Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 2, 395–415

Persistent URL: http://dml.cz/dmlcz/128265

Czechoslovak Mathematical Journal, 58 (133) (2008), 395–415

GENERALIZATIONS OF PSEUDO MV-ALGEBRAS AND GENERALIZED PSEUDO EFFECT ALGEBRAS

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(Received February 22, 2006)

Abstract. We deal with unbounded dually residuated lattices that generalize pseudo MV-algebras in such a way that every principal order-ideal is a pseudo MV-algebra. We describe the connections of these *generalized pseudo MV-algebras* to generalized pseudo effect algebras, which allows us to represent every generalized pseudo MV-algebra A by means of the positive cone of a suitable ℓ-group G_A. We prove that the lattice of all (normal) ideals of A and the lattice of all (normal) convex ℓ-subgroups of G_A are isomorphic. We also introduce the concept of Archimedeanness and show that every Archimedean generalized pseudo MV-algebra is commutative.

Keywords: pseudo MV-algebra, $DR\ell$-monoid, generalized pseudo effect algebra

MSC 2000: 06F05, 03G25

Introduction

The recent research on algebras connected to fuzzy logic is concerned, among others, with their non-commutative generalizations, i.e., the truth functions of strong conjunction and disjunction are not assumed to be commutative. This began with pseudo MV-algebras (see [12], [24]), a non-commutative version of the well-known MV-algebras which are the algebraic semantics of the Łukasziewicz many valued propositional calculus.

Pseudo MV-algebras can be equivalently treated as bounded dually residuated lattices ($DR\ell$-monoids) satisfying simple additional identities, and it is therefore natural to view certain $DR\ell$-monoids as "unbounded" pseudo MV-algebras. Of course, this can be equally done in the setting of residuated lattices, but we favour

Supported by the Research and Development Council of the Czech Govenrment via the project MSM6198959214.

dually residuated ones since the initial definition of pseudo MV-algebras is closer to dually residuated lattices.

In [20] we studied many properties of the lattice of all ideals (= convex subalgebras) of these $DR\ell$-monoids which turned out to be markedly similar to the properties of ideal lattices of pseudo MV-algebras. Taking into account the fact that the ideal lattice of any pseudo MV-algebra is isomorphic to the lattice of all convex ℓ-subgroups of a suitable ℓ-group, the question arises whether the same holds for our "unbounded" pseudo MV-algebras. In the present paper, we give the affirmative answer by means of the so-called generalized pseudo effect algebras (see [10]) that are an extension of effect algebras provided we drop the commutativity of the partial addition as well as the existence of a greatest element.

The paper is organized as follows. In Section 1 we recall the basic properties of pseudo MV-algebras and dually residuated ℓ-monoids. We also prove that every generalized pseudo MV-algebra ($GPMV$-algebra) embeds into an ultraproduct of a family of pseudo MV-algebras. Section 2 is devoted to the relations between our $GPMV$-algebras and generalized pseudo effect algebras, which allows us to give a representation of $GPMV$-algebras as lattice ideals in the positive cones of ℓ-groups. In Section 3 we prove that the lattice of (normal) ideals of every $GPMV$-algebra is isomorphic to the lattice of all (normal) convex ℓ-subgroups of some ℓ-group. This is applied in Section 4 to obtain simple alternative proofs of our earlier results from [20]. Finally, in Section 5 we deal with the Archimedean property of $GPMV$-algebras.

1. Pseudo MV-algebras and dually residuated lattices

Definition 1.1. A *pseudo MV-algebra* is an algebra $(A, \oplus, ^-, ^\sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0\rangle$ that satisfies the identities

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(A2) $x \oplus 0 = x = 0 \oplus x$,

(A3) $x \oplus 1 = 1 = 1 \oplus x$,

(A4) $1^- = 0 = 1^\sim$,

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,

(A6) $x \oplus (y \odot x^\sim) = y \oplus (x \odot y^\sim) = (y^- \odot x) \oplus y = (x^- \odot y) \oplus x$,

(A7) $(x^- \oplus y) \odot x = y \odot (x \oplus y^\sim)$,

(A8) $(x^-)^\sim = x$,

where the supplementary binary operation $\odot$ is defined by[1]

$$x \odot y := (x^- \oplus y^-)^\sim.$$

[1] In [12], $x \odot y$ was defined as $(y^- \oplus x^-)^\sim$.

As we have pointed out at the beginning, pseudo MV-algebras were introduced by G. Georgescu and A. Iorgulescu [12] and independently by J. Rachůnek [24] as a non-commutative generalization of MV-algebras. Actually, if the addition $\oplus$ is commutative then the unary operations $^-$ and $^\sim$ coincide and the resulting algebra becomes an MV-algebra.

The above definition is that by G. Georgescu and A. Iorgulescu, while J. Rachůnek's one arising from C. C. Chang's original definition of MV-algebras was more complicated. Nevertheless, both concepts are equivalent.

Like MV-algebras, pseudo MV-algebras are very close to ℓ-groups:

Example 1.2. Let $(G, +, -, 0, \vee, \wedge)$ be an ℓ-group and $u \in G$ an order-unit.[2] Then $\Gamma(G, u) := ([0, u], \oplus, ^-, ^\sim, 0, u)$ is a pseudo MV-algebra, where $[0, u] = \{x \in G\colon\ 0 \leqslant x \leqslant u\}$ and

$$x \oplus y := (x + y) \wedge u,\ \ x^- := u - x \ \text{ and } \ x^\sim := -x + u$$

for $x, y \in [0, u]$.

A. Dvurečenskij [5] enhanced D. Mundici's famous result on MV-algebras and Abelian ℓ-groups [23] and proved that every pseudo MV-algebra is obtained in that form; i.e., for every pseudo MV-algebra A there exists an ℓ-group G with an order-unit u such that A and $\Gamma(G, u)$ are isomorphic.

As proved in [24], pseudo MV-algebras can be considered as a particular case of the so-called $DR\ell$-monoids that were introduced and studied by K. L. N. Swamy [26] as a common abstraction of Abelian ℓ-groups and Boolean algebras. The definition we use here is adopted from T. Kovář's thesis [21].

First of all, by an *ℓ-monoid* we mean an algebra $(A, \oplus, 0, \vee, \wedge)$, where $(A, \oplus, 0)$ is a monoid, $(A, \vee, \wedge)$ is a lattice and $\oplus$ distributes over $\vee$, i.e., A fulfils the equations

$$(x \vee y) \oplus z = (x \oplus z) \vee (y \oplus z), \quad x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z).$$

Definition 1.3. An algebra $(A, \oplus, 0, \vee, \wedge, \oslash, \otimes)$ of type $\langle 2, 0, 2, 2, 2, 2\rangle$ is called a *dually residuated ℓ-monoid* or briefly a *$DR\ell$-monoid* if

(a) $(A, \oplus, 0, \vee, \wedge)$ is an ℓ-monoid;

(b) for any $x, y \in A$, $x \oslash y$ is the least element $z \in A$ such that $z \oplus y \geqslant x$, and $x \otimes y$ is the least element $z \in A$ such that $y \oplus z \geqslant x$;

[2] We call $u \geqslant 0$ an *order-unit* of G if for every $x \in G$ there exists $n \in \mathbb{N}$ such that $-nu \leqslant x \leqslant nu$; this is equivalent to saying that the convex ℓ-subgroup of G generated by u is G.

(c) A satisfies the identities

$$((x \oslash y) \vee 0) \oplus y \leqslant x \vee y, \quad y \oplus ((x \otimes y) \vee 0) \leqslant x \vee y,$$
$$x \oslash x \geqslant 0, \quad x \otimes x \geqslant 0.$$

A $DR\ell$-monoid is called *lower bounded* provided 0 is its least element. A *bounded $DR\ell$-monoid* is an algebra $(A, \oplus, \vee, \wedge, \oslash, \otimes, 0, 1)$ such that $(A, \oplus, 0, \vee, \wedge, \oslash, \otimes)$ is a $DR\ell$-monoid with a greatest element 1.

Lemma 1.4. *The following assertions hold in any $DR\ell$-monoid:*
(1) $x \oplus y \geqslant z$ *iff* $x \geqslant z \oslash y$ *iff* $y \geqslant z \otimes x$,
(2) $x \vee y = ((x \oslash y) \vee 0) \oplus y = y \oplus ((x \otimes y) \vee 0)$,
(3) $x \oslash 0 = x \otimes 0 = x, \; x \oslash x = x \otimes x = 0$,
(4) $(x \vee y) \oslash z = (x \oslash z) \vee (y \oslash z), \; (x \vee y) \otimes z = (x \otimes z) \vee (y \otimes z)$,
(5) $x \oslash (y \wedge z) = (x \oslash y) \vee (x \oslash z), \; x \otimes (y \wedge z) = (x \otimes y) \vee (x \otimes z)$,
(6) $x \oslash (y \oplus z) = (x \oslash z) \oslash y, \; x \otimes (y \oplus z) = (x \otimes y) \otimes z$,
(7) $(x \oslash y) \otimes z = (x \otimes z) \oslash y$,
(8) $(x \oslash y) \oplus (y \oslash z) \geqslant x \oslash z, \; (y \otimes z) \oplus (x \otimes y) \geqslant x \otimes z$,
(9) $(x \oplus z) \oslash (y \oplus z) \leqslant x \oslash y, \; (x \oplus y) \otimes (x \oplus z) \leqslant y \otimes z$.

Remark 1.5. Seeing the definition and basic properties of $DR\ell$-monoids, it should be evident that our $DR\ell$-monoids are dual to residuated lattices satisfying the divisibility identities. To be more precise, a *residuated lattice* is an algebra $(L, \vee, \wedge, \cdot, \rightarrow, \rightsquigarrow, e)$, where $(L, \vee, \wedge)$ is a lattice, $(L, \cdot, e)$ is a monoid and

$$x \cdot y \leqslant z \quad \text{iff} \quad x \leqslant y \rightarrow z \quad \text{iff} \quad y \leqslant x \rightsquigarrow z$$

for all $x, y, z \in L$. If, moreover, e is the greatest element of L then L is called an *integral residuated lattice*. A residuated lattice that fulfils the divisibility identities

$$x \wedge y = ((y \rightarrow x) \wedge e) \cdot y = y \cdot ((y \rightsquigarrow x) \wedge e)$$

is called a *GBL-algebra* (see [11], [17]).

It is plain that given any $DR\ell$-monoid $(A, \oplus, 0, \vee, \wedge, \oslash, \otimes)$, then the dual structure $(A, \sqcup, \sqcap, \cdot, \rightarrow, \rightsquigarrow, e)$ defined by $x \sqcup y := x \wedge y$, $x \sqcap y := x \vee y$, $x \cdot y := x \oplus y$, $x \rightarrow y := y \oslash x$, $x \rightsquigarrow y := y \otimes x$ and $e := 0$ is a GBL-algebra.

The converse need not be evident at once. As known, the multiplication in residuated lattices distributes over joins and it can be proved that in the case of GBL-algebras it distributes over meets, too. This was shown in [7] for integral GBL-algebras, but with minor modifications the proof still works for arbitrary GBL-algebras. Finally, any GBL-algebra verifies $x \rightarrow x = x \rightsquigarrow x = e$ (see [11]), and therefore, if $(L, \vee, \wedge, \cdot, \rightarrow, \rightsquigarrow, e)$ is a GBL-algebra then defining $x \oplus y := x \cdot y$, $0 := e$, $x \sqcup y := x \wedge y$, $x \sqcap y := x \vee y$, $x \oslash y := y \rightarrow x$ and $x \rightsquigarrow y := y \otimes x$ we get a $DR\ell$-monoid $(A, \oplus, 0, \sqcup, \sqcap, \oslash, \otimes)$.

Altogether, the class of $DR\ell$-monoids is termwise equivalent to the class of GBL-algebras.

Now, we turn back to pseudo MV-algebras. Let $(A, \oplus, ^-, ^\sim, 0, 1)$ be a pseudo MV-algebra and define

$$(1.1)\qquad \begin{aligned} x \vee y &:= x \oplus (y \odot x^\sim) = (x^- \odot y) \oplus x, \\ x \wedge y &:= x \odot (y \oplus x^\sim) = (x^- \oplus y) \odot x, \\ x \oslash y &:= y^- \odot x, \\ x \otimes y &:= x \odot y^\sim. \end{aligned}$$

Observe that for $A = \Gamma(G, u)$ the lattice operations $\vee$ and $\wedge$ in A given by (1.1) are the restrictions of those in G to the interval $[0, u]$ and we have $x \oslash y = (x - y) \vee 0$ and $x \otimes y = (-y + x) \vee 0$. A straightforward verification yields that $(A, \oplus, \vee, \wedge, \oslash, \otimes, 0, 1)$ is a bounded $DR\ell$-monoid satisfying

$$(1.2)\qquad x \wedge y = x \otimes (x \oslash y) = x \oslash (x \otimes y),$$

and conversely, given a bounded $DR\ell$-monoid that fulfils (1.2), the algebra $(A, \oplus, ^-, ^\sim, 0, 1)$—where $x^- := 1 \oslash x$ and $x^\sim := 1 \otimes x$—is a pseudo MV-algebra.

Remark 1.6. The identities (1.2) can be even replaced by the seemingly weaker equations

$$(1.3)\qquad x = 1 \otimes (1 \oslash x) = 1 \oslash (1 \otimes x).$$

Indeed, in any bounded $DR\ell$-monoid satisfying (1.3) we have

$$\begin{aligned} x \wedge y &= (1 \oslash (1 \otimes x)) \wedge (1 \oslash (1 \otimes y)) \\ &= 1 \oslash ((1 \otimes x) \vee (1 \otimes y)) \\ &= 1 \oslash (((1 \otimes y) \oslash (1 \otimes x)) \oplus (1 \otimes x)) \\ &= 1 \oslash (((1 \oslash (1 \otimes x)) \otimes y) \oplus (1 \otimes x)) \\ &= 1 \oslash ((x \otimes y) \oplus (1 \otimes x)) \\ &= (1 \oslash (1 \otimes x)) \oslash (x \otimes y) \\ &= x \oslash (x \otimes y) \end{aligned}$$

and similarly $x \wedge y = x \otimes (x \oslash y)$. This observation is essentially due to A. Iorgulescu [16].

Summarizing, pseudo MV-algebras are termwise equivalent to bounded $DR\ell$-monoids verifying (1.2), and hence the $DR\ell$-monoids that satisfy (1.2) are the desired generalization of pseudo MV-algebras.

Note that though a $DR\ell$-monoid A satisfying (1.2) need not have a greatest element, it is always lower bounded because $x \wedge 0 = x \odot (x \oslash 0) = x \odot x = 0$ for all $x \in A$.

Definition 1.7. A *generalized pseudo MV-algebra*, in short: a *$GPMV$-algebra*, is a $DR\ell$-monoid satisfying the identities (1.2).

Residuated lattices that are equivalent to our $GPMV$-algebras appear in literature on residuated lattices under the name (*integral*) *GMV-algebras* (see [2], [11], [17]). Another equivalent counterpart are *Wajsberg pseudo hoops* (see [13]).

It is easy to see that $GPMV$-algebras extend pseudo MV-algebras in such a way that every principal order-ideal is a pseudo MV-algebra:

Lemma 1.8. *Let $(A, \oplus, 0, \vee, \wedge, \oslash, \odot)$ be a $GPMV$-algebra and $a \in A$. If we define*

$$x \oplus_a y := (x \oplus y) \wedge a$$

for $x, y \in [0, a]$, then $A[a] := ([0, a], \oplus_a, \vee, \wedge, \oslash, \odot, 0, a)$ is a bounded $GPMV$-algebra.

It is worth noticing that for arbitrary $x, y, a \in A$ we have

$$(x \wedge a) \oplus_a (y \wedge a) = (x \oplus y) \wedge a.$$

We close this section with proving that every $GPMV$-algebra embeds into a pseudo MV-algebra:

Theorem 1.9. *Every $GPMV$-algebra can be isomorphically embedded into a bounded $GPMV$-algebra.*

Proof. Let A be a $GPMV$-algebra. We shall show that A can be embedded into an ultraproduct of $\{A[a]\colon a \in A\}$.

It is easy to see that $[a) \cap [b) = [a \vee b) \neq \emptyset$ for all $a, b \in A$, so the set $\{[a)\colon a \in A\}$ has the finite intersection property and hence there exists an ultrafilter U in the Boolean algebra 2^A of all subsets of A such that $\{[a)\colon a \in A\} \subseteq U$. Let

$$B = \prod_{a \in A} A[a]/U$$

be the ultraproduct of $\{A[a]\colon a \in A\}$ over U. Clearly, B is a bounded $GPMV$-algebra. Recall that the ultraproduct B is the quotient algebra $\prod\limits_{a \in A} A[a]/\theta_U$, where

θ_U is the congruence on the direct product $\prod_{a\in A} A[a]$ given by $(\alpha,\beta)\in\theta_U$ iff $\{a\in A\colon \alpha(a)=\beta(a)\}\in U$; the elements of B are denoted α/U or, in more detail, $(\alpha(a)\colon a\in A)/U$.

Now, we define a mapping $f\colon A\to B$ via

$$f(x) := (x\wedge a\colon a\in A)/U,$$

which turns out to be the desired isomorphic embedding.

f is injective: Note that for any $x,y\in A$, $f(x)=f(y)$ iff $\{a\in A\colon x\wedge a=y\wedge a\}\in U$. Assume that $x\neq y$. It is clear that whenever $a\geqslant x\vee y$ then $x\wedge a=x\neq y=y\wedge a$, and hence $[x\vee y)\subseteq\{a\in A\colon x\wedge a\neq y\wedge a\}$. Since $[x\vee y)\in U$, also $\{a\in A\colon x\wedge a\neq y\wedge a\}\in U$. But $\{a\in A\colon x\wedge a\neq y\wedge a\}$ is the complement of $\{a\in A\colon x\wedge a=y\wedge a\}$ in the Boolean algebra 2^A, and consequently, $\{a\in A\colon x\wedge a=y\wedge a\}\notin U$ since U is an ultrafilter in 2^A. This shows that $f(x)\neq f(y)$ provided $x\neq y$.

f preserves $\oplus$: We have $f(x\oplus y)=((x\oplus y)\wedge a\colon a\in A)/U$ on the one hand and $f(x)\oplus f(y)=(x\wedge a\colon a\in A)/U\oplus(y\wedge a\colon a\in A)/U=((x\wedge a)\oplus_a(y\wedge a)\colon a\in A)/U=((x\oplus y)\wedge a\colon a\in A)/U$ on the other, so that $f(x\oplus y)=f(x)\oplus f(y)$.

f preserves $\oslash$: We have $f(x\oslash y)=((x\oslash y)\wedge a\colon a\in A)/U$ and $f(x)\oslash f(y)=(x\wedge a\colon a\in A)/U\oslash(y\wedge a\colon a\in A)/U=((x\wedge a)\oslash(y\wedge a)\colon a\in A)/U$, thus $f(x\oslash y)=f(x)\oslash f(y)$ iff $\{a\in A\colon (x\oslash y)\wedge a=(x\wedge a)\oslash(y\wedge a)\}\in U$. Let $x\geqslant a$. Then $(x\oslash y)\wedge a=x\oslash y$ and $(x\wedge a)\oslash(y\wedge a)=x\oslash y$. This yields $[x)\subseteq\{a\in A\colon (x\oslash y)\wedge a=(x\wedge a)\oslash(y\wedge a)\}$ and hence $\{a\in A\colon (x\oslash y)\wedge a=(x\wedge a)\oslash(y\wedge a)\}\in U$ as desired.

It can be shown analogously that f preserves $\obslash$ as well as both $\vee$ and $\wedge$. $\square$

Since bounded $GPMV$-algebras are de facto pseudo MV-algebras that can be represented as intervals in ℓ-groups, we immediately obtain:

Corollary 1.10. *For every $GPMV$-algebra $(A,\oplus,0,\vee,\wedge,\oslash,\obslash)$ there exists an ℓ-group $(G,+,-,0,\vee,\wedge)$ and an element $0<u\in G$ such that $(A,\oplus,0,\vee,\wedge,\oslash,\obslash)$ is isomorphic to a subalgebra of $([0,u],\oplus,0,\vee,\wedge,\oslash,\obslash)$, where*

$$x\oplus y:=(x+y)\wedge u,\ \ x\oslash y:=(x-y)\vee 0\ \ \text{and}\ \ x\obslash y:=(-y+x)\vee 0.$$

2. Generalized pseudo effect algebras

Generalized pseudo effect algebras were invented by A. Dvurečenskij and T. Vetterlein [10] as a generalization of effect algebras—partial additive structures related to the logic of quantum mechanics (see e.g. [6])—omitting both commutativity and boundedness:

A *generalized pseudo effect algebra* or simply a *GPE-algebra* is a structure $(E, +, 0)$, where 0 is an element of E and $+$ is a partial binary operation on E satifying the following axioms, for all $a, b, c \in E$:

(E1) $a + b$ and $(a + b) + c$ exist iff $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$;

(E2) if $a + b$ exists then $a + b = x + a = b + y$ for some $x, y \in E$;

(E3) if $a + c$ and $b + c$ exist and are equal then $a = b$, if $c + a$ and $c + b$ exist and are equal then $a = b$;

(E4) if $a + b$ exists and equals 0 then $a = b = 0$;

(E5) $a + 0$ and $0 + a$ exist and $a + 0 = a = 0 + a$.

We define a partial order $\leqslant$ on E by $a \leqslant b$ iff $b = x + a$ for some $x \in E$, which is equivalent to $b = a + y$ for some $y \in E$. Clearly, 0 is the least element of $(E, \leqslant)$. If $(E, \leqslant)$ is a lattice then $(E, +, 0)$ is called a *lattice-ordered* GPE-algebra.

A *pseudo effect algebra* is a structure $(E, +, 0, 1)$ such that $(E, +, 0)$ is a GPE-algebra having a greatest element 1. In other words, pseudo effect algebras are bounded GPE-algebras. Moreover, if the partial addition $+$ is commutative then $(E, +, 0, 1)$ is an *effect algebra* (see [8], [9]).

Natural examples of GPE-algebras arise from positive cones of partially ordered groups:

Example 2.1 [10]**.** Let $(G, +, -, 0, \leqslant)$ be a partially ordered group and let X be a non-empty subset of its positive cone $G^+ = \{g \in G\colon\ 0 \leqslant g\}$ such that whenever $a, b \in X$ and $a \leqslant b$ then $b - a, -a + b \in X$. Then $(X, +, 0)$ is a GPE-algebra, where $+$ is the restriction of the group addition to those pairs of elements of X whose sum belongs to X. Thus, in particular, $(G^+, +, 0)$ is a GPE-algebra.

Given a pseudo MV-algebra $(A, \oplus, ^-, ^\sim, 0, 1)$, one defines a partial addition $+$ making A a pseudo effect algebra as follows (see [6], [5]): $a + b$ is defined and equal to $a \oplus b$ iff $a \leqslant b^-$ (alternatively, iff $b \leqslant a^\sim$). If we view A as a bounded $GPMV$-algebra, then $a \wedge b^- = (1 \oslash b) \odot ((1 \oslash b) \oslash a) = (1 \oslash b) \odot (1 \oslash (a \oplus b)) = (1 \odot (1 \oslash (a \oplus b))) \oslash b = (a \oplus b) \oslash b$, and hence $a \leqslant b^-$ is equivalent to $(a \oplus b) \oslash b = a$.

This observation allows one to introduce a partial addition also in any $GPMV$-algebra $(A, \oplus, 0, \vee, \wedge, \oslash, \odot)$ in the following way:

$$a + b \ \text{ is defined iff } \ (a \oplus b) \oslash b = a, \quad \text{in which case } \ a + b := a \oplus b,$$

or equivalently,

$$a + b \text{ is defined iff } \ (a \oplus b) \odot a = b, \quad \text{in which case } \ a + b := a \oplus b.$$

The two definitions are easily seen to be equivalent. Indeed, if $(a \oplus b) \oslash b = a$ then $(a \oplus b) \odot a = (a \oplus b) \odot ((a \oplus b) \oslash b) = (a \oplus b) \wedge b = b$, and vice versa.

We say that a *GPE*-algebra $(E, +, 0)$ satisfies the *Weak Riesz Decomposition Property* (RDP_0), if for all $a, b, c \in E$, $a \leqslant b + c$ implies the existence of $b_1, c_1 \in E$ such that $b_1 \leqslant b$, $c_1 \leqslant c$ and $a = b_1 + c_1$.

Proposition 2.2. *For any GPMV-algebra* $(A, \oplus, 0, \vee, \wedge, \oslash, \odot)$*, the structure* $(A, +, 0)$ *is a lattice-ordered GPE-algebra satisfying* (RDP_0)*. Moreover, for every* $a, b \in A$*,*

(a) $a \oplus b = \max\{a_1 + b_1 \colon a_1 \leqslant a, b_1 \leqslant b$ *and* $a_1 + b_1$ *is defined*$\}$*,*

(b) $a \oslash b$ *is the unique* $x \in A$ *with* $x + (a \wedge b) = a$ *and* $a \odot b$ *is the unique* $y \in A$ *with* $(a \wedge b) + y = a$*.*

Proof. (E1) Let $a + b$ and $(a + b) + c$ exist in A. Then

$$c = ((a \oplus b) \oplus c) \odot (a \oplus b) = (a \oplus (b \oplus c)) \odot (a \oplus b) \leqslant (b \oplus c) \odot b \leqslant c$$

by (9) of Lemma 1.4, thus $(b \oplus c) \odot b = c$ and $b + c$ is defined. Further, by Lemma 1.4 (6), $(a \oplus (b \oplus c)) \oslash (b \oplus c) = (((a \oplus b) \oplus c) \oslash c) \oslash b = (a \oplus b) \oslash b = a$, so $a + (b + c)$ is also defined.

(E2) Let $a + b$ be defined. Then $((a \oplus b) \oslash a) \oplus a = (a \oplus b) \vee a = a \oplus b$, whence $(((a \oplus b) \oslash a) \oplus a) \oslash a = (a \oplus b) \oslash a$, so that $((a \oplus b) \oslash a) + a$ exists. We have shown that $a + b = c + a$, where $c = (a \oplus b) \oslash a$. Similarly $a + b = b + d$ for $d = (a \oplus b) \odot b$.

(E3) Assume that $a + c$ and $b + c$ exist and are equal. From $a + c = b + c$ it follows that $a = (a + c) \oslash c = (b + c) \oslash c = b$.

(E4) If $a + b$ is defined then clearly $a = b = 0$ whenever $a + b = 0$.

(E5) We have $(a \oplus 0) \oslash 0 = 0$, so $a + 0 = a$.

For (RDP_0), let $a \leqslant b + c$ and denote $b_1 = a \wedge b$ and $c_1 = a \odot b_1$. Then $c_1 = a \odot (a \wedge b) = a \odot b \leqslant c$, whence $b_1 \oplus c_1 = b_1 \oplus (a \odot b_1) = a \vee b_1 = a$, and consequently, $b_1 + c_1$ is defined since $(b_1 \oplus c_1) \odot b_1 = a \odot b_1 = c_1$.

To prove (a) is suffices to note that either $a \oplus b = ((a \oplus b) \oslash b) + b$ or $a \oplus b = a + ((a \oplus b) \odot a)$.

Finally, for (b), $(a \oslash b) + (a \wedge b)$ is defined and equal to a since $(a \oslash b) \oplus (a \wedge b) = (a \oslash (a \wedge b)) \oplus (a \wedge b) = a \vee (a \wedge b) = a$ and hence $((a \oslash b) \oplus (a \wedge b)) \oslash (a \wedge b) = a \oslash (a \wedge b) = a \oslash b$. Thus $a \oslash b$ is the unique x with $x + (a \wedge b) = a$. Analogously, $a \odot b$ is the unique y with $(a \wedge b) + y = a$. $\square$

For the reverse passage from certain GPE-algebras to $GPMV$-algebras we need the following technical lemma:

Lemma 2.3 [10]. *Let $(E,+,0)$ be a GPE-algebra and $a,b,c \in E$.*
(i) *If $a+b$ exists then a_1+b_1 exists for every $a_1 \leqslant a$, $b_1 \leqslant b$.*
(ii) *If $b+c$ exists then $a \leqslant b$ iff $a+c$ exists and $a+c \leqslant b+c$. Similarly, if $c+b$ exists then $a \leqslant b$ iff $c+a$ exists and $c+a \leqslant c+b$.*

Proposition 2.4. *Let $(E,+,0)$ be a lattice-ordered GPE-algebra satisfying (RDP$_0$) such that for every $a,b \in E$ there exists*

$$a \oplus b := \max\{a_1+b_1\colon\ a_1 \leqslant a, b_1 \leqslant b \text{ and } a_1+b_1 \text{ is defined}\}.$$

Then $(E,\oplus,0,\vee,\wedge,\oslash,\odot)$—where $a \oslash b$ is the unique $x \in E$ with $x+(a\wedge b)=a$ and $a \odot b$ is the unique $y \in E$ with $(a\wedge b)+y=a$—is a $GPMV$-algebra.

Proof. First, we show that the operation $\oplus$ is associative. We have

$$(a\oplus b)\oplus c = \max\{d_1+c_1\colon\ d_1 \leqslant a\oplus b, c_1 \leqslant c \text{ and } d_1+c_1 \text{ exists}\}.$$

But if $d_1 \leqslant a\oplus b$ then due to the definition of $\oplus$ and (RDP$_0$) there are $a_1 \leqslant a$ and $b_1 \leqslant b$ such that $d_1 = a_1+b_1$. Hence

$$\begin{aligned}(a\oplus b)\oplus c &= \max\{(a_1+b_1)+c_1\colon\ a_1 \leqslant a, b_1 \leqslant b, c_1 \leqslant c \text{ and } (a_1+b_1)+c_1 \text{ exists}\}\\ &= \max\{a_1+b_1+c_1\colon\ a_1 \leqslant a, b_1 \leqslant b, c_1 \leqslant c \text{ and } a_1+b_1+c_1 \text{ exists}\}.\end{aligned}$$

Analogously,

$$a\oplus(b\oplus c) = \max\{a_1+b_1+c_1\colon\ a_1 \leqslant a, b_1 \leqslant b, c_1 \leqslant c \text{ and } a_1+b_1+c_1 \text{ exists}\},$$

so that $(a\oplus b)\oplus c = a\oplus(b\oplus c)$.

Obviously, $a\oplus 0 = a = 0\oplus a$, thus $(E,\oplus,0)$ is a monoid.

Now, we prove that $c \geqslant a\oslash b$ iff $c\oplus b \geqslant a$. If $a\oslash b \leqslant c$ then $a \leqslant c\oplus b = \max\{c_1+b_1\colon\ c_1 \leqslant c, b_1 \leqslant b, c_1+b_1 \text{ exists}\}$ since $a=(a\oslash b)+(a\wedge b)$, where $a\oslash b \leqslant c$ and $a\wedge b \leqslant b$. Conversely, let $a \leqslant c\oplus b$. Then $a=c_1+b_1$ for some $c_1 \leqslant c$, $b_1 \leqslant b$. Note that $b_1 \leqslant a$ and so $b_1 \leqslant a\wedge b$. Since $(a\oslash b)+(a\wedge b)$ exists, it follows that so does $(a\oslash b)+b_1$ and we have $(a\oslash b)+b_1 \leqslant (a\oslash b)+(a\wedge b) = a = c_1+b_1$, which implies $a\oslash b \leqslant c_1 \leqslant c$ as desired. Similarly, $c \geqslant a\odot b$ is equivalent to $b\oplus c \geqslant a$. Thus $(A,\oplus,0,\vee,\wedge,\oslash,\odot)$ is a dually residuated lattice.

It remains to verify that $a\wedge b = a\oslash(a\odot b) = a\odot(a\oslash b)$ for all $a,b\in E$. We have $a\odot b = x$, where $(a\wedge b)+x=a$, and $a\oslash(a\odot b)=a\oslash x = y$, where $y+(a\wedge x)=a$. But $a\wedge x = x$, so $y+x=a=(a\wedge b)+x$ whence $y=a\wedge b$ follows. Analogously, $a\odot(a\oslash b)=a\wedge b$. □

Combining Propositions 2.2 and 2.4, $GPMV$-algebras are equivalent to those lattice-ordered GPE-algebras satisfying the Weak Riesz Decomposition Property (RDP_0) where

$$a \oplus b := \max\{a_1 + b_1 \colon\ a_1 \leqslant a, b_1 \leqslant b \text{ and } a_1 + b_1 \text{ is defined}\}$$

exists for all a, b.

By [9], Theorem 8.8, pseudo MV-algebras (= bounded $GPMV$-algebras) are in a one-to-one correspondence with lattice-ordered pseudo effect algebras (= bounded GPE-algebras) satisfying (RDP_0). Hence, if a given GPE-algebra has an upper bound 1, then $a \oplus b$ exists and

$$a \oplus b = (a \wedge (1 \oslash b)) + b = a + ((1 \varobslash a) \wedge b),$$

where $1 \oslash b$ and $1 \varobslash a$ are the unique x, y such that $x + b = 1$ and $a + y = 1$, respectively.

Many GPE-algebras are obtained as in Example 2.1:

Proposition 2.5 [10]. *Every GPE-algebra $(E, +, 0)$ which is a meet-semilattice and satisfies* (RDP_0) *can be isomorphically embedded into the positive cone $(G_E^+, +, 0)$ of an ℓ-group $(G_E, +, -, 0, \vee, \wedge)$ such that finite infima and existing finite suprema are preserved, and moreover, assuming $E \subseteq G_E$, E is a convex subset of G_E^+ that generates G_E^+ as a semigroup.*

Let $(E, +, 0)$ be a lattice-ordered GPE-algebra that obeys (RDP_0) as in Proposition 2.4 and let $(G_E, +, -, 0, \vee, \wedge)$ be the ℓ-group with the positive cone G_E^+ into which $(E, +, 0)$ can be embedded as in Proposition 2.5. Assume that $E \subseteq G_E^+$. Then, for every $a, b \in E$,

$$a \oplus b = \max\{a_1 + b_1 \colon\ a_1 \leqslant a, b_1 \leqslant b \text{ and } a_1 + b_1 \in E\} \tag{2.1}$$

and

$$\begin{aligned} a \oslash b &= a - (a \wedge b) = (a - b) \vee 0, \\ a \varobslash b &= -(a \wedge b) + a = (-b + a) \vee 0. \end{aligned} \tag{2.2}$$

Now, by Propositions 2.5 and 2.2 we obtain:

Theorem 2.6. *For every $GPMV$-algebra A there exists a lattice-ordered group G_A such that A can be embedded into G_A^+ in such a way that finite suprema and infima are preserved, and assuming $A \subseteq G_A^+$, the operations $\oslash$ and $\otimes$ are given by* (2.2) *and A is a lattice ideal which generates G_A^+ as a semigroup.*

Another important observation concerns morphisms of GPE-algebras. We recall from [10] that, given GPE-algebras E and F, a mapping $f\colon E \to F$ is called a *GPE-homomorphism* if $f(0) = 0$ and $f(a+b) = f(a) + f(b)$ provided $a + b$ exists in E.

Proposition 2.7 [10]. *Let E and G_E be as in Proposition* 2.5, *assume that $E \subseteq G_E$. Every meet-preserving GPE-homomorphism f of E into the positive cone H^+ of a ℓ-group H can be uniquely extended to an ℓ-group homomorphism of G_E into H.*

Let f be a homomorphism of a $GPMV$-algebra A into a $GPMV$-algebra B. Trivially, $f(0) = 0$. Suppose that $a + b$ is defined in A, i.e., $(a \oplus b) \oslash b = a$. Then $(f(a) \oplus f(b)) \oslash f(b) = f((a \oplus b) \oslash b) = f(a)$ showing that $f(a) + f(b)$ is defined in B. Thus f is a GPE-homomorphism which evidently preserves infima. Hence we get:

Corollary 2.8. *Let A and B be $GPMV$-algebras, G_A and G_B their representing ℓ-groups from Theorem* 2.6, *and assume $A \subseteq G_A$, $B \subseteq G_B$. Then every homomorphism $f\colon A \to B$ extends uniquely to an ℓ-group homomorphism $\hat{f}\colon G_A \to G_B$.*

3. The ideal lattice

The concept of an ideal of a general $DR\ell$-monoid was introduced and studied in [18]. Here we restrict ourselves to the case of $GPMV$-algebras (which are necessarily lower bounded):

An *ideal* of a $GPMV$-algebra A is a non-empty subset I such that

(I1) $a \oplus b \in I$ for all $a, b \in I$,

(I2) if $a \in I$ and $b \leqslant a$ then $b \in I$.

It is easy to prove that for every $\emptyset \neq I \subseteq A$, the following assertions are equivalent:

1. I is an ideal,
2. I is a convex subalgebra of A,
3. for all $a, b \in A$, if $a \in I$ and $b \oslash a \in I$ then $b \in I$,
4. for all $a, b \in A$, if $a \in I$ and $b \otimes a \in I$ then $b \in I$.

We use $\mathfrak{I}(A)$ to denote the set of all ideals of A; it is an algebraic distributive lattice when ordered by set-inclusion. For any $\emptyset \neq X \subseteq A$, the set

$$I(X) = \{a \in A\colon\ a \leqslant x_1 \oplus \ldots \oplus x_n \text{ for some } x_1, \ldots, x_n \in X, n \in \mathbb{N}\}$$

is the smallest ideal containing X.

An ideal $I \in \mathfrak{I}(A)$ is called *normal* if, for all $a, b \in A$,

$$a \oslash b \in I \quad \text{iff} \quad a \otimes b \in I.$$

This is equivalent to saying that[3] $a \oplus I = I \oplus a$ for every $a \in A$. There is a one-to-one correspondence between the normal ideals of A and its congruences. Namely, given a normal ideal I, the relation Θ_I defined by

$$(a, b) \in \Theta_I \quad \text{iff} \quad (a \oslash b) \vee (b \oslash a) \in I$$

is a congruence whose kernel $[0]_{\Theta_I} = \{a \in A\colon\ (a, 0) \in \Theta_I\}$ is I, and conversely, given a congruence Θ, $I = [0]_\Theta$ is the normal ideal such that $\Theta_I = \Theta$.

We write simply a/I instead of $[a]_{\Theta_I} = \{b \in A\colon\ (a, b) \in \Theta_I\}$ and, accordingly, the quotient algebra A/Θ_I is denoted by A/I.

From now on, we assume that A is a $GPMV$-algebra, G_A the ℓ-group from Theorem 2.6, and $A \subseteq G_A$.

Proposition 3.1. *If I is an ideal in A then*[4]

$$\varphi_A(I) := G_A(I)$$

is a convex ℓ-subgroup of G_A such that $I = \varphi_A(I) \cap A$.

If K is a convex ℓ-subgroup of G_A then

$$\psi_A(K) := K \cap A$$

is an ideal in A such that $K = G_A(\psi_A(K))$.

Proof. It is clear that $I \subseteq \varphi_A(I) \cap A$ for every $I \in \mathfrak{I}(A)$. Conversely, if $x \in \varphi_A(I) \cap A$ then $x \geqslant 0$ and so $x = a_1 + \ldots + a_n$ for some $a_1, \ldots, a_n \in I$. Since $x \in A$, it follows that $x \in I$, proving $\varphi_A(I) \cap A \subseteq I$.

For the latter claim, let $K \in \mathfrak{C}(G_A)$. We first prove that $\psi_A(K)$ is an ideal in A. Obviously, $0 \in \psi_A(K)$. Take $a, b \in A$ and suppose that $a \oslash b, b \in \psi_A(K)$. Then

[3] We write $a \oplus I$ and $I \oplus a$ for $\{a \oplus x\colon\ x \in I\}$ and $\{x \oplus a\colon\ x \in I\}$, respectively.

[4] For $X \subseteq G_A$, $G_A(X)$ is the convex ℓ-subgroup of G_A generated by X.

$0 \leqslant a \leqslant a \vee b = (a \oslash b) \oplus b = (a \oslash b) + b \in K \cap A$, so $a \in K \cap A = \psi_A(K)$. Thus $\psi_A(K) \in \mathfrak{I}(A)$.

Further, we prove that the convex ℓ-subgroup of G_A generated by $\psi_A(K)$ is just K. If $x \in K$, $x \geqslant 0$, then $x = a_1 + \ldots + a_n$ for some $a_1, \ldots, a_n \in A$. But $0 \leqslant a_i \leqslant x$ implies $a_i \in K \cap A$ for all $i = 1, \ldots, n$, and hence $x \in G_A(\psi_A(K))$. If x is an arbitrary element of K then $0 \leqslant |x| = x \vee -x \in K$ and the same argument yields $|x| \in G_A(\psi_A(K))$, so that $x \in G_A(\psi_A(K))$. This shows $K \subseteq G_A(\psi_A(K))$. The other inclusion is evident. □

Next, we focus our attention on congruence kernels—normal ideals of generalized pseudo MV-algebras and ℓ-ideals of ℓ-groups.

Proposition 3.2. *For any $I \in \mathfrak{I}(A)$, I is a normal ideal of A if and only if $\varphi_A(I)$ is an ℓ-ideal of G_A. For any $K \in \mathfrak{C}(G_A)$, K is an ℓ-ideal if and only if $\psi_A(K)$ is a normal ideal of A.*

P r o o f. Let K be an ℓ-ideal of G_A, i.e., a normal convex ℓ-subgroup. Observe that $x - (x \wedge y) \in K$ iff $-(x \wedge y) + x \in K$ for all $x, y \in G_A$. Indeed, if $x - (x \wedge y) \in K$ then $x = (x - (x \wedge y)) + (x \wedge y) \in K + (x \wedge y) = (x \wedge y) + K$ since K is a normal subgroup of G_A. This means $x = (x \wedge y) + z$ for some $z \in K$, so that $-(x \wedge y) + x = z \in K$. Analogously $-(x \wedge y) + x \in K$ yields $x - (x \wedge y) \in K$.

Consequently, if $a \oslash b \in \psi_A(K) = K \cap A$ for $a, b \in A$, then also $a \mathbin{\varnothing} b \in \psi_A(K)$, and vice versa. Thus $\psi_A(K)$ is a normal ideal in A provided K is an ℓ-ideal in G_A.

Conversely, let I be a normal ideal of A. Let f be the canonical homomorphism of A onto the quotient algebra A/I given by $f(a) := a/I$. By Theorem 2.6, A/I may be embedded into the positive cone of an ℓ-group $G_{A/I}$ as a lattice ideal that generates $G^+_{A/I}$. By Corollary 2.8, f extends to an ℓ-group homomorphism $\hat{f}\colon G_A \to G_{A/I}$, i.e., $\hat{f}(a) = a/I$ for each $a \in A$. We are going to show that $G_A(I) = \operatorname{Ker}(\hat{f})$.

Let $x \in G_A(I)$. If $x \geqslant 0$ then $x = a_1 + \ldots + a_n$ for some $a_1, \ldots, a_n \in I$, whence we obtain $\hat{f}(x) = \hat{f}(a_1) + \ldots + \hat{f}(a_n) = a_1/I + \ldots + a_n/I = I$ since $a_i \in I$ for every $i = 1, \ldots, n$. Thus $x \in \operatorname{Ker}(\hat{f})$. If $x \in G_A(I)$ is arbitrary then similarly $|x| \in \operatorname{Ker}(\hat{f})$, which yields $x \in \operatorname{Ker}(\hat{f})$. Hence $G_A(I) \subseteq \operatorname{Ker}(\hat{f})$.

On the other hand, let $x \in \operatorname{Ker}(\hat{f})$, i.e., $\hat{f}(x) = I$. If $x \geqslant 0$ then $x = a_1 + \ldots + a_n$ for some $a_1, \ldots, a_n \in A$. But $0 \leqslant a_i \leqslant x$ implies $I = \hat{f}(0) \leqslant \hat{f}(a_i) \leqslant \hat{f}(x) = I$, so $\hat{f}(a_i) = I$ and hence $a_i \in I$ for all $i = 1, \ldots, n$. This means $x = a_1 + \ldots + a_n \in G_A(I)$. The parallel argument shows that $|x| \in G_A(I)$ for an arbitrary $x \in \operatorname{Ker}(\hat{f})$, and thus $x \in G_A(I)$. Altogether, $G_A(I) = \operatorname{Ker}(\hat{f})$, which certainly is an ℓ-ideal of G_A. □

Let us denote the lattice of all normal ideals of A by $\mathfrak{N}\mathfrak{I}(A)$ and the lattice of all ℓ-ideals of G_A by $\mathfrak{N}\mathfrak{C}(G_A)$. We have proved:

Theorem 3.3. *The ideal lattice $\mathfrak{I}(A)$ of A is isomorphic to the lattice $\mathfrak{C}(G_A)$ of all convex ℓ-subgroups of G_A under the mapping φ_A whose inverse is ψ_A. In addition, the restriction $\varphi_A \restriction_{\mathfrak{N}\mathfrak{I}(A)}$ is an isomorphism of $\mathfrak{N}\mathfrak{I}(A)$ onto $\mathfrak{N}\mathfrak{C}(G_A)$ the inverse of which is the restriction $\psi_A \restriction_{\mathfrak{N}\mathfrak{C}(G_A)}$.*

Corollary 3.4. *A $GPMV$-algebra A is linearly ordered if and only if G_A is a linearly ordered group.*

Proof. One readily sees that if A is linearly ordered then its ideal lattice $\mathfrak{I}(A)$, and hence likewise the lattice $\mathfrak{C}(G_A)$ of convex ℓ-subgroups of G_A, is a chain with respect to set-inclusion. But in this case G_A is a linearly ordered group. □

4. Values and complete distributivity

By Zorn's lemma, the set of all ideals that do not contain a given $a \in A \setminus \{0\}$ has a maximal element; such an ideal is called a *value* of a in A. We use $\Gamma_A(a)$ to denote the set of all values of a in A. It is easily seen that if $V \in \Gamma_A(a)$ for some $a \in A \setminus \{0\}$ then V has a unique cover V^* in the lattice $\mathfrak{I}(A)$. Of course, $a \in V^* \setminus V$. A value V is *normal* provided it is a normal ideal in its cover V^*. If all values are normal then A is called a *normal-valued* $GPMV$-algebra.

It is also worth noticing that V is a value in A if and only if it is a completely meet-irreducible element of the ideal lattice $\mathfrak{I}(A)$, and hence, since $\mathfrak{I}(A)$ is algebraic, it follows that every ideal equals the intersection of all values containing it.

An element $a \in A$ is said to be *special* if it has a unique value; the only value of a special element is called the *special value.*

A $GPMV$-algebra A is *finite-valued* if $\Gamma_A(a)$ is finite for all $a \in A \setminus \{0\}$.

Let now A be a $GPMV$-algebra, G_A its representing ℓ-group and let $A \subseteq G_A$. In view of Theorem 3.3 it is obvious that an ideal V is a value of $a \in A \setminus \{0\}$ if and only if $\varphi_A(V)$ is a value of a in G_A, and moreover, $\varphi_A(V^*)$ is the cover of $\varphi_A(V)$ in the lattice $\mathfrak{C}(G_A)$. As known, an ℓ-group is finite-valued if and only if every value is special, therefore we get (cf. [19]):

Theorem 4.1. *A $GPMV$-algebra A is finite-valued if and only if every value in A is special.*

Further, for any ideal $I \in \mathfrak{I}(A)$, $\varphi_A(I) = G_A(I)$ is precisely its representing ℓ-group G_I. This entails that a value V in A is normal in its cover V^* if and only if $\varphi_A(V)$ is normal in its cover $\varphi_A(V)^* = \varphi_A(V^*)$. Indeed, V is normal in V^* if and only if $\varphi_{V^*}(V) = G_{V^*}(V) = G_A(V) = \varphi_A(V)$ is normal in $G_{V^*} = \varphi_A(V^*)$.

As a corollary we have that A is normal-valued if and only if so is the ℓ-group G_A. Using the fact that in ℓ-groups special values are normal, we obtain:

Theorem 4.2. *Let A be a $GPMV$-algebra. Then every special value is normal. Consequently, if A is finite-valued then it is normal-valued.*

Let $X \subseteq A$. It is plain that the embedding of A into G_A preserves arbitrary existing infima, i.e., $\inf_A X$ exists iff so does $\inf_{G_A} X$, in which case they are equal. The analogue for suprema holds, too.

Lemma 4.3. *For any $X \subseteq A$, if $\sup_A X$ exists then $\sup_A X = \sup_{G_A} X$; if $\sup_{G_A} X$ exists and belongs to A then $\sup_A X = \sup_{G_A} X$.*

P r o o f. Denote $x_0 := \sup_A X$. Let $a \in G_A$ be another upper bound of X. Then $x_0 \wedge a \in A$ and $x_0 \wedge a \geqslant x$ for every $x \in X$, hence $a \geqslant x_0$, proving that x_0 is the l.u.b. of X.

The latter claim is obvious. □

An ideal $I \in \mathfrak{I}(A)$ is defined to be *closed* if $\sup_A X \in I$ for every $X \subseteq I$ whose supremum exists in A.

We call an ideal $P \in \mathfrak{I}(A)$ *prime* if it is a prime element of the ideal lattice $\mathfrak{I}(A)$, i.e., for any $I, J \in \mathfrak{I}(A)$, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. Equivalently, P is prime if and only if $a \wedge b \in P$ entails $a \in P$ or $b \in P$ for all $a, b \in A$. Note that every value is a prime ideal.

Proposition 4.4. *Let P be a prime ideal of A. Then P is closed if and only if $\varphi_A(P)$ is a closed prime subgroup of G_A.*

P r o o f. First note that P is a prime ideal iff $\varphi_A(P)$ is a prime subgroup of G_A, so we may assume that $P \neq A$.

Let P be closed, let $X \subseteq \varphi_A(P) \cap G_A^+$ and $x_0 := \sup_{G_A} X$. Take any $a \in A \setminus P$. Then $a \wedge x_0 \in A$ and $a \wedge x \in P$ for every $x \in X$. Since P is closed, we have $a \wedge x_0 = \bigvee\limits_{x \in X} (a \wedge x) \in P$. However, $a \notin \varphi_A(P)$ and $\varphi_A(P)$ is a prime subgroup of G_A, and so $x_0 \in P$.

Conversely, P is easily seen to be closed whenever $\varphi_A(P)$ is a closed prime subgroup. □

As a consequence we have (cf. [20]):

Proposition 4.5. *Given $P, Q \in \mathfrak{I}(A)$ with $P \subseteq Q$, if P is closed prime then so is Q.*

Proof. This follows from the fact that $\varphi_A(Q) \supseteq \varphi_A(P)$ is a closed prime subgroup of G_A whenever so is $\varphi_A(P)$. □

A value V in A is called *essential* if it contains all values of some $a \in A \setminus \{0\}$. Evidently, V is an essential value in A iff so is $\varphi_A(V)$ in G_A. Since essential values in ℓ-groups are closed, by the previous proposition we obtain (cf. [20]):

Proposition 4.6. *Let A be a $GPMV$-algebra. Every essential value is closed; in particular, every special value is closed. If, moreover, A is normal-valued then every closed value is essential.*

Proof. We have to justify the latter statement. For that purpose, suppose that V is a closed value of some $a \in A \setminus \{0\}$. Then $\varphi_A(V)$ is a closed value of a in the ℓ-group G_A which is normal-valued. It is known that in the case of normal-valued ℓ-groups closed values are essential, hence $\varphi_A(V)$ contains all values of some $x \in G_A^+ \setminus \{0\}$. It is clear now that every value $W \in \mathit{\Gamma}_A(a \wedge x)$ is contained in V, so V is essential. □

Let A be a $GPMV$-algebra. The *distributive radical* of A is the intersection of all closed prime ideals of A. Since any closed prime ideal is the intersection of the values exceeding it every one of which is closed, it can be easily seen that $D(A)$ equals the intersection of all closed values in A. Observe that $a \in D(A)$ if and only if a has no closed value.

Proposition 4.7. $\varphi_A(D(A)) = D(G_A)$.

Proof. Let $x \in \varphi_A(D(A))$, $x \geqslant 0$, i.e., $x = a_1 + \ldots + a_n$ where $a_1, \ldots, a_n \in D(A)$. Since a_i's have no closed values in A, they have no closed values in G_A either, which yields that $a_i \in D(G_A)$ for all $i = 1, \ldots, n$. Consequently, $x \in D(G_A)$.

Conversely, if $x \in D(G_A)$, $x \geqslant 0$, then $x = a_1 + \ldots + a_n$ for some $a_1, \ldots, a_n \in A$, and x has no closed value in G_A. If $V \in \mathit{\Gamma}_A(a_i)$, then $x \notin \varphi_A(V)$, and so $\varphi_A(V) \subseteq M$ for some $M \in \mathit{\Gamma}_{G_A}(x)$. Therefore $\varphi_A(V)$, and hence V, is not closed. This yields $a_i \in D(A)$ for any $i = 1, \ldots, n$, so that $x \in \varphi_A(D(A))$. □

Note that the distributive radical $D(A)$ of A is a (closed) normal ideal since $D(G_A)$ is an ℓ-ideal of G_A (see e.g. [3], 6.2.2).

We say that a $GPMV$-algebra A is *completely distributive* if

$$\bigwedge_{s\in S}\bigvee_{t\in T} a_{st} = \bigvee_{f\colon S\to T}\bigwedge_{s\in S} a_{sf(s)}$$

for all $\{a_{st}\colon s \in S, t \in T\} \subseteq A$ for which the indicated infima and suprema exist.

It is well-known that an ℓ-group G is completely distributive if and only if $D(G) = \{0\}$.

Before proving the analogue for $GPMV$-algebras, we remark that for any ideal $I \in \mathfrak{I}(A)$, there exists the smallest closed ideal exceeding I; it is denoted by $\mathrm{cl}(I)$ and consists of those elements a that can be written as $a = \bigvee_{t \in T} a_t$, where $\{a_t \colon t \in T\} \subseteq I$.

Theorem 4.8 (cf. [20]). *A $GPMV$-algebra A is completely distributive if and only if $D(A) = \{0\}$.*

Proof. If $D(A) = \{0\}$ then by the previous proposition we have $D(G_A) = \{0\}$, hence G_A is a completely distributive ℓ-group, so in view of Lemma 4.3, A is completely distributive.

Assume that A is completely distributive but there exists $a \in D(A) \setminus \{0\}$. Let $\{P_s \colon s \in S\}$ be the set of all prime ideals. Since $\mathrm{cl}(P_s)$ is a closed prime ideal for every $s \in S$, it follows that $a \in \mathrm{cl}(P_s)$ for all $s \in S$, and a can be written in the form $a = \bigvee_{t \in T} a_{st}$ for some $\{a_{st} \colon t \in T\} \subseteq P_s$ (for each $s \in S$ we take the same T). For any $f \colon S \to T$ we have $\bigwedge_{s \in S} a_{sf(s)} = 0$ as $\bigcap_{s \in S} P_s = \{0\}$. However, then $a = \bigwedge_{s \in S} \bigvee_{t \in T} a_{st} = \bigvee_{f \colon S \to T} \bigwedge_{s \in S} a_{sf(s)} = 0$, a contradiction. □

Since A is finite-valued if and only if every value in A is special, and special values are closed, we get

Corollary 4.9. *If A is finite-valued then it is completely distributive.*

5. Archimedean GPMV-algebras

In analogy with ℓ-groups, we write $a \ll b$ if, for every $n \in \mathbb{N}$, $n \cdot a = a + \ldots + a$ (n-times) exists and $n \cdot a \leqslant b$. A $GPMV$-algebra A is said to be *Archimedean* if $a \not\ll b$ for all $a, b \in A \setminus \{0\}$.

The ℓ-group representation of $GPMV$-algebras allows one to prove that any Archimedean $GPMV$-algebra is commutative.

Theorem 5.1. *Let A be a $GPMV$-algebra. Then A is Archimedean if and only if G_A is an Archimedean ℓ-group.*

Proof. Let G_A be Archimedean, i.e., for any $a, b \in G_A^+$, if $n \cdot a \leqslant b$ for all $n \in \mathbb{N}$, then $a = 0$. If $a, b \in A$ and $a \ll b$, then $n \cdot a \leqslant b$ for each positive integer n, which entails $a = 0$. Thus A is Archimedean, too.

Conversely, let A be an Archimedean $GPMV$-algebra, let $x, y \in G_A^+$ and assume that $n \cdot x \leqslant y$ for all $n \in \mathbb{N}$. Since A generates G_A^+, there exist $a_1, \ldots, a_m \in A$ such that $y = a_1 + \ldots + a_m$. We proceed by induction on m.

(a) Let $m = 1$, i.e., $n \cdot x \leqslant a_1$ for all $n \in \mathbb{N}$. Then obviously $x \leqslant a_1$, and so $x \in A$. Now, for every $n \in \mathbb{N}$, $n \cdot x$ is defined in A and is less than or equal to a_1, whence $x = 0$ follows.

(b) Suppose that the statement holds for every positive integer $k \leqslant m$. Let $n \cdot x \leqslant a_1 + \ldots + a_m + a_{m+1}$ for all $n \in \mathbb{N}$; then $n \cdot x - a_{m+1} \leqslant a_1 + \ldots + a_m$. It can be easily seen that in any ℓ-group G, $n \cdot (x \vee 0) = n \cdot x \vee (n-1) \cdot x \vee \ldots \vee x \vee 0$ for every $x \in G$ and $n \in \mathbb{N}$. Furthermore, if $x, y \in G^+$ then $n \cdot (x - y) \leqslant n \cdot x - y$. Therefore for any $r \in \mathbb{N}$,

$$\begin{aligned} & r \cdot ((n \cdot x - a_{m+1}) \vee 0) \\ & \quad = r \cdot (n \cdot x - a_{m+1}) \vee (r-1) \cdot (n \cdot x - a_{m+1}) \vee \ldots \vee (n \cdot x - a_{m+1}) \vee 0 \\ & \quad \leqslant (rn \cdot x - a_{m+1}) \vee ((r-1)n \cdot x - a_{m+1}) \vee \ldots \vee (n \cdot x - a_{m+1}) \vee 0 \\ & \quad \leqslant a_1 + \ldots + a_m. \end{aligned}$$

By the induction hypothesis we obtain $(n \cdot x - a_{m+1}) \vee 0 = 0$, so $n \cdot x \leqslant a_{m+1}$ for all $n \in \mathbb{N}$, which yields $x = 0$. □

Corollary 5.2. *Every Archimedean GPMV-algebra is commutative.*

Proof. It is well-known that any Archimedean ℓ-group is Abelian (e.g. [14], Theorem 4.B). Hence if A is Archimedean then G_A is Abelian and so $a \oslash b = a \otimes b$ for all $a, b \in A$. This entails the commutativity of A since $a \geqslant (b \oplus a) \otimes b = (b \oplus a) \oslash b$ whence $a \oplus b \geqslant b \oplus a$, and similarly $a \oplus b \leqslant b \oplus a$. □

An *Archimedean lattice* (see [22]) is an algebraic lattice L such that for each compact element $c \in L$, the meet of all maximal elements in the interval $[0, c]$ is 0 (where 0 is the least element of L). As known, an Abelian ℓ-group G is Archimedean if and only if the lattice $\mathfrak{C}(G)$ of its convex ℓ-subgroups is an Archimedean lattice. The proof can be easily done by observing that the compact elements of $\mathfrak{C}(G)$ are just the principal convex ℓ-subgroups $G(a)$, $a \in G$, and using the fact that in each ℓ-group $G(a)$ which has a strong order unit a, the intersection of all maximal ℓ-ideals equals the set $\{x \in G(a)\colon\ x \ll a\}$.

Since A is Archimedean exactly if G_A is an Archimedean ℓ-group, it follows that $\mathfrak{I}(A)$ is an Archimedean lattice if and only if so is $\mathfrak{C}(G_A)$. Hence

Theorem 5.3. *A commutative GPMV-algebra A is Archimedean if and only if its ideal lattice $\mathfrak{I}(A)$ is an Archimedean lattice.*

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