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ON THE UNCOMPLEMENTED SUBSPACE $K(X,Y)$

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1. Introduction

Let us consider the long standing conjecture: The space $K(X,Y)$ of compact linear operators either equals the space $L(X,Y)$ of all bounded linear operators or is uncomplemented in $L(X,Y)$. A number of authors have treated this problem and have given an affirmative answer in certain cases [1, 3, 6, 7, 8, 9, 11, 13, 16, 17, 18, 19]. The most important of these positive results affirm that the answer is yes if one of the Banach spaces X or Y have an unconditional basis. Here we show (Theorem 1) that if the space $K(X,Y)$ contains a copy of c_0 then there is no projection of $L(X,Y)$ onto $K(X,Y)$ (unless X or Y have finite dimensions). Next, we give a sufficient condition for $K(X,Y)$ to contain c_0, namely that there is a noncompact operator $f\colon X \to Y$ which factors through a Banach space Z, Z having an unconditional basis (Theorem 2). Actually a more general types of (bases in) Z are sufficient (Remark 3a)). These two theorems (and Remark 3a) combine to give unification and generalization of probably all the known results on the problem. We feel that further generalizations might be possible along this line.

The methods of proofs are basically contained in [5, 11, 15, 19]. Generally the argument is as follows: The embedding of c_0 into $K(X,Y)$ may be extended to an embedding J of l_∞ into $L(X,Y)$,and if composed with a projection P of $L(X,Y)$ onto $K(X,Y)$ we would get an operator $PJ\colon l_\infty \to K(X,Y)$, which is an isomorphism on $c_0 \subset l_\infty$. Rosenthal's results yield that this operator is more or less actually isomorphism. But this is not possible because of good properties of the space of compact operators: $\sum PJ(e_i)$ would be converging—a contradiction to the assumption that the projection P exists. Several cases should be distinguished, e.g., when Y contains or does not contain c_0.

2. Notation and necessary background

Throughout the paper X, Y will be general Banach spaces. We will say that the Banach space X contains (a copy of) the Banach space X_1 if X contains an isomorphic copy of X_1. By an operator we always mean a bounded linear operator. Similarly, by a projection we understand a bounded linear projection. $K(X,Y)$ and $L(X,Y)$ denote the spaces of all compact and all bounded operators $T: X \to Y$, the spaces being equipped with the usual sup norm. The weak operator topology on $L(X,Y)$ is defined by the linear functionals $T \to y^*(Tx)$ for $x \in X$ and $y^* \in Y^*$. Following [11] we denote by w' the dual weak operator topology which is defined by the functionals $T \to x^{**}(T^*y^*)$ for $x^{**} \in X^{**}$ and $y^* \in Y^*$. The strong operator topology is given by the set of seminorms $T \to \|Tx\|$ for $x \in X$.

We will say that the series $\sum_{i=1}^{\infty} x_i$ of elements of X is weakly uncondionally Cauchy (w. u. C.) if $\sum_{i=1}^{\infty} |x^*(x_i)| < \infty$ for all $x^* \in X^*$ or, which is the same, if

$$\sup\left\{\left\|\sum_{i\in F} x_i\right\|;\ F \text{ finite}\right\} < \infty.$$

c_0, l_∞ denote the usual sequence spaces, and if M is a subset of natural numbers then $l_\infty(M)$ will denote the subspace formed only by the elements $\{x_i\}$ such that $x_i = 0$ outside M. By $e_n = \{\delta_{ni}\}_i$ we denote the usual basis elements of $c_0 \subset l_\infty$. $B_X{}^{**}$ and $B_Y{}^*$ will denote the closed unit balls in X^{**} and Y^* taken in its w^* topologies, respectively. If $T \in L(X,Y)$ is an operator then by $\tilde{T}$ we denote the function on $K = B_X{}^{**} \times B_Y{}^*$ given by

$$\tilde{T}(x^{**}, y^*) = x^{**}(T^*y^*).$$

Then $\tilde{T} \in C(K)$ iff $T \in K(X,Y)$ by [11, 20] and by the Lebesgue dominated convergence theorem we have:

Proposition 1 ([11]). *If $T_n \in K(X,Y)$ and $T \in K(X,Y)$, then: $T_n \to T$ in the weak topology of $K(X,Y)$ iff $T_n \to T$ in the w' topology iff $\tilde{T}_n \to \tilde{T}$ pointwise on $K = B_X{}^{**} \times B_Y{}^*$.*

We will make use also of the following results:

Proposition 2 ([15, 10]). *If T is an operator from l_∞ into a Banach space Y, then either $\|Te_n\| \to 0$ or there is an infinite index subset M such that $T|l_\infty(M)$ is an isomorphism.*

Proposition 3 ([11]). *Let $T: l_\infty \to l_\infty$ be an operator. Suppose that $Te_n = 0$ for all n. Then there exists an infinite index subset M such that $T|l_\infty(M) = 0$.*

Proposition 4 ([11]). *Let X, Y be Banach spaces. The following are equivalent:*
(i) *$K(X,Y)$ contains an isomorphic copy of l_∞.*
(ii) *Either Y contains a copy of l_∞ or X contains a complemented copy of l_1.*

Lemma 1 ([11]). *Suppose that X contains a complemented copy of l_1 and that Y is inifite dimensional. Then $K(X,Y)$ is uncomplemented in $L(X,Y)$.*

Remark 1. Notice that the above lemma will also be a consequence of Theorem 2. Indeed, let $Y_0 \subset Y$ be any separable infinite dimensional subspace, let P be the projection in X onto the subspace l_1 and let $Q\colon l_1 \to Y_0$ be a surjection. Then $T = QP\colon X \to Y$ is not compact and T is factorizable through l_1.

3. The main results

J. Johnson [9, Theorem 4] proved that if c_0 is a complemented subspace of Y (and X is infinite dimensional), then $K(X,Y)$ is not complemented in $L(X,Y)$. We start with a more general

Lemma 2. *Let the space Y contain a copy of c_0 and let X be an infinite dimensional Banach space. Then $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. Let I be the isomorphic embedding of $L(X,c_0)$ into $L(X,Y)$ which is canonically given by the embedding of c_0 into Y. As in [9] we choose, using the well known result of Josefson and Nissenzweig (cf. e.g. [4]), a sequence $\{x_n^*\} \subset X^*$ converging w^* to 0 so that $\|x_n^*\| = 1$ for each n. Given $x \in X$ and $\alpha = \{\alpha_i\} \in l_\infty$ define $J(\alpha)(x) = \{\alpha_n x_n^*(x)\}$. It is easy to verify that J is an isomorphism of l_∞ into $L(X,c_0)$ which sends c_0 into $K(X,c_0)$.

Now suppose that there is a projection P of $L(X,Y)$ onto $K(X,Y)$. Then the composition $S = PIJ\colon l_\infty \to K(X,Y)$ is obviously an isomorphism on c_0.

Next we consider an operator $S_1 = IJ\colon l_\infty \to L(X,Y)$ and observe that for every $\{\alpha_i\} \in l_\infty$

$$(*)\qquad \sum_{i=1}^{\infty} \alpha_i S_1 e_i \quad \text{converges in the } w' \text{ operator topology in } L(X,Y) \text{ to } S_1(\{\alpha_i\}).$$

Indeed, if $x'' \in X''$ and if $y' \in Y'$ is such that $y'|c_0 = \{y_i\} \in l_1$, then evidently $J(e_i)(x) = \{x_i^*(x)\delta_{ij}\}_j \in c_0$ and $J(\{\alpha_i\}) = \{\alpha_j x_j^*(x)\} \in c_0$. Thus $(S_1(e_i))^* y^* = y_i x_i^* \in X^*$ and $(S_1(\{\alpha\}))^* y^* = \sum_i \alpha_i y_i x_i^* \in X^*$ which implies that $x^{**}(S_1 e_i)^* y^* = y_i x^{**}(x_i)$ and $x^{**}(S_1(\{\alpha_i\}))^* y^* = \sum_i \alpha_i y_i x^{**}(x_i^*)$. This finally yields that

$$\lim_n \left(x^{**} \left(\left(\sum_{i=1}^{n} \alpha_i S_1(e_i) \right) \right)^* y^* \right) = x^{**}(S_1(\{\alpha_i\}))^* y^*.$$

Because evidently $S_1(e_i) = S(e_i)$ and $\|J(e_i)\| = \|x_i^*\| = 1$, Proposition 2 applied twice ensures that there is an infinite subset M_1 of natural numbers such that $S_1|l_\infty(M_1)$ and also $S|l_\infty(M_1)$ are isomorphisms. Now Kalton's Proposition 3 applied to $l_\infty(M_1)$ yields another infinite index set $M \subset M_1$ such that $S|l_\infty(M) = S_1|l_\infty(M)$. The above observations combine to give the folowing result: Let $e \in l_\infty$ denote the sequence $\{\alpha_i\}$ which is 1 for $i \in M$ and 0 otherwise. Then

$$\sum_{i \in M} S(e_i) = \sum_{i \in M} S_1(e_i) = S_1(e) = S(e)$$

and the sum converges in the w' operator topology by $(*)$. Now, because $S(e)$ is a compact operator, Proposition 1 gives that $\sum_{i \in M} S(e_i)$ converges to $S(e)$ weakly. Having in mind that S is an isomorphism on $l_\infty(M)$ and thus also a weak isomorphism, we get that $\sum_{i \in M} e_i$ converges to e in the weak topology of $l_\infty(M)$. Howver, this is impossible as one may easily see by considering the element $\varphi \in l_\infty(M)'$ of the form $\varphi(\{\alpha_i\}) =$ *Banach* $\lim_{i \in M} \alpha_i$. This completes our proof. □

Remark. Using the Orlicz-Pettis theorem we could have even proved that the sum $\sum_{i \in M} Se_i$ converges.

Theorem 1. *Let X, Y be arbitrary infinite dimensional Banach spaces and let $K(X,Y)$ contain an isomorphic copy of c_0. Then there is no projection of $L(X,Y)$ onto $K(X,Y)$ (nor $L(X,Y) = K(X,Y)$).*

Proof. In view of Lemmas 1 and 2 we may suppose that X does not contain a complemented copy of l_1 and Y does not contain a copy of c_0. Then $K(X,Y)$ does not contain a copy of l_∞ by Proposition 4. Now let $J\colon c_0 \to K(X,Y)$ be the isomorphic embedding. Then $\sum_i J(e_i)$ and thus also $\sum_i e_i$ are w. u. C. This in turn implies that $\sum J(e_i)(x)$ is again a w. u. C. series in Y for every $x \in X$. But because we suppose that Y does not contain c_0 we conclude using the classical result of Bessaga and Pelczynski [2 or 4] that $\sum_n J(e_i)(x)$ is unconditionally converging. Therefore J may be extended to $J\colon l_\infty \to L(X,Y)$ by the formula

$$J(\alpha)(x) = \sum_i \alpha_i J(e_i)(x)$$

where, as we have observed, the series $\sum_i \alpha_i J(e_i)$ converges in the strong operator topology in $L(X,Y)$. To show that J is continuous we again follow [11]. Indeed, we have already observed that $\sum_i \alpha_i J(e_i)(x)$ is w. u. C. for all x and this means that J is continuous for the topologies $\sigma(l_\infty, l_1)$ and the weak operator topology on $L(X,Y)$. This fact implies that J has a closed graph and thus J is continuous. If we now assumed again that P is a projection of $L(X,Y)$ onto $K(X,Y)$ we would get the continuous operator $S = PJ: l_\infty \to K(X,Y)$. Now S being an isomorphism on $c_0 \subset l_\infty$ we conclude again by Rosenthal (Proposition 2) that $S|l_\infty(M)$ is an isomorphism for some infinite M and thus $K(X,Y)$ would contain a copy of $l_\infty(M) \sim l_\infty$. But we have already observed that under our assumptions on X and Y the space $K(X,Y)$ contains no copy of l_∞. This contradiction shows that continuous the projection P cannot exist. □

Remark 2. A weaker form of Lemma 2 would also be sufficient for the proof of the theorem. If Y contains an isomorphic copy of l_∞ and X is infinite dimensional, then $K(X,Y)$ is not complemented in $L(X,Y)$. The proof of this weaker form of the Lemma 2 would be slightly easier spearing one usage of Proposition 2. This weaker form is also sufficient for the proof of Theorem 1. Indeed, the use of Kalton's Proposition 4 remains untouched and thus again $l_\infty \not\subset K(X,Y)$. If now c_0 were contained in Y, then $S = PIJ: l_\infty \to K(X,Y)$ defined as in the proof of Lemma 2 would be an isomorphism on some $l_\infty(M)$—a contradiction.

Theorem 2. *Let X, Y be arbitrary Banach spaces and $T: X \to Y$ a non-compact operator. Suppose that T admits a factorization $T = AB$ through a Banach space Z with an unconditional basis (countable or uncountable). Then the space $K(X,Y)$ of compact operators contains an isomorphic copy of c_0 and thus $K(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. Let $T = AB$ be the factorization of T through the Banach space Z and let $\{P_K\}_{K\in\mathcal{F}}$ be the finite dimensional projections in Z defined by the uconditional basis $\{u_i\}_{i\in I}$ with the unconditional basis constant C. Here $\mathcal{F}$ stands for the directed set of all finite subsets of the index set I. We follow the usual argument used in less general situations: Let $T_K: X \to Y$ be the compact operators $T_K = AP_KB$. Then $T_K \to T$ in the strong operator topology but not in the norm topology (otherwise T would be compact). This implies that $\{T_K\}$ is not a norm Cauchy net and thus there is a subsequence $\{T_{K_k}\} = \{S_k\}$ and $\varepsilon > 0$ such that $\|S_{2k} - S_{2k-1}\| > \varepsilon$. Now let $S_{[k]} = S_{2k} - S_{2k-1}$. We observe that $\sum_k S_{[k]}$ is w. u. C. In fact, let F be a finite subset of natural numbers. Then

$$\Big\|\sum_{k\in F} S_{[k]}\Big\| \leqslant \|A\|\cdot\|B\|\cdot\Big\|\sum_{k\in F} P_{K_{2k}} - P_{K_{2k-1}}\Big\| \leqslant \|A\|\cdot\|B\|\cdot C,$$

which shows that $\sum_k S_{[k]}$ is a w. u. C. series in $K(X,Y)$. This series is of course not converging because $\|S_{[k]}\| > \varepsilon$. Again by the result of Bessaga and Pelczynski [2] $\overline{\mathrm{sp}\{S_{[k]}\}} \subset K(X,Y)$ contains a copy of c_0. □

Remark 3. a) It is easy to see that we do not have actually to suppose in Theorem 2 that Z has an unconditional basis. In fact it is sufficient to suppose the existence of an unconditional expansion of identity $\{Z_\alpha\}$ in Z (this means that $\sum_\alpha Z_\alpha(z)$ converge to z unconditionally for every $z \in Z$) and such that all $AZ_\alpha B's$ are compact. This remark also applies to all the next remarks though for the sake of simplicity we state them only in terms of unconditional bases.

b) In [12] an operator $T\colon X \to Y$ is called linear with an unconditionally converging image decomposition (lucid) if it is of the form

$$T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n$$

and the series converges unconditionally for all $x \in X$. The authors show that such operators are exactly those factorizable through a Banach space Z, Z having an unconditional basis. Thus Theorem 2 may be rephrased: If there is a lucid non-compact operator $T\colon X \to Y$, then $c_0 \subset K(X,Y)$ and $K(X,Y)$ is not complemented in $L(X,Y)$.

c) If there is a non-compact $T \in L(X,Y)$ and the range of $T(X)$ is contained in a subspace $Z \subset Y$, Z having an unconditional basis, then $K(X,Y)$ is not complemented in $L(X,Y)$.

d) If there is a non-compact $T \in L(X,Y)$ factorizable through a Hilbert space, then $K(X,Y)$ is not complemented in $L(X,Y)$.

This remark applies to get [8]: Let P be a Pisier space (cf. [14, 8]). Then either $K(P,P^*) = L(P,P^*)$ or $K(P,P^*)$ is not complemented in $L(P,P^*)$.

e) Suppose that X and Y are infinite-dimensional Banach spaces and that each (non-compact) operator $T \in L(X,Y)$ is factorizable through a Banach space Z such that Z has an unconditional basis. Then the following are equivalent:

1) $K(X,Y)$ contains a copy of c_0,
2) $L(X,Y)$ contains a copy of c_0,
3) $L(X,Y)$ contains a copy of l_∞,
4) $K(X,Y) \neq L(X,Y)$,
5) $K(X,Y)$ is not complemented in $L(X,Y)$.

The assumptions of this remark apply particulary if X has an unconditional basis [3, 10, 18] or if Y is a complemented subspace of a space with an unconditional basis [3, 6, 19].

Proof. 1) $\Rightarrow$ 2) is trivial.

2) $\Rightarrow$ 3) holds even generally for any infinite dimensional X and any Y, as was actually shown in [11, Theorem 6, iii) $\Rightarrow$ ii)]. (We have to use Josefson and Nissenzweig in the first part of the proof.)

3) $\Rightarrow$ 4) again holds for any infinite dimensional Banach spaces X and Y: Suppose $l_\infty \subset L(X,Y)$ and $K(X,Y) = L(X,Y)$. Then $c_0 \subset l_\infty \subset K(X,Y)$ and thus $K(X,Y) \neq L(X,Y)$ by Theorem 1.

4) $\Rightarrow$ 1) follows from our assumptions and Theorem 2.

1) $\Rightarrow$ 5) again by Theorem 1.

5) $\Rightarrow$ 4) is always trivially true. □

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