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ON ELEMENTARY SYMMETRIC FUNCTIONS OF THE EIGENVALUES OF THE SUM AND PRODUCT OF NORMAL MATRICES

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1. INTRODUCTION

Throughout this paper, we let $A, B \in \mathbb{C}^{n\times n}$ be normal with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively. For $1 \leqq m \leqq n$, denote by $e_m(x_1, \ldots, x_n)$ the m'th elementary symmetric function of $x_1, \ldots, x_n \in \mathbb{C}$, and by $E_m(C)$ the m'th elementary symmetric function of the eigenvalues of $C \in \mathbb{C}^{n\times n}$. We study the conjectures

$$(\mathrm{E}_m) \qquad E_m(A + B) \in \operatorname{co}\{e_m(\alpha_1 + \beta_{\sigma(1)}, \ldots, \alpha_n + \beta_{\sigma(n)}) \mid \sigma \in S_n\},$$

$$(\mathrm{F}_m) \qquad E_m(AB) \in \operatorname{co}\{e_m(\alpha_1\beta_{\sigma(1)}, \ldots, \alpha_n\beta_{\sigma(n)}) \mid \sigma \in S_n\}.$$

Here co denotes the convex hull.

(E_1) is trivially true $(\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B)$.

Marcus [4] and de Oliveira [7] conjectured

$$(\mathrm{E}_n) \qquad \det(A + B) \in \operatorname{co}\{\prod_i (\alpha_i + \beta_{\sigma(i)}) \mid \sigma \in S_n\},$$

which is still open. It is true if A and B are Hermitian, i.e., if all the α_i's and β_i's are real [2]. It is also true in certain other special cases, see [6] and the references therein.

(E_2) is known to be true in the Hermitian case [2].

We will show (E_2) and (E_3).

Due to de Oliveira [7], and in fact tracing back to Horn [3, Theorem 7], we have

$$(\mathrm{F}_1) \qquad \operatorname{tr} AB \in \operatorname{co}\{\sum_i \alpha_i\beta_{\sigma(i)} \mid \sigma \in S_n\}.$$

(F_n) is clearly true $(\det AB = \det A \det B)$. We will show (F_{n-1}).

By a unitary similarity transformation, (E_m) and (F_m) can be seen equivalent to

$$(\mathrm{E}'_m) \qquad E_m(A' + UB'U^H) \in \operatorname{co}\{e_m(\alpha_1 + \beta_{\sigma(1)}, \ldots, \alpha_n + \beta_{\sigma(n)}) \mid \sigma \in S_n\},$$

$$(\mathrm{F}'_m) \qquad E_m(A'UB'U^H) \in \operatorname{co}\{e_m(\alpha_1\beta_{\sigma(1)}, \ldots, \alpha_n\beta_{\sigma(n)}) \mid \sigma \in S_n\}.$$

Here $A' = \operatorname{diag}(\alpha_i)$, $B' = \operatorname{diag}(\beta_i)$, and $U \in \mathbb{C}^{n\times n}$ is unitary.

2. (E_2)

We prove (E_2). Let us denote by $\gamma_1, \ldots, \gamma_n$ the eigenvalues of $C = A + B$ and by $C^{(m)}$ the m'th compound of C. Since

$$2 \sum_{i<j}\sum \gamma_i \gamma_j = (\sum_i \gamma_i)^2 - \sum_i \gamma_i^2 ,$$

i.e.,

$$2 \operatorname{tr} C^{(2)} = (\operatorname{tr} C)^2 - \operatorname{tr} C^2 ,$$

we have

(1) $$2 \operatorname{tr}(A + B)^{(2)} = (\operatorname{tr}(A + B))^2 - \operatorname{tr}(A + B)^2 = = f - \operatorname{tr}(AB + BA) = f - 2 \operatorname{tr} AB .$$

Here

$$f = (\operatorname{tr}(A + B))^2 - \operatorname{tr}(A^2 + B^2) .$$

On the other hand, denoting

$$\eta_i^\sigma = \alpha_i + \beta_{\sigma(i)} ,$$

we have

(2) $$2e_2(\eta_1^\sigma, \ldots, \eta_n^\sigma) = (\sum_i \eta_i^\sigma)^2 - \sum_i (\eta_i^\sigma)^2 = = (\sum_i (\alpha_i + \beta_{\sigma(i)}))^2 - \sum_i (\alpha_i + \beta_{\sigma(i)})^2 = f - 2 \sum_i \alpha_i \beta_{\sigma(i)} .$$

Now (E_2) follows from (1) and (2) by (F_1).

3. (E_3)

Analogously, let us start from

$$6 \sum_{i<j<k}\sum\sum \gamma_i \gamma_j \gamma_k = (\sum_i \gamma_i)^3 - 3(\sum_i \gamma_i)(\sum_i \gamma_i^2) + 2 \sum_i \gamma_i^3 ,$$

i.e.,

$$6 \operatorname{tr} C^{(3)} = (\operatorname{tr} C)^3 - 3 \operatorname{tr} C \operatorname{tr} C^2 + 2 \operatorname{tr} C^3 ,$$

which implies

$$6 \operatorname{tr}(A + B)^{(3)} = (\operatorname{tr}(A + B))^3 - 3 \operatorname{tr}(A + B) \operatorname{tr}(A + B)^2 + + 2 \operatorname{tr}(A + B)^3 = g - 3 \operatorname{tr}(A + B) \operatorname{tr}(AB + BA) + + 2 \operatorname{tr}(A^2B + ABA + BA^2 + AB^2 + BAB + B^2A) = = g - 6 \operatorname{tr}(A + B) \operatorname{tr} AB + 6 \operatorname{tr}(A^2B + AB^2) .$$

Here

$$g = (\operatorname{tr}(A + B))^3 - 3 \operatorname{tr}(A + B) \operatorname{tr}(A^2 + B^2) + 2 \operatorname{tr}(A^3 + B^3) .$$

On the other hand,

$$6e_3(\eta_1^\sigma, \ldots, \eta_n^\sigma) = (\sum_i \eta_i^\sigma)^3 - 3(\sum_i \eta_i^\sigma) \sum_i (\eta_i^\sigma)^2 + 2 \sum_i (\eta_i^\sigma)^3 =$$

$$= g - 6 \operatorname{tr}(A + B) \sum_i \alpha_i \beta_{\sigma(i)} + 6 \sum_i \alpha_i^2 \beta_{\sigma(i)} + 6 \sum_i \alpha_i \beta_{\sigma(i)}^2 .$$

By (F_1), there exists a convex combination

$$\operatorname{tr} AB = \sum_{\sigma \in S_n} t_\sigma \sum_i \alpha_i \beta_{\sigma(i)} \quad (t_\sigma \geqq 0, \sum_\sigma t_\sigma = 1), \tag{3}$$

and the t_σ's are obtained as follows [3, 7]: For some unitary U,

$$\operatorname{tr} AB = \operatorname{tr} A' U B' U^H . \tag{4}$$

By Birkhoff's theorem, there exists a convex combination

$$|U|^2 = \sum_\sigma t_\sigma P_\sigma \tag{5}$$

where $|\cdot|^2$ is understood elementwise and the P_σ's are permutation matrices with rows corresponding to σ. Since this U satisfies also $\operatorname{tr} A^2 B = \operatorname{tr} (A')^2 U B' U^H$, $\operatorname{tr} AB^2 = \operatorname{tr} A' U (B')^2 U^H$, these same t_σ's satisfy also

$$\operatorname{tr} A^2 B = \sum_\sigma t_\sigma \sum_i \alpha_i^2 \beta_{\sigma(i)}, \quad \operatorname{tr} AB^2 = \sum_\sigma t_\sigma \sum_i \alpha_i \beta_{\sigma(i)}^2 . \tag{6}$$

Now (E_3) follows.

4. (E_4)

By Newton's formula,

$$24 \sum_{i<j<k<l} \sum \sum \sum \gamma_i \gamma_j \gamma_k \gamma_l = \ldots + 3(\sum_i \gamma_i^2)^2 + \ldots,$$

and so

$$24 \operatorname{tr}(A + B)^{(4)} = \ldots + 3(\operatorname{tr}(A + B)^2)^2 + \ldots = \ldots + 12(\operatorname{tr} AB)^2 + \ldots .$$

On the other hand,

$$24 e_4(\eta_1^\sigma, \ldots, \eta_n^\sigma) = \ldots + 3(\sum_i (\alpha_i + \beta_{\sigma(i)})^2)^2 + \ldots =$$

$$= \ldots + 12(\sum_i \alpha_i \beta_{\sigma(i)})^2 + \ldots .$$

Let U be as in (4) and the t_σ's as in (5). Using (3), (6), and the analogous results for $\operatorname{tr} A^3 B$ etc., we could do as before if these same t_σ's would satisfy also

$$(\operatorname{tr} AB)^2 = \sum_\sigma t_\sigma (\sum_i \alpha_i \beta_{\sigma(i)})^2 ,$$

which is obviously not true in general. Therefore the case $m \geqq 4$ remains open.

5. (E_n)

Let $U \in \mathbb{C}^{n \times n}$ be unitary. Consider the following condition:

(P) *There exist* t_σ*'s* $(\sigma \in S_n)$ *with* $t_\sigma \geqq 0$, $\sum_\sigma t_\sigma = 1$, *satisfying*

$$|U^{(m)}|^2 = \sum_\sigma t_\sigma |P_\sigma^{(m)}|$$

for all $m = 1, \ldots, n$.

It is easy to see that (P) is true for all U if $n \leqq 3$. Drury [1] proved that (P) is not generally valid if $n \geqq 4$. (Dropping out the requirement $t_\sigma \geqq 0$, we obtain a weaker condition, which always holds [6].) Now let $C = A' + UB'U^H$ be as in (E_n'). We claim that

$$U \textit{ satisfies } (P) \Rightarrow C \textit{ satisfies } (E_n') .$$

For the proof, let $N = \{1, \ldots, n\}$, $1 \leqq m \leqq n$, $S = \{I \subset N \mid |I| = m\}$, $|\cdot| =$ = card. Order S lexicographically. For $J, K \in S$, the matrix

$$Q = (q_{JK}) = \sum_\sigma t_\sigma |P_\sigma^{(m)}|$$

has ($\delta(I, K) = 0$ if $I \neq K$, and $\delta(I, I) = 1$)

$$q_{JK} = \sum_\sigma t_\sigma \delta(\sigma(J), K) = \sum_{\sigma, \sigma(J) = K} t_\sigma .$$

Now, for $\emptyset \neq J$, $K \subset N$, denote U_{JK} = the corresponding submatrix of U, $\alpha_J =$ $= \alpha_{j_1} \ldots \alpha_{j_p}$ if $J = \{j_1, \ldots, j_p\}$, β_J respectively, and $J^\sim = N \setminus J$. Since

$$d = \det C = \det(A' + UB'U^H) = \sum_{l=0}^{n} \sum_{J, |J| = l} \sum_{K, |K| = l} |\det U_{JK}|^2 \alpha_{J^\sim} \beta_K$$

[7] and, by (P),

$$|\det U_{JK}|^2 = q_{JK} = \sum_{\sigma, \sigma(J) = K} t_\sigma ,$$

we have

$$d = \sum_{l=0}^{n} \sum_{J, |J| = l} \sum_{K, |K| = l} \sum_{\sigma, \sigma(J) = K} t_\sigma \alpha_{J^\sim} \beta_K = \sum_l \sum_{|J| = l} \sum_\sigma t_\sigma \alpha_{J^\sim} \beta_{\sigma(J)} =$$

$$= \sum_J \sum_\sigma t_\sigma \alpha_{J^\sim} \beta_{\sigma(J)} = \sum_\sigma t_\sigma \sum_J \alpha_{J^\sim} \beta_{\sigma(J)} = \sum_\sigma t_\sigma \prod_j (\alpha_j + \beta_{\sigma(j)}) .$$

Thus C satisfies (E_n').

6. (F_{n-1})

We prove (F_{n-1}). For $1 \leqq i \leqq n$, denote $a_i = \alpha_{N \setminus \{i\}}$, $b_i = \beta_{N \setminus \{i\}}$. Now $A^{(n-1)}$ and $B^{(n-1)}$ are normal with eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively. Applying (F_1) to $A^{(n-1)}$, $B^{(n-1)}$, we have

$$\operatorname{tr}(AB)^{(n-1)} = \operatorname{tr} A^{(n-1)} B^{(n-1)} \in \operatorname{co}\{a_1 b_{\sigma(1)} + \ldots + a_n b_{\sigma(n)} \mid \sigma \in S_n\} .$$

Since

$$a_1 b_{\sigma(1)} + \dots + a_n b_{\sigma(n)} = e_{n-1}(\alpha_1 \beta_{\sigma(1)}, \dots, \alpha_n \beta_{\sigma(n)}),$$

(F_{n-1}) follows.

7. (F_m), $1 \leqq m \leqq n$

Let $U \in \mathbb{C}^{n \times n}$ be unitary, $1 \leqq m \leqq n$. Consider the following condition:

(P_m) *There exist* t_σ*'s* $(\sigma \in S_n)$ *with* $t_\sigma \geqq 0$, $\sum_\sigma t_\sigma = 1$, *satisfying*

$$|U^{(m)}|^2 = \sum_\sigma t_\sigma |P_\sigma^{(m)}| .$$

Drury [1] proved that (P_m) is not generally valid if $n \geqq 4$ and $2 \leqq m \leqq n-2$. Let $G = A'UB'U^H$ be as in (F'_m). We claim that, for $1 \leqq m \leqq n$,

$$U \text{ satisfies } (\mathrm{P}_m) \Rightarrow G \text{ satisfies } (\mathrm{F}'_m) .$$

For the proof, denoting by su the sum of elements, we have

$$\begin{aligned} E_m(G) &= \operatorname{tr}(A'UB'U^H)^{(m)} = \operatorname{tr}((A')^{(m)} U^{(m)} (B')^{(m)} (U^H)^{(m)}) = \\ &= \operatorname{su}((A')^{(m)} |U^{(m)}|^2 (B')^{(m)}) = \operatorname{su}((A')^{(m)} (\sum_\sigma t_\sigma |P_\sigma^{(m)}|) (B')^{(m)}) = \\ &= \sum_\sigma t_\sigma \operatorname{su}((A')^{(m)} |P_\sigma^{(m)}|^2 (B')^{(m)}) = \\ &= \sum_\sigma t_\sigma \operatorname{tr}((A')^{(m)} P_\sigma^{(m)} (B')^{(m)} (P_\sigma^T)^{(m)}) = \\ &= \sum_\sigma t_\sigma \operatorname{tr}(A' P_\sigma B' P_\sigma^T)^{(m)} = \sum_\sigma t_\sigma e_m(\alpha_1 \beta_{\sigma(1)}, \dots, \alpha_n \beta_{\sigma(n)}) . \end{aligned}$$

Therefore (F'_m) holds.

Let us add one remark. For $1 \leqq m \leqq p \leqq n$, let $A_p \in C^{p \times p}$ be the principal submatrix of A corresponding to the p first rows and columns. Marcus and Sandy ([5], see also [4]) proved that

$$E_m(A_p) \in \operatorname{co} \{ e_m(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)}) \mid \sigma \in S_n \} .$$

It is easy to see that this is a special case of (F_m) where

$$B = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} .$$

Thus (F_m) holds in this special case.

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