < +\infty,\ \dot{\varphi}_{\Lambda_1,0}(T_0(\Lambda_1, v), v) > 0. \tag{44}$$

Lemma 1,(iii) implies $T_0(\lambda_n, v) \to T_0(\Lambda_1, v)$ whenever $\lambda_n \to \Lambda_1$. Therefore $T_0(\lambda, v) < +\infty$ for all λ sufficiently close to Λ_1 and (41) implies (i).

(ii) $|U_{\lambda,0}(T_0(\lambda,v),v)| < |v|$ for all λ sufficiently close to Λ_1.

We use (43) to obtain $T_0(\Lambda_1,v) = 2\pi/\beta(\Lambda_1)$ and (14) together with Remark 4 to find $U_{\lambda,0}(T_0(\lambda,v),v) = \mathrm{e}^{\frac{2\pi\alpha(\lambda)}{\beta(\lambda)}} v$ for $\lambda = \Lambda_1$. By the assumptions (14), (16) we get $|U_{\lambda,0}(T_0(\lambda,v),v)| = \mathrm{e}^{\frac{2\pi\alpha(\lambda)}{\beta(\lambda)}} |v| < |v|$ provided $\lambda = \Lambda_1$. The statement now follows from (44) and from Lemma 1, (i), (iii).

(iii) If $T_0(\delta,v) < +\infty$ then $|U_{\delta,0}(t,v)| > |v|$ for $t > 0$.

We shall first prove $\delta = \Lambda_2$.

Because of (i) and (15) we have $\alpha(\delta)+\nu(\delta) > 0$ and Lemma 3 implies $g(U_{\delta,0}(t,v)) \leqslant \tau$ for $t \geqslant 0$. Consequently, by Remark 5,(i)

$$U_{\delta,0}(T_0(\delta,v),v) = kv \text{ with some } k > 0.$$

Hence

$$\dot\varphi_{\delta,0}(T_0(\delta,v),v) = \dot\varphi_{\delta,0}(0,kv) = \dot\varphi_{\delta,0}(0,v).$$

According to Lemma 2,(V) the assumption $T_0(\delta,v) < +\infty$ implies $\dot\varphi_{\delta,0}(0,v) > 0$. Thus $\dot\varphi_{\delta,0}(T_0(\delta,v),v) > 0$ and from Lemma 1,(iii) we obtain that $T_0(\lambda,v) < +\infty$ for all λ sufficiently close to δ. Thus (41) implies $\delta = \Lambda_2$.

Furthermore, by virtue of (17) we can use Lemma 4 to obtain

$$0 \neq U_{\delta,0}(t,v) \in \partial K \text{ for } t \geqslant 0. \tag{45}$$

Thus we can use (18) together with Remark 11 to obtain

$$\begin{aligned}|U_{\delta,0}(t,v)|^2 - |v|^2 &= |U_{\delta,0}(t,v)|^2 - |U_{\delta,0}(0,v)|^2\\ &= \int_0^t 2(\dot U_{\delta,0}(s,v), U_{\delta,0}(s,v))\mathrm{d}s\\ &= \int_0^t 2(A_\delta U_{\delta,0}(s,v), U_{\delta,0}(s,v))\mathrm{d}s > 0,\ t > 0.\end{aligned}$$

(iv) There exists a real constant B such that

$$\frac{(\dot U_{\lambda,0}(t,v), U_{\lambda,0}(t,v))}{\dot\varphi_{\lambda,0}(t,v)} \geqslant B\,|U_{\lambda,0}(t,v)|^2$$

for all $\lambda \in [\Lambda_1,\delta)$, $t \in [0,T_0(\lambda,v))$.

Assume that, on the contrary, there exist sequences $\lambda_n \in [\Lambda_1,\delta)$, $t_n \in [0,T_0(\lambda_n,v))$ satisfying

$$\frac{(\dot U_{\lambda_n,0}(t_n,v), U_{\lambda_n,0}(t_n,v))}{\dot\varphi_{\lambda_n,0}(t_n,v)} \leqslant -n|U_{\lambda_n,0}(t_n,v)|^2,\ n = 1,2,\ldots \tag{46}$$

Since $U_{\lambda_n,0}(t_n,v) \neq 0$ (see Lemma 2,(I)) we can rewrite (46) as

$$\frac{(\dot{U}_{\lambda_n,0}(0,u_n),u_n)}{\dot{\varphi}_{\lambda_n,0}(0,u_n)} \leqslant -n,\ n=1,2,\ldots \tag{47}$$

where

$$u_n = \frac{U_{\lambda_n,0}(t_n,v)}{|U_{\lambda_n,0}(t_n,v)|}.$$

We may assume $u_n \to u \in K$, $\lambda_n \to \lambda \in [\Lambda_1,\delta]$ and, since $|P_{u_n}A_{\lambda_n}u_n| \leqslant |A_{\lambda_n}u_n| \leqslant C$, also

$$\dot{U}_{\lambda_n,0}(0,u_n) = P_{u_n}A_{\lambda_n}u_n \to w \in \mathbf{R}^3 \tag{48}$$

(see Remark 8). Moreover, Remark 11 yields

$$(\dot{U}_{\lambda_n,0}(0,u_n),u_n) = (A_{\lambda_n}u_n,u_n) \to (A_\lambda u,u). \tag{49}$$

On the other hand, considering (41) we obtain from Lemma 2,(V)

$$\dot{\varphi}_{\lambda,0}(t,v) > 0 \text{ for all } \lambda \in [\Lambda_1,\delta),\ t \in [0,T_0(\lambda,v)). \tag{50}$$

Hence

$$\dot{\varphi}_{\lambda_n,0}(0,u_n) = \dot{\varphi}_{\lambda_n,0}(t_n,v) > 0. \tag{51}$$

On the other hand, (47), (49) imply

$$\dot{\varphi}_{\lambda_n,0}(0,u_n) \to 0 \text{ as } n \to +\infty. \tag{52}$$

Using Remark 2 we get

$$\langle \dot{U}_{\lambda_n,0}(0,u_n), x_{n2}\bar{u} - x_{n1}\bar{v}\rangle = (x_{n1}^2 + x_{n2}^2)\dot{\varphi}_{\lambda_n,0}(0,u_n) \to 0,$$

where $Lx_n = u_n$, $Lx = u$, and consequently, (48) yields

$$\langle w, x_2\bar{u} - x_1\bar{v}\rangle = 0. \tag{53}$$

Furthermore, we have

$$P_{u_n}A_{\lambda_n}u_n \in \partial T_K(u_n) \text{ for all } n \text{ sufficiently large.} \tag{54}$$

Indeed, if $P_{u_n}A_{\lambda_n} \in \operatorname{int} T_K(u_n)$ we would get by Observation 4 that $\dot{U}_{\lambda_n,0}(0,u_n) = P_{u_n}A_{\lambda_n}u_n = A_{\lambda_n}u_n$ and by Remark 10 $\dot{\varphi}_{\lambda_n,0}(0,u_n) = \beta(\lambda_n)$. But (52) would imply $\beta(\lambda) = 0$ for some λ in $[\Lambda_1, \Lambda_2]$, which contradicts (16). By Observation 3,(i) we conclude that (54), (48) imply $w \in \partial T_i(u)$ for some $i, 1 \leqslant i \leqslant N$. Recalling Observation 1 we obtain from (53) $w = \mu u, \mu \in \mathbf{R}$. Remark 8 and (48) yield

$$0 \leqslant (\dot{U}_{\lambda_n,0}(0, u_n) - A_{\lambda_n} u_n, v - u_n) \to (\mu u - A_\lambda u, v - u) \text{ for all } v \in K$$

and therefore u is an eigenvector of (7). Moreover, $u \in \partial K$ because $u \in \operatorname{int} K$ would imply $\dot{\varphi}_{\lambda_n,0}(0, u_n) = \beta(\lambda_n) \to \beta(\lambda) > 0$ (see Remark 10), which would contradict (52). By the assumption of Theorem 2 the eigenvalue μ is positive. Finally, recalling Remark 9 we have $(A_\lambda u, u) = \mu |u|^2 > 0$ and therefore (49) yields $(\dot{U}_{\lambda_n,0}(0, u_n), u_n) > 0$ for n large. This inequality together with (51) contradicts (47).

(v) The function $\varphi_{\delta,0}(t, v)$ is nondecreasing on $[0, T_0(\delta, v))$.

Assume there exist $0 \leqslant t_1 < t_2 < T_0(\delta, v)$ such that $\varphi_{\delta,0}(t_1, v) > \varphi_{\delta,0}(t_2, v)$. By Lemma 1

$$\varphi_{\lambda,0}(t_1, v) > \varphi_{\lambda,0}(t_2, v), \quad 0 \leqslant t_1 < t_2 < T_0(\lambda, v) \tag{55}$$

for all λ sufficiently close to δ. As we have proved in (i) the interval $[\Lambda_1, \delta)$ is nonempty and therefore we conclude from (55) that $\dot{\varphi}_{\lambda_0,0}(t_0, v) \leqslant 0$ for some $\lambda_0 \in [\Lambda_1, \delta)$ and $t_0 \in [0, T_0(\lambda_0, v))$. This contradicts (50) and (v) is proved.

(vi) If $T_0(\delta, v) = +\infty$ then $\lim\limits_{t \to +\infty} |U_{\delta,0}(t, v)| = +\infty$.

Lemma 2,(I) implies $U_{\delta,0}(t, v) \notin S$ for all $t > 0$. Thus we get from the definition of $T_0(\delta, v)$ that $\varphi_{\delta,0}(t, v) < 2\pi$ for all $t > 0$. It follows from (v) that the function $\varphi_{\delta,0}(t, v)$ has a proper limit as $t \to +\infty$. Set $U(t) = U_{\delta,0}(t, v)$. Then Lemma 2,(III) yields

$$\frac{U(t)}{|U(t)|} \to u \text{ as } t \to +\infty, \tag{56}$$

where $u \in \partial K$ is an eigenvector of (7). By Remark 11 we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|U(t)|^2 &= 2(\dot{U}(t), U(t)) = 2(A_\lambda U(t), U(t)) \\ &= 2|U(t)|^2 \left(A_\lambda \frac{U(t)}{|U(t)|}, \frac{U(t)}{|U(t)|}\right), \end{aligned} \tag{57}$$

and by (56)

$$\left(A_\lambda \frac{U(t)}{|U(t)|}, \frac{U(t)}{|U(t)|}\right) \to (A_\lambda u, u) \text{ as } t \to +\infty. \tag{58}$$

Let μ be the eigenvalue of (7) corresponding to u. By the last assumption of Theorem 2, μ is positive and Remark 9 yields $(A_\lambda u, u) = \mu |u|^2 > 0$. Consequently, (vi) follows from (57) and (58).

(vii) If $T_0(\delta, v) = +\infty$ then $|U_{\lambda_n,0}(T_0(\lambda_n, v), v)| \to +\infty$ for a sequence $\lambda_n \to \delta-$.

Since $T_0(\delta, v) = +\infty$ we use Lemma 1,(i),(ii) to conclude from (i) and (vi) that there exist sequences $\lambda_n \to \delta-$, $t_n \in [0, T_0(\lambda_n, v))$ satisfying

$$|U_{\lambda_n,0}(t_n, v)| \to +\infty, \ n \to +\infty. \tag{59}$$

To prove $|U_{\lambda_n,0}(T_0(\lambda_n, v), v)| \to +\infty$ we define for each $\lambda \in [\Lambda_1, \delta)$ a function $V_\lambda : [0, 2\pi] \to K$ as follows:

$$V_\lambda(\varphi) = U_{\lambda,0}(t, v) \text{ for } \varphi = \varphi_{\lambda,0}(t, v), \ t \in [0, T_0(\lambda, v)].$$

It follows from (50) that $V_\lambda(\varphi)$ is correctly defined. Moreover, $V_\lambda(\varphi)$ is absolutely continuous and right differentiable on $[0, 2\pi)$ (see Remark 8). Thus we obtain from (iv)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varphi}|V_\lambda(\varphi)|^2 &= 2\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}V_\lambda(\varphi), V_\lambda(\varphi)\right) \\ &= 2\left(\frac{\dot{U}_{\lambda,0}(t, v)}{\dot{\varphi}_{\lambda,0}(t, v)}, U_{\lambda,0}(t, v)\right) \geqslant 2B|U_{\lambda,0}(t, v)|^2 = 2B|V_\lambda(\varphi)|^2 \end{aligned} \tag{60}$$

for some $B < 0$ and all $\varphi \in [0, 2\pi)$. Now Gronwall's lemma yields

$$|V_\lambda(2\pi)|^2 \geqslant |V_\lambda(\varphi)|^2 \mathrm{e}^{2B(2\pi-\varphi)}, \ \varphi \in [0, 2\pi).$$

We set $\varphi_n = \varphi_{\lambda_n,0}(t_n, v) \in [0, 2\pi)$ and obtain

$$\begin{aligned} |U_{\lambda_n,0}(T_0(\lambda_n, v), v)|^2 &= |V_{\lambda_n}(2\pi)|^2 \\ &\geqslant \mathrm{e}^{2B(2\pi-\varphi_n)}|V_{\lambda_n}(\varphi_n)|^2 \geqslant \mathrm{e}^{-4\pi|B|}|U_{\lambda_n,0}(t_n, v)|^2. \end{aligned} \tag{61}$$

The statement now follows from (59).

We shall complete the proof of Theorem 2 by finding values $\lambda_1 < \lambda_2$ in the interval $[\Lambda_1, \Lambda_2]$ such that the conditions (9)-(13) are valid. To do this we need to consider two cases: $T_0(\delta, v) < +\infty$ and $T_0(\delta, v) = +\infty$.

When $T_0(\delta, v) = +\infty$ we use (i), (ii) and (vii) to conclude that the conditions (12), (13) hold for some $\Lambda_1 < \lambda_1 < \lambda_2 < \delta$. In addition, (41) implies (9) and the conditions (10), (11) are guaranteed by (15), (16).

In the case $T_0(\delta, v) < +\infty$ we find $\lambda_1 \in (\Lambda_1, \delta)$ satisfying (12) by (i), (ii). Further, we set $\lambda_2 = \delta$ to obtain (13) from (iii). The conditions (9), (10), (11) are obtained as above. □

5. Example

Lemma 5. *Suppose that $0 \neq u \in \partial K$, $\lambda \in \mathbb{R}$ and there is j such that*

$$P_u A_\lambda u = P_u^j A_\lambda u. \tag{62}$$

Set $x = L^{-1}u, y = L^* n_j(u)$ (the inner normal to $L^{-1}K_j$), $z = L^{-1}n_j(u)$. If

$$z_3 > 0, \quad x_3 > 0, \quad \beta(\lambda) - |\nu(\lambda)| \frac{\sqrt{y_1^2 + y_2^2}\sqrt{z_1^2 + z_2^2}}{y_3 z_3} > 0, \quad \beta(\lambda) > 0 \tag{63}$$

and u is an eigenvector of (7) then the corresponding eigenvalue μ of (7) is positive.

Proof. We can suppose without loss of generality that $x_1^2 + x_2^2 = 1$ and we shall write n instead of $n_j(u)$. Realize that $0 = (u, n) = (x, L^* n) = (x, y)$, i.e.

$$-x_3 y_3 = x_1 y_1 + x_2 y_2. \tag{64}$$

We have $A_\lambda u \notin T_j(u)$ because otherwise (62) would yield $P_u A_\lambda u = A_\lambda u$ and therefore u would be an eigenvector of A_λ by Remark 9. However, A_λ has no eigenvectors on ∂K under the assumption (4). Hence, formula (62) yields

$$P_u A_\lambda u = A_\lambda u - (A_\lambda u, n)n \tag{65}$$

and by Remark 9

$$\mu u = A_\lambda u - (A_\lambda u, n)n,$$

which is equivalent to

$$\mu x = B_\lambda x - (B_\lambda x, y)z.$$

Multiplying this equation successively by $[x_1, x_2, 0]$, $[x_2, -x_1, 0]$ and using (64) we obtain

$$\mu = \alpha - [(\alpha + \nu)(x_1 y_1 + x_2 y_2) + \beta(x_2 y_1 - x_1 y_2)](x_1 z_1 + x_2 z_2), \tag{66}$$

$$0 = \beta - [(\alpha + \nu)(x_1 y_1 + x_2 y_2) + \beta(x_2 y_1 - x_1 y_2)](x_2 z_1 - x_1 z_2), \tag{67}$$

where we write α, β, ν instead of $\alpha(\lambda)$, $\beta(\lambda)$, $\nu(\lambda)$. Set $a = x_1 y_1 + x_2 y_2$, $b = x_2 y_1 - x_1 y_2$, $c = x_2 z_1 - x_1 z_2$, $d = x_1 z_1 + x_2 z_2$.

Let us show that

$$c < 0, \quad y_3 > 0, \quad \frac{a}{y_3} < 0. \tag{68}$$

The first inequality can be obtained from (67) by using the inequalities $\beta > 0$, $(\alpha + \nu)a + \beta b = (B_\lambda x, y) = (A_\lambda u, n) < 0$ (because $A_\lambda u \notin T_j(u)$). The second follows from the assumption (4) and from the fact that y is the normal to the cone $L^{-1}K_j$ at the point x. Finally, formulas (63), (64) imply $a/y_3 = -x_3 < 0$. Calculating α from (67) and substituting in (66) we get

$$\alpha = \frac{\beta - \nu ac - \beta bc}{ac},$$

$$\mu = \beta\frac{1 - bc - ad}{ac} - \nu. \tag{69}$$

Also, $(y, z) = (L^*n, L^{-1}n) = (n, n) = 1$ and by a simple calculation we get

$$1 - bc - ad = 1 - y_1z_1 - y_2z_2 = 1 - (y, z) + y_3z_3 = y_3z_3.$$

Hence, we use (68), (63) to obtain from (69)

$$\mu = \beta\frac{y_3z_3}{ac} - \nu = \frac{y_3z_3}{ac}(\beta - \frac{ac}{y_3z_3}\nu)$$
$$\geqslant \frac{y_3z_3}{ac}\left(\beta - |\nu|\frac{\sqrt{y_1^2 + y_2^2}\sqrt{z_1^2 + z_2^2}}{y_3z_3}\right) > 0.$$

□

Example. Consider the matrix A_λ and the cone K in $\mathbb{R}^3$ defined by

$$A_\lambda = \frac{1}{6}\begin{pmatrix} 5\lambda + 17 & -\lambda + 17 & -\lambda - 19 \\ -2\lambda - 50 & 4\lambda - 14 & -2\lambda + 22 \\ -3\lambda + 27 & -3\lambda - 9 & 3\lambda - 9 \end{pmatrix},$$

$$K = \{u \in \mathbb{R}^3;\ u_j \geqslant 0, j = 1, 2, 3\}.$$

The eigenvalues $\alpha(\lambda) \pm i\beta(\lambda) = \lambda \pm 6i$, $-\nu(\lambda) = -1$ clearly satisfy (14), (15), (16) with $\Lambda_1 = -1$, $\Lambda_2 > -1$ arbitrary. The corresponding eigenvectors are $\bar{u} \pm i\bar{v} = [1, -3, 2] \pm i[2, -1, -1]$, $\bar{w} = [1, 2, 3]$. Hence,

$$L = \begin{pmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 2 & -1 & 3 \end{pmatrix}, \qquad L^{-1} = \frac{1}{30}\begin{pmatrix} -1 & -7 & 5 \\ 13 & 1 & -5 \\ 5 & 5 & 5 \end{pmatrix}.$$

Our cone can be described as

$$K = \{u = Lx;\ (Lx)_j \geqslant 0, j = 1, 2, 3\}$$
$$= \{u = x_1\bar{u} + x_2\bar{v} + x_3\bar{w};\ x_3 \geqslant f_j(x_1, x_2), j = 1, 2, 3\},$$

where f_j are defined by $x_3 - f_j(x_1, x_2) = (Lx)_j$.

Suppose that $u \in \partial K$ is an eigenvector of (7) with some $\lambda \geqslant -1$. We shall prove that then the corresponding eigenvalue must be positive. Consider successively points $u \in \partial K$ of two types (see the notation from Section 3):

(a) $u \in \partial K_3 \cap \operatorname{int} K_1 \cap \operatorname{int} K_2$, i.e. $u = [u_1, u_2, 0], u_1 > 0,\ u_2 > 0$. Then $T_K(u) = T_3(u) = K_3$ and therefore (62) holds with $j = 3$. We have $n_3(u) = [0, 0, 1]$, $y = [2, -1, 3]$, $z = \frac{1}{6}[1, -1, 1]$, $x = \frac{1}{30}[-u_1 - 7u_2, 13u_1 + u_2, 5u_1 + 5u_2]$ and (63) is fulfilled. Lemma 5 implies $\mu > 0$.

(b) $u \in \partial K_3 \cap \partial K_1$; we can suppose $u = [0, 1, 0]$. Then $T_K(u) = K_3 \cap K_1$, $A_\lambda u = \frac{1}{6}[-\lambda + 17,\ 4\lambda - 14,\ -3\lambda - 9]$. If $\lambda \leqslant 17$ then $P_u A_\lambda u = P_u^3 A_\lambda u$ and the same argument as in (a) can be used to prove $\mu > 0$. On the other hand, we use Remark 9 to obtain $\mu = (A_\lambda u, u) = \frac{1}{6}[4\lambda - 14] > 0$ when $\lambda > 17$. The cases $u \in \partial K_1 \cap \operatorname{int} K_2 \cap \operatorname{int} K_3$, $u \in \partial K_2 \cap \operatorname{int} K_1 \cap \operatorname{int} K_3$ and $u \in \partial K_1 \cap \partial K_2$, $u \in \partial K_2 \cap \partial K_3$ can be treated as (a) and (b), respectively. Summarizing all possible cases we can see that (7) can have only positive eigenvalues corresponding to eigenvectors $u \in \partial K$ if $\lambda \geqslant -1 = \Lambda_1$. Furthermore, considering as above the separate regions of the cone K, we find that the condition (17) is fulfilled with $\Lambda_2 = 20$. For instance, in the region (a) we have $A_{20} u = \frac{1}{6}[117u_1 - 3u_2, -90u_1 + 66u_2, -33u_1 - 69u_2]$ and therefore $A_{20} u \notin T_K(u) = K_3$ because $-33u_1 - 69u_2 < 0$ for points under consideration. For the points u belonging to the region (b) the condition (17) for any $\lambda > -3$ follows from the expression for A_λ written above. The other cases can be treated similarly. The assumption (18) with $\Lambda_2 = 20$ is also satisfied. For instance in the case (a) we obtain $(A_{20} u, u) = \frac{1}{6}[117u_1^2 + 66u_2^2 - 93u_1 u_2] > 0$ for all $u_1 \neq 0$, $u_2 \neq 0$.

References

[1] *J. P. Aubin, A. Cellina*: Differential Inclusions, Springer-Verlag, Berlin, 1984.

[2] *M. Degiovanni, A. Marino*: Non-smooth variational bifurcation, Atti Acc. Lincei Rend. Fis. (8) *LXXXI* (1987), 259–270.

[3] *J. Eisner, M. Kučera*: Hopf bifurcation and ordinary differential inequalities, to appear.

[4] *M. Kučera*: Bifurcation of periodic solutions to ordinary differential inequalities, to appear in Proceedings of Colloquium "Differential Equations and Applications", Budapest 1991 (series Colloquia Mathematica Societatis János Bolyai, North-Holland, 1992).

[5] *M. Kučera*: Bifurcation points of variational inequalities, Czechoslovak Math. J. *32 (107)* (1982), 208–226.

[6] *E. Miersemann*: Über höhere Verzweigungspunkte nichtlinearen Variationsungleichungen., Math. Nachr. *85* (1978), 195–213.

[7] *M. Pazy*: Semi-groups of nonlinear contractions in Hilbert space, Problems in Nonlinear Analysis (C.I.M.E., IV Ciclo, Varenna, 1970), Edizioni Cremonese, Rome, 1971, pp. 343–430.

[8] *P. Quittner*: Spectral analysis of variational inequalities, Comment. Math. Univ. Carol. 27 (1986), 605–629.

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