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DOUBLE COVERS AND LOGICS OF GRAPHS II

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This paper is a continuation of results from [4]. The considered graphs are undirected graphs without loops and multiple edges.

The main concepts of this topic are the logic of a graph (introduced in [2] and based on a more general concept from [1]) and the double cover of a graph [3].

Let $V(G)$ be the vertex set of a graph G. If A is a subset of $V(G)$, then by $A^\perp$ we denote the set of all vertices of $V(G)$ which are adjacent to all vertices of A in G. Further we denote $A^{\perp\perp}=(A^\perp)^\perp$ and for a one-element subset $\{u\}$ of $V(G)$ we write $u^\perp$ and $u^{\perp\perp}$ instead of $\{u\}^\perp$ and $\{u\}^{\perp\perp}$.

Obviously $A\subseteq A^{\perp\perp}$ for each subset A of $V(G)$ and $A\subseteq B$ implies $B^\perp\subseteq A^\perp$ for any two subsets A, B of $V(G)$. For each subset A of $V(G)$ we have $(A^{\perp\perp})^\perp=(A^\perp)^{\perp\perp}=A^\perp$. If $A=\emptyset$, then $A^\perp=V(G)$, $A^{\perp\perp}=\emptyset$. If $A=A^{\perp\perp}$, we say that A is $\perp\perp$-closed. The $\perp\perp$-closed subsets of $V(G)$ form a complete lattice with respect to the set inclusion. This lattice together with the unary operation assigning $A^\perp$ to A (this operation is an operation of complementation on this lattice) is called the logic of the graph G and denoted by $\mathscr{L}(G)$. The least element of $\mathscr{L}(G)$ is the empty set, its greatest element is $V(G)$. For each $A\in\mathscr{L}(G)$ we have $A=\bigcap_{a\in A^\perp} a^\perp=\bigcup_{a\in A} a^{\perp\perp}$.

Also the following two assertions are evident. For any $A\subseteq V(G)$ we have $A^\perp=\bigcap_{a\in A} a^\perp$. For any system $\{A_i\}_{i\in I}$ of subsets of $V(G)$, where I is a subscript set, we have

$$\left(\bigcup_{i\in I} A_i\right)^\perp=\bigcap_{i\in I} A_i^\perp.$$

We shall not reproduce the general definition of the double cover of a graph. We shall study only a particular case of double covers — the bipartite double covers.

If G is a graph with the vertex set $V(G)$, then the bipartite double cover $B(G)$ of G is the bipartite graph on the (disjoint) sets $V=V(G)$ and $V'=\{v'\mid v\in V(G)\}$ such that if u is adjacent to v in G, then u is adjacent to v' and u' is adjacent to v in $B(G)$ and no other edges in $B(G)$ exist.

We shall consider some properties of graphs concerning the sets $A^\perp$.

Property $P1$. A graph G has no vertices of the degree 0 or 1 and $|u^\perp \cap v^\perp| \leqq 1$ for any two distinct vertices u, v of G.

Property $P2$. For any two vertices u, v of G the inclusion $u^\perp \subseteq v^\perp$ implies $u = v$.

Property $P3$. For any two vertices u, v of G the inclusion $u^\perp \subseteq v^\perp$ implies $u^\perp = v^\perp$.

Property $P4$. For any two vertices u, v of G the equality $u^\perp = v^\perp$ implies $u = v$.

Property $P5$. For each vertex $x \in V(G)$ and each subset $Y \subseteq V(G)$ the equality $x^\perp = Y^\perp$ implies $x \in Y$.

Property $P6$. For each vertex $x \in V(G)$ the element $x^\perp$ is completely meet--irreducible in $\mathscr{L}(G)$.

Evidently $P1 \Rightarrow P2 \Rightarrow P3$, but not conversely, $P2 \Rightarrow P4$, but not conversely, and $P2 \Leftrightarrow P3 \,\&\, P4$.

Proposition 1. *A graph G has the property $P5$ if and only if it has the properties $P4$ and $P6$.*

Proof. Let G have the properties $P4$ and $P6$. If $x \in V(G)$ and $Y \subseteq V(G)$ are such that $x^\perp = Y^\perp$, then $x^\perp = \bigcap_{y \in Y} y^\perp$. As G has the property $P6$, we have $x^\perp = y^\perp$ for an element $y \in Y$. According to the property $P4$ this implies $x = y \in Y$.

Conversely, let G have the property $P5$. Evidently it has also the property $P4$. If $x \in V(G)$ and $x^\perp = \bigcap_{i \in I} A_i$ for a family $\{A_i\}_{i \in I}$ of elements of $\mathscr{L}(G)$, then $x^\perp = \bigcap_{i \in I} A_i^{\perp\perp} = \left(\bigcup_{i \in I} A_i^\perp\right)^\perp$. According to the property $P5$ this implies $x \in A_i^\perp$ for some $i \in I$ and therefore $A_i \subseteq x^\perp$, which together with $x^\perp \subseteq A_i$ implies $x^\perp = A_i$. Hence $x^\perp$ is completely meet-irreducible in $\mathscr{L}(G)$. ■

Proposition 2. *If G, H are graphs with the property $P5$ and $\mathscr{L}(G) \cong \mathscr{L}(H)$, then $G \cong H$.*

Proof. Let φ: $\mathscr{L}(G) \to \mathscr{L}(H)$ be an isomorphism. According to the property $P6$ for each $x \in V(G)$ there exists $y \in V(H)$ such that $\varphi(x^\perp) = y^\perp$. According to the property $P4$ such an element y is unique. Define the mapping ψ: $V(G) \to V(H)$, $x \mapsto y$ in such a way that $\varphi(x^\perp) = y^\perp$. Evidently ψ is a bijection. If $x \in V(G)$, $y \in V(G)$, then $\{x, y\} \in E(G) \Leftrightarrow x \in y^\perp \Leftrightarrow x^{\perp\perp} \subseteq y^\perp \Leftrightarrow \varphi(x^{\perp\perp}) \subseteq \varphi(y^\perp) \Leftrightarrow \varphi(x^\perp)^\perp \subseteq \psi(y)^\perp \Leftrightarrow \psi(x)^{\perp\perp} \subseteq \psi(y)^\perp \Leftrightarrow \{\psi(x), \psi(y)\} \in E(H)$. Hence ψ is an isomorphism of the graphs G and H. ■

Now let G be a graph and let $A \in \mathscr{L}(G)$. If A is an atom in $\mathscr{L}(G)$, then $A = x^{\perp\perp}$ for an element $x \in V(G)$. If A is a dual atom in $\mathscr{L}(G)$, then $A = x^\perp$ for an element $x \in V(G)$. These as ertions are evident.

Proposition 3. *Let G be a graph. Then the following three assertions are equivalent:*
(i) *G has the property $P2$.*
(ii) *For any vertex u of G, $u^{\perp\perp} = \{u\}$.*
(iii) *The set of atoms of $\mathscr{L}(G)$ is equal to the set of all one-element subsets of $V(G)$.*

Proof. (i) $\Rightarrow$ (ii). If $u \in V(G)$ and $v \in u^{\perp\perp}$, then $u^{\perp} = u^{\perp\perp\perp} \subseteq v^{\perp}$ and according to the property $P2$ this implies $u = v$.

(ii) $\Rightarrow$ (i). If $u \in V(G)$, $v \in V(G)$ and $u^{\perp} \subseteq v^{\perp}$, then by (ii) we have $\{v\} = v^{\perp\perp} \subseteq u^{\perp\perp} = \{u\}$, hence $u = v$.

(ii) $\Leftrightarrow$ (iii). This is now evident. ∎

Proposition 4. *Let G be a graph. Then the following three assertions are equivalent:*
(i) *G has the property $P3$.*
(ii) *The set of atoms of $\mathscr{L}(G)$ is equal to the set of all sets $u^{\perp\perp}$ for $u \in V(G)$.*
(iii) *The set of dual atoms of $\mathscr{L}(G)$ is equal to the set of all sets $u^{\perp}$ for $u \in V(G)$.*

Proof. (i) $\Rightarrow$ (ii). If $u \in V(G)$, $\emptyset \neq A \in \mathscr{L}(G)$ and $A \subseteq u^{\perp\perp}$, then for each $a \in A$ we have $u^{\perp} \subseteq A^{\perp} \subseteq a^{\perp}$ and according to the property $P3$ this implies $u^{\perp} = A^{\perp} = a^{\perp}$. Then $A = u^{\perp\perp}$, because $A^{\perp\perp} = A$.

(ii) $\Rightarrow$ (iii). If $u \in V(G)$, $A \in \mathscr{L}(G)$, $A \neq V(G)$ and $u^{\perp} \subseteq A$, then $A^{\perp} \subseteq u^{\perp\perp}$ and according to (ii) this implies $A^{\perp} = u^{\perp\perp}$ and hence $A = u^{\perp}$.

(iii) $\Rightarrow$ (i). If $u \in V(G)$, $v \in V(G)$ and $u^{\perp} \subseteq v^{\perp}$, then according to (iii) we have $u^{\perp} = v^{\perp}$. Hence G has $P3$. ∎

Proposition 5. *Let G be a graph, $|V(G)| \geqq 2$. Then the following two assertions are equivalent:*
(i) *G has the property $P1$.*
(ii) *The logic $\mathscr{L}(G)$ of G consists of the least element $\emptyset$, the set of atoms equal to the set of all one-element subsets of $V(G)$, the set of dual atoms equal to the set of all sets $u^{\perp}$ for $u \in V(G)$ and the greatest element $V(G)$ and no atom of $\mathscr{L}(G)$ is equal to a dual atom of $\mathscr{L}(G)$.*

Proof. (i) $\Rightarrow$ (ii). Let G have the property $P1$. Then it has also the properties $P2$ and $P3$. Hence the set of atoms of $\mathscr{L}(G)$ is the set of all one-element subsets of $V(G)$ (by Proposition 3) and the set of dual atoms of $\mathscr{L}(G)$ is the set of all sets $u^{\perp}$ for $u \in V(G)$ (by Proposition 4). Let $A \in \mathscr{L}(G)$, $A \neq V(G)$. Since $A = \bigcap_{a \in A^{\perp}} a^{\perp}$, A is either a dual atom of $\mathscr{L}(G)$, or the intersection of at least two dual atoms of $\mathscr{L}(G)$. The property $P1$ implies that in the latter case $|A| \leqq 1$, hence either $A = \emptyset$, or A is a one-element subset of $V(G)$, i.e. an atom of $\mathscr{L}(G)$. As G has no vertex of the degree 0 or 1, each dual atom of $\mathscr{L}(G)$ contains at least two vertices and it cannot be equal to an atom of $\mathscr{L}(G)$.

(ii) $\Rightarrow$ (i). Let (ii) hold. Then the meet (i.e. the intersection) of any two dual atoms is either the least element (i.e. $\emptyset$), or an atom (i.e. a one-element set). Hence $|u^{\perp} \cap v^{\perp}| \leqq 1$ for any two distinct vertices u, v of G. As no dual atom is equal to an atom and as $|V(G)| \geqq 2$, we have $|u^{\perp}| \geqq 2$ for each $u \in V(G)$ and there is no vertex of the degree 0 or 1. ■

Theorem 1. *Let G be a graph. Let H be the ordered subset of $\mathscr{L}(G)$ consisting of all $x^{\perp}$ and all $x^{\perp\perp}$ for $x \in V(G)$ with the ordering induced by that of $\mathscr{L}(G)$. Let $\leqq$ be the following ordering on the vertex set $V(B(G))$ of the bipartite double cover $B(G)$ of G: for $x, y \in V(B(G))$, $x \leqq y \Leftrightarrow x = y$ or $x \in V'$, $y \in V$ and $\{x, y\}$ is an edge in $B(G)$. Let φ: $V(B(G)) \to H$ be such that $\varphi(x) = x^{\perp}$, $\varphi(x') = x^{\perp\perp}$ for all $x \in V(G)$. Then φ is an isomorphism of ordered sets if and only if G has the property $P2$ and has no vertices of the degree 0 or 1.*

Proof. Evidently the mapping φ is a surjection and $x \leqq y$ implies $\varphi(x) \subseteq \varphi(y)$ for any x, y from $V(B(G))$. If φ is an isomorphism, $x \in V(G)$, $y \in V(G)$ and $x^{\perp} \subseteq y^{\perp}$, then $x \leqq y$, because $x^{\perp} = \varphi(x)$ and $y^{\perp} = \varphi(y)$. As both x, y are in V, there cannot be $x < y$ and we have $x = y$. We have the property $P2$. If $x^{\perp} = \emptyset$ for an element $x \in V$, then, according to the property $P2$, $|V(G)| = 1$, which is a contradiction. If $x^{\perp} = \{y\}$ for some $x \in V(G)$, $y \in V(G)$, then $\varphi(x') = x^{\perp\perp} = y^{\perp} = \varphi(y)$, which is a contradiction, because $x' \neq y$. Conversely, let G have the property $P2$ and let it have no vertex of the degree 0 or 1. Let x, y be two vertices of $B(G)$ and let $\varphi(x) \subseteq \varphi(y)$. We shall consider all possible cases. If both x, y belong to V, then $x^{\perp} \subseteq y^{\perp}$, which implies $x = y$ according to the property $P2$. If x, y belong to V', then $x = z'$, $y = t'$ for some vertices z, t of G. Then $\{z\} = z^{\perp\perp} \subseteq t^{\perp\perp} = \{t\}$ and hence $x = y$. If $x \in V'$, $y \in V$, then $x = z'$ for $z \in V(G)$ and $\{z\} = z^{\perp\perp} \subseteq y^{\perp}$, which implies $x \leqq y$. If $x \in V$, $y \in V'$, then $y = t'$ for $t \in V(G)$ and $x^{\perp} \subseteq t^{\perp\perp} = \{t\}$, which is a contradiction. Thus we have proved that φ is an isomorphism.

Corollary. *Let G be a graph with the property $P1$. Let H be the graph obtained from the Hasse diagram of $\mathscr{L}(G)$ by deleting the vertices corresponding to $V(G)$ and $\emptyset$. Then H is isomorphic to $B(G)$.*

By the symbol $\operatorname{Aut} G$ the automorphism group of a graph G will be denoted. For each $\alpha \in \operatorname{Aut} G$ we define the mapping α' such that $\alpha'(A) = \{\alpha(a) | a \in A\}$ for each subset A of $V(G)$. Then evidently for each $A \in \mathscr{L}(G)$ we have $\alpha'(A) \in \mathscr{L}(G)$. Further $\alpha'(A)^{\perp} = \alpha'(A^{\perp})$ for each $A \subseteq V(G)$. If A, B are two subsets of $V(G)$, then $A \subseteq B \Leftrightarrow \alpha'(A) \subseteq \alpha'(B)$. The restriction α^* of α' onto $\mathscr{L}(G)$ belongs to the automorphism group $\operatorname{Aut} \mathscr{L}(G)$ of $\mathscr{L}(G)$. The mapping φ: $\operatorname{Aut} G \to \operatorname{Aut} \mathscr{L}(G)$, $\alpha \mapsto \alpha^*$ is evidently a homomorphism of groups.

Theorem 2. *Let G be a graph. Then φ is an imbedding if and only if G has the property $P4$. If G has the property $P2$, then φ is an isomorphism.*

Proof. If φ is not a surjection, then there exist mappings α, β from $\operatorname{Aut} G$ such that $\alpha \neq \beta$ and $\alpha^* = \beta^*$ and therefore there exists $x \in V(G)$ such that

$u=\alpha(x)\neq v=\beta(x)$. Then $u^{\perp}=\alpha(x)^{\perp}=\alpha^*(x^{\perp})=\beta^*(x^{\perp})=\beta(x)^{\perp}=v^{\perp}$. Hence G has not the property $P4$. Conversely, let u, v be two vertices of G such that $u\neq v$ and $u^{\perp}=v^{\perp}$. Let α be a mapping of $V(G)$ onto $V(G)$ such that $\alpha(u)=v$, $\alpha(v)=u$, $\alpha(x)=x$ for any x distinct from u and v. Then $\alpha\in\operatorname{Aut}G$. If ω is the identity automorphism of G, then $\alpha\neq\omega$, $\alpha^*=\omega^*$ and φ is not an injection.

Now suppose that the graph G has the property $P2$. Then it has also the property $P4$ and φ is an injection. Evidently $\{u\}\in\mathscr{L}(G)$ for each vertex $u\in V(G)$. Let $\beta\in\operatorname{Aut}\mathscr{L}(G)$. Define the mapping α: $V(G)\to V(G)$, $x\mapsto y$ so that $\beta(\{x\})=\{y\}$. Evidently $\alpha\in\operatorname{Aut}G$. If $A\in\mathscr{L}(G)$, then $A=\bigvee_{a\in A}\{a\}$; this implies $\beta(A)=\bigvee_{a\in A}\beta(\{a\})=\bigvee_{a\in A}\{\alpha(a)\}=\alpha^*(A)$. Hence $\beta=\alpha^*$ and φ is a surjection. ■

Now consider a graph G with the property $P2$. If G has a vertex of the degree 0, then G consists only of this vertex. If G has a vertex of the degree 1, then there exists a connected component of G isomorphic to the complete graph K_2 with two vertices; other connected components of G are either isomorphic to K_2, or with the property $P2$ and without vertices of the degree 0 or 1.

Theorem 3. *Let G be a graph with the property $P2$. Then the group of all automorphism of $B(G)$ which map V onto V and V' onto V' is isomorphic to the group of all lattice automorphisms of $\mathscr{L}(G)$.*

Remark. By a lattice automorphism of $\mathscr{L}(G)$ we mean a bijection of $\mathscr{L}(G)$ onto itself which preserves the lattice operations, but need not preserve the mapping $A\mapsto A^{\perp}$.

Proof. First suppose that G has no vertices of the degree 0 or 1. Then we may take the mapping φ from Theorem 1 and consider its inverse φ^{-1}. This is an isomorphism of H onto $V(B(G))$ (as ordered set) which maps the set $\mathscr{A}$ of atoms of $\mathscr{L}(G)$ onto V and the set $\mathscr{D}$ of dual atoms of $\mathscr{L}(G)$ onto V'. Therefore it suffices to prove that each automorphism of H which maps $\mathscr{A}$ onto $\mathscr{A}$ and $\mathscr{D}$ onto $\mathscr{D}$ can be uniquely extended to an automorphism of $\mathscr{L}(G)$. Let α be an automorphism of H which maps $\mathscr{A}$ onto $\mathscr{A}$ and $\mathscr{D}$ onto $\mathscr{D}$. Evidently this is not only an automorphism of H, but also an order automorphism of $\mathscr{A}\cup\mathscr{D}$. If $u\in V(G)$, then let $\alpha_0(u)$ be the vertex v of G such that $\{v\}=\alpha(\{u\})$. If A is a dual atom of $\mathscr{L}(G)$, then evidently $\alpha(A)=\{\alpha_0(u)\mid u\in A\}$; therefore the images of dual atoms in α are uniquely determined by the images of atoms. As each element of $\mathscr{L}(G)$ distinct from $V(G)$ is an intersection of dual atoms, evidently the unique possible extension of α to a lattice automorphism of $\mathscr{L}(G)$ is given by $\alpha(A)=\{\alpha_0(u)\mid u\in A\}$ for each $A\in\mathscr{L}(G)$. This extension is the image of α in an isomorphism of the group of all automorphisms of H which map $\mathscr{A}$ onto $\mathscr{A}$ and $\mathscr{D}$ onto $\mathscr{D}$ onto the group of all lattice automorphisms of $\mathscr{L}(G)$.

If G has the vertices of the degree 0 or 1, the proof can be easily made using the assertions which were written above this theorem. If G has a vertex of the degree 0, the proof is trivial. In the case when G has vertices of the degree 1 we take into account that the logic of a disconnected graph is isomorphic to the algebra obtained from the logics of its connected components by identifying all least elements and all greatest elements.

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ДВОЙНЫЕ ПОКРЫТИЯ И ЛОГИКИ ГРАФОВ II

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Резюме

Логика графа есть решетка определенных подмножеств множества вершин графа. Двойное покрытие графа есть определенный граф, соответствующий заданному графу. Исследуются соотношения между этими двумя понятиями.