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Filip Švrček

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INTERIOR AND CLOSURE OPERATORS ON BOUNDED RESIDUATED LATTICE ORDERED MONOIDS

Filip Švrček, Olomouc

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Abstract. GMV-algebras endowed with additive closure operators or with its duals-multiplicative interior operators (closure or interior GMV-algebras) were introduced as a non-commutative generalization of topological Boolean algebras. In the paper, the multiplicative interior and additive closure operators on DRl-monoids are introduced as natural generalizations of the multiplicative interior and additive closure operators on GMV-algebras.

Keywords: GMV-algebra, DRl-monoid, filter

MSC 2000: 06D35, 06F05, 03G25

1. Introduction

In 1965, the commutative dually residuated lattice-ordered semigroups (DRl-semigroups) were introduced by K. L. N. Swamy [28] as a common generalization of abelian l-groups and Brouwerian algebras. Bounded commutative DRl-monoids are in a close connection with algebras of fuzzy logic. For example, each BL-algebra (or more precisely the dual to each BL-algebra) and each MV-algebra can be considered as a special case of a bounded commutative DRl-semigroup.

The non-commutative extension of DRl-semigroups was introduced first by K. Swamy and later in 1996 T. Kovář dealt with them in his Ph.D. Thesis [15].

Definition 1.1. An algebra $\mathscr{M} = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0\rangle$ is called a *bounded residuated lattice ordered monoid (bounded Rl-monoid)* iff for each $x, y, z \in M$

(i) $(M; \odot, 1)$ is a monoid,

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,

(iii) $x \odot y \leqslant z \Longleftrightarrow x \leqslant y \to z \Longleftrightarrow y \leqslant x \rightsquigarrow z$,
(iv) $x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y)$.

Bounded Rl-monoids are in fact the so called bounded integral generalized BL-algebras-a special class of residuated lattices studied in [1] and [2]. One can show that the lattice $(M; \vee, \wedge)$ is distributive and also that the binary operation $\odot$ distributes over $\vee$ and $\wedge$-see [9].

Bounded Rl-monoids form a variety of type $\langle 2,2,2,2,2,0,0\rangle$. For example, every GMV-algebra (pseudo MV-algebra) and every pseudo BL-algebra are special cases of Rl-monoids. In the sequel, an *Rl-monoid* will mean a bounded Rl-monoid.

The aim of the paper is to generalize the results of the paper [26] (where one works with additive closure and multiplicative interior operators on GMV-algebras) to the wider class of algebras, to the class of Rl-monoids. In the second section of the paper we will introduce multiplicative interior and additive closure operators on Rl-monoids as natural generalization of the same operators on GMV-algebras and we will describe their mutual relation. In the final third section we will study operators of interior on algebras derived from Rl-monoids, for example derived by factorization by their filters.

In the next lemma we will show some of the basic properties of Rl-monoids.

Lemma 1.1 ([15], [23]). *Let $\mathscr{M} = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ be an Rl-monoid. Then the following assertions hold for every $x, y, z \in M$:*

(i) $x \odot y \leqslant x \wedge y \leqslant x, y$;
(ii) *if* $x \leqslant y$ *then* $x \odot z \leqslant y \odot z$ *and* $z \odot x \leqslant z \odot y$;
(iii) *if* $x \leqslant y$ *then* $z \to x \leqslant z \to y$ *and* $z \rightsquigarrow x \leqslant z \rightsquigarrow y$;
(iv) *if* $x \leqslant y$ *then* $y \to z \leqslant x \to z$ *and* $y \rightsquigarrow z \leqslant x \rightsquigarrow z$;
(v) $x \leqslant y$ *iff* $x \to y = 1$ *iff* $x \rightsquigarrow y = 1$;
(vi) $x \to x = x \rightsquigarrow x = 1$;
(vii) $1 \to x = 1 \rightsquigarrow x = x$;
(viii) $y \leqslant x \to y$ *and* $y \leqslant x \rightsquigarrow y$;
(ix) $x \to 1 = x \rightsquigarrow 1 = 1$;
(x) *if* $x \leqslant y$ *then* $y \to 0 \leqslant x \to 0$ *and* $y \rightsquigarrow 0 \leqslant x \rightsquigarrow 0$.

2. Multiplicative interior and additive closure operators on Rl-monoids

Definition 2.1. Let $\mathscr{M} = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ be an Rl-monoid and $f\colon M \rightarrow M$ a mapping. Then f is called a *multiplicative interior operator* (or *mi-operator*) on $\mathscr{M}$ iff for each $x, y \in M$

1. $f(x \odot y) = f(x) \odot f(y)$,
2. $f(x) \leqslant x$,
3. $f(f(x)) = f(x)$,
4. $f(1) = 1$.

Lemma 2.1. *Each mi-operator on an Rl-monoid $\mathscr{M}$ is isotone.*

P r o o f. Let us consider an mi-operator f on $\mathscr{M}$ and $x, y \in M$ such that $x \leqslant y$. Then

$$f(x) = f(y \wedge x) = f((y \rightarrow x) \odot y) = f(y \rightarrow x) \odot f(y).$$

Since $f(y) \odot f(y \rightarrow x) \leqslant f(y)$, we have also $f(x) \leqslant f(y)$. That means, f is isotone on $\mathscr{M}$. $\square$

Hence (by 2 and 3 from Definition 2.1) f is an interior operator on the lattice $(M; \vee, \wedge)$ of the Rl-monoid $\mathscr{M}$.

Lemma 2.2. *For an mi-operator f on an Rl-monoid $\mathscr{M}$ and for each $x, y \in M$,*

$$f(x \rightarrow y) \leqslant f(x) \rightarrow f(y),$$
$$f(x \rightsquigarrow y) \leqslant f(x) \rightsquigarrow f(y).$$

P r o o f. Let $x, y \in M$. Then

$$(x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y \leqslant y$$

and by Lemma 2.1

$$f(x \rightarrow y) \odot f(x) = f(x) \odot f(x \rightsquigarrow y) \leqslant f(y).$$

So, by Definition 1.1, the inequalities we are proving, hold on $\mathscr{M}$. $\square$

On an arbitrary Rl-monoid $\mathscr{M}$ we define two unary operations, *negations* $^{-}\colon M \rightarrow M$ and $^{\sim}\colon M \rightarrow M$ by

$$x^{-} := x \rightarrow 0,$$
$$x^{\sim} := x \rightsquigarrow 0$$

for each element $x \in M$.

We can characterize GMV-algebras by means of the negations, because by [22], the class of GMV-algebras is a subvariety of the variety of Rl-monoids determined by the identities $x^{-\sim} = x = x^{\sim-}$ and $(x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$.

Let us show now some properties of the two operations of negation.

Lemma 2.3. *In every Rl-monoid $\mathscr{M}$ the following assertions hold for each elements $x, y \in M$:*

(i) $0^{\sim-} = 0 = 0^{-\sim}$, $1^{\sim-} = 1 = 1^{-\sim}$;

(ii) $x \leqslant x^{-\sim}, x^{\sim-}$;

(iii) $x^- = x^{-\sim-}$, $x^\sim = x^{\sim-\sim}$;

(iv) $x \leqslant y \Longrightarrow x^- \geqslant y^-, x^\sim \geqslant y^\sim$.

Proof. See [23]. □

Let us consider a mapping $f\colon M \to M$ and two new mappings

$$f_-^\sim\colon M \to M, \quad f_\sim^-\colon M \to M$$

such that for each $x \in M$

$$f_-^\sim(x) := (f(x^-))^\sim$$

and

$$f_\sim^-(x) := (f(x^\sim))^-.$$

Proposition 2.4. *If f is an mi-operator on an Rl-monoid $\mathscr{M}$ then both the mappings $f_-^\sim, f_\sim^-$ are isotone.*

Proof. Let us consider elements $x, y \in M$ such that $x \leqslant y$. Then $y^- \leqslant x^-$ (see Lemma 3.3(iv)), so $f(y^-) \leqslant f(x^-)$. Therefore $(f(x^-))^\sim \leqslant (f(y^-))^\sim$, or equivalently $f_-^\sim(x) \leqslant f_-^\sim(y)$. Analogously for $f_\sim^-$. □

Proposition 2.5. *If f is an mi-operator on an Rl-monoid $\mathscr{M}$ then for each element $x \in M$ we have*

$2'$. $x \leqslant f_-^\sim(x)$,

$3'$. $f_-^\sim(f_-^\sim(x)) = f_-^\sim(x)$,

$4'$. $f_-^\sim(0) = 0$.

Proof. Let us consider an arbitrary element $x \in M$.

$2'$: $f_-^\sim(x) = (f(x^-))^\sim \geqslant x^{-\sim} \geqslant x$.

$3'$: By $2'$ we have

$$f_-^\sim(x) \leqslant f_-^\sim(f_-^\sim(x)).$$

Further we know that

$$f(x^-) \leqslant (f(x^-))^{\sim -}$$

and so

$$f(x^-) = f(f(x^-)) \leqslant f((f(x^-))^{\sim -}).$$

Therefore

$$f_-^\sim(f_-^\sim(x)) = f((f(x^-))^{\sim -})^\sim \leqslant f(x^-)^\sim = f_-^\sim(x).$$

4′: $f_-^\sim(0) = (f(0^-))^\sim = (f(1))^\sim = 1^\sim = 0.$ □

Remark 2.6.

a) Of course, relations 2′–4′ from the preceding proposition are satisfied also for the operator $f_\sim^-$.

b) By Propositions 2.4 and 2.5 and Remark 2.6 a), $f_-^\sim$ and $f_\sim^-$ are closure operators on the lattice $(M; \vee, \wedge)$.

c) By the proof of part 2′ of Proposition 2.5 the stronger inequality $x^{-\sim} \leqslant f_-^\sim(x)$ is satisfied in $\mathscr{M}$.

Definition 2.2. An *Rl*-monoid $\mathscr{M}$ is said to be *good* if and only if

(G) $$x^{-\sim} = x^{\sim -}$$

holds for each element $x \in M$.

Remark 2.7. The identity (G) holds for example in every GMV-algebra. On the other hand, the situation is not so clear for the case of pseudo BL-algebras. It was proved [5], [6] that every linearly ordered pseudo BL-algebra, hence every representable pseudo BL-algebra is good. Anyway, the general problem is still open-see [12], Open problem 3.21.

Moreover, we can show (see [23]) that every good BL-algebra and every Heyting algebra satisfy the identity

(N1) $$(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}$$

or the equivalent form

(N2) $$(x \odot y)^{\sim -} = x^{\sim -} \odot y^{\sim -}.$$

Therefore, it is clear that the class of *Rl*-monoids satisfying (N1) and (N2) is really wide, which leads us to the following definition.

Definition 2.3. An *Rl*-monoid $\mathscr{M}$ is said to be *normal* if and only if it satisfies both the identities (N1) and (N2).

We can define a new binary operation "$\oplus$" on every *Rl*-monoid $\mathscr{M} = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$. For arbitrary elements $x, y \in M$ we put

$$x \oplus y := (x^- \odot y^-)^\sim.$$

Then this new binary operation has the following properties.

Lemma 2.8. *If $\mathscr{M}$ is a good Rl-monoid and $x, y, z \in M$, then*

(a) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(b) $x \oplus y = (x^\sim \odot y^\sim)^-$,
(c) $x, y \leqslant x \vee y \leqslant x \oplus y$,
(d) $x \oplus 0 = x^{-\sim} = 0 \oplus x$,
(e) $x \oplus 1 = 1 = 1 \oplus x$.

P r o o f. See [8]. □

Lemma 2.9. *The following equalities hold in every good normal Rl-monoid $\mathscr{M}$:*

(i) $(x \oplus y)^- = x^- \odot y^-$;
(ii) $(x \oplus y)^\sim = x^\sim \odot y^\sim$;
(iii) $(x \odot y)^- = x^- \oplus y^-$;
(iv) $(x \odot y)^\sim = x^\sim \oplus y^\sim$.

P r o o f. Let us choose arbitrary $x, y \in M$. Then we have

(i): $(x \oplus y)^- = (x^- \odot y^-)^{\sim -} = x^{-\sim -} \odot y^{-\sim -} = x^- \odot y^-$ (we have used (N2) and Lemma 2.3 (iii));
(ii): $(x \oplus y)^\sim = (x^\sim \odot y^\sim)^{-\sim} = x^{\sim -\sim} \odot y^{\sim -\sim} = x^\sim \odot y^\sim$ (we have used Lemma 2.8(b), (N1) and Lemma 3.3(iii));
(iii): $(x \odot y)^- = (x \odot y)^{-\sim -} = (x^{-\sim} \odot y^{-\sim})^- = x^- \oplus y^-$ (we have used Lemma 2.3(iii), (N1) and Lemma 3.8(b));
(iv): $(x \odot y)^\sim = (x \odot y)^{\sim -\sim} = (x^{\sim -} \odot y^{\sim -})^\sim = x^\sim \oplus y^\sim$ (we have used Lemma 2.3(iii) and (N2)). □

Definition 2.4. If $\mathscr{M}$ is an *Rl*-monoid and $g\colon M \to M$ a mapping then g is called an *additive closure operator* (*ac-operator*) on $\mathscr{M}$ iff for each $x, y \in M$

1′. $g(x \oplus y) = g(x) \oplus g(y)$,
2′. $x \leqslant g(x)$,
3′. $g(g(x)) = g(x)$,
4′. $g(0) = 0$.

Theorem 2.10. *If $\mathscr{M}$ is a good normal Rl-monoid and f is an mi-operator on $\mathscr{M}$ then the mappings $f_{-}^{\sim}$ and $f_{\sim}^{-}$ are ac-operators on $\mathscr{M}$, which are moreover isotone.*

P r o o f. Thanks to Proposition 2.6 it is enough to check the identity $1'$ from the definition of an *ac*-operator. Let us do it for $f_{-}^{\sim}$, for $f_{\sim}^{-}$ it is analogous. Let $x, y \in M$. Then by Lemma 2.9(i)

$$f_{-}^{\sim}(x \oplus y) = (f((x \oplus y)^{-}))^{\sim} = (f(x^{-} \odot y^{-}))^{\sim} = (f(x^{-}) \odot f(y^{-}))^{\sim}.$$

By Lemma 2.9(iv) we further get

$$(f(x^{-}) \odot f(y^{-}))^{\sim} = (f(x^{-}))^{\sim} \oplus (f(y^{-}))^{\sim} = f_{-}^{\sim}(x) \oplus f_{-}^{\sim}(y).$$

The isotony of the operators $f_{-}^{\sim}$ and $f_{\sim}^{-}$ is a direct consequence of the isotony of f and Lemma 2.3(iv). □

Now, let us consider the converse situation. We choose an *ac*-operator g on an *Rl*-monoid $\mathscr{M}$ and we will study properties of the mappings $g_{-}^{\sim}$ and $g_{\sim}^{-}$.

Lemma 2.11. *Every ac-operator g on a good Rl-monoid $\mathscr{M}$ satisfies the equality*

$$g(x^{-\sim}) = (g(x))^{-\sim}.$$

P r o o f. We have

$$g(x^{-\sim}) = g(x \oplus 0) = g(x) \oplus g(0) = g(x) \oplus 0 = (g(x))^{-\sim}.$$

□

Theorem 2.12. *Let g be an ac-operator on a good normal Rl-monoid $\mathscr{M}$. Then the mappings $g_{-}^{\sim}$ and $g_{\sim}^{-}$ satisfy identities 1, 3, 4 from Definition 2.1. Moreover, if g is isotone then both $g_{-}^{\sim}$ and $g_{\sim}^{-}$ are also isotone.*

P r o o f. Let us choose arbitrary elements $x, y \in M$. Then for example for $g_{-}^{\sim}$ we have

1: $g_{-}^{\sim}(x \odot y) = (g((x \odot y)^{-}))^{\sim} = (g(x^{-} \oplus y^{-}))^{\sim} = (g(x^{-}) \oplus g(y^{-}))^{\sim} = (g(x^{-}))^{\sim} \odot (g(y^{-}))^{\sim} = g_{-}^{\sim}(x) \odot g_{-}^{\sim}(y)$;

3: $g_{-}^{\sim}(g_{-}^{\sim}(x)) = (g(((g(x^{-}))^{\sim -}))^{\sim} = (g(g(x^{-\sim -})))^{\sim} = (g(g(x^{-})))^{\sim} = (g(x^{-}))^{\sim} = g_{-}^{\sim}(x)$ (we have used Lemma 3.10);

4: $g_{-}^{\sim}(1) = (g(1^{-}))^{\sim} = (g(0))^{\sim} = 0^{\sim} = 1$.

The second part is obvious. □

Remark 2.13. Axiom 2. from Definition 2.1 need not be satisfied by $g_{-}^{\sim}$ (or $g_{\sim}^{-}$) in general. Only the following weaker inequality holds for arbitrary $x \in M$:

$$g_{-}^{\sim}(x) = (g(x^{-}))^{\sim} \leqslant x^{-\sim}.$$

Theorem 2.14. *Let us consider a good normal Rl-monoid $\mathscr{M}$ and an operator h on $\mathscr{M}$ which satisfies identities 1, 3, 4 from Definition 2.1 and the ineqality $h(x) \leqslant x^{-\sim}$ for arbitrary $x \in M$. Then the mappings $h_{-}^{\sim}$ and $h_{\sim}^{-}$ are ac-operators on $\mathscr{M}$.*

Proof. We must check axioms $1'$–$4'$ from Definition 2.4 for our mappings $h_{-}^{\sim}$ and $h_{\sim}^{-}$ and the Rl-monoid $\mathscr{M}$. So, for an arbitrary element $x \in M$ we have

$$2'\colon\ h_{-}^{\sim}(x) = (h(x^{-}))^{\sim} \geqslant ((x^{-})^{\sim-})^{\sim} = x^{-\sim} \geqslant x.$$

For the other three identities $1'$, $3'$ and $4'$ we have now the same situation as in Proposition 2.5 and Theorem 2.10. □

3. Operators on algebras derived from Rl-monoids

Let us have an Rl-monoid $\mathscr{M}$ and its mi-operator f. In this chapter, the algebra $(\mathscr{M}, f) = (M; \odot, \vee, \wedge, \rightarrow, 0, 1, f)$ will be called an *interior Rl-monoid* (analogously to the GMV-algebras in [26]).

Definition 3.1. If $\mathscr{M}$ is an Rl-monoid then a non-empty subset F of M is called a *filter* in $\mathscr{M}$ iff

(F1) $x, y \in F \Longrightarrow x \odot y \in F$,

(F2) $x \in F,\ y \in M,\ x \leqslant y \Longrightarrow y \in F$.

A filter F is called *normal* iff

(F3) $x \rightarrow y \in F \Longleftrightarrow x \rightsquigarrow y \in M$ for each $x, y \in M$.

It is known (see [2] or [18]) that normal filters of Rl-monoids coincide with kernels of their congruences. If F is a normal filter of an Rl-monoid $\mathscr{M}$ then F is the kernel of the unique congruence $\Theta(F)$ such that

$$\langle x, y\rangle \in \Theta(F) \Longleftrightarrow (x \rightarrow y), (y \rightarrow x) \in F$$

for each $x, y \in M$. Therefore, for each Rl-monoid M we can consider the quotient Rl-monoid $\mathscr{M}/F$ by its filter F.

Definition 3.2. Let F be a filter in an interior Rl-monoid $(\mathscr{M}, f)$. Then F is called an *i-filter* (or *interior filter*) iff

(F4) $x \in F \Longrightarrow f(x) \in F$.

Theorem 3.1. *Let $(\mathscr{M}, f)$ be an interior Rl-monoid and let F be its normal i-filter. Further, let us consider the mapping $\tilde{f}\colon \mathscr{M}/F \to \mathscr{M}/F$ such that for each $x \in M$,*

$$\tilde{f}(x/F) := f(x)/F.$$

Then the Rl-monoid $\mathscr{M}/F$ endowed with $\tilde{f}$ is an interior Rl-monoid.

P r o o f. Let us consider $x, y \in M$ such that $x/F = y/F$. So we have $\langle x, y\rangle \in \Theta(F)$ or equivalently $(x \to y), (y \to x) \in F$ and further $f(x \to y), f(y \to x) \in F$ with regard to (F4). Acording to Lemma 2.2,

$$f(x \to y) \leqslant f(x) \to f(y), \quad f(y \to x) \leqslant f(y) \to f(x),$$

therefore also $f(x) \to f(y), f(y) \to f(x) \in F$ and $\langle f(x), f(y)\rangle \in \Theta(F)$. This means that the unary operation $\tilde{f}$ is correctly defined on $\mathscr{M}/F$. We have to check conditions 1–4 from the definition of the *mi*-operator on the *Rl*-monoid for $\tilde{f}$ and the proof will be done. Let x, y be arbitrary elements from M.

1: $\tilde{f}(x/F) \odot \tilde{f}(y/F) = f(x)/F \odot f(y)/F = (f(x) \odot f(y))/F == f(x \odot y)/F = \tilde{f}((x \odot y)/F) = \tilde{f}((x/F) \odot (y/F))$;

2: $\tilde{f}(x/F) = f(x)/F \leqslant x/F$;

3: $\tilde{f}(\tilde{f}(x/F)) = \tilde{f}(f(x)/F) = f(f(x))/F = f(x)/F = \tilde{f}(x/F)$;

4: $\tilde{f}(1/F) = f(1)/F = 1/F$. □

Corollary 3.2. *There is a one-to-one correspondence between the normal i-filters and the congruences of the interior Rl-monoids.*

We will denote by $D(\mathscr{M}) = \{x \in M\colon\ x^{-\sim} = 1\}$ the set of all *dense* elements of a good *Rl*-monoid $\mathscr{M}$.

Proposition 3.3. *For every good Rl-monoid $\mathscr{M}$ the set $D(\mathscr{M})$ is a normal filter in $\mathscr{M}$.*

P r o o f. See [23], Theorem 10. □

Similarly to the commutative case, we can show (see [23], Theorems 9, 10) that for a good *Rl*-monoid $\mathscr{M}$ the quotient *Rl*-monoid $M/D(\mathscr{M})$ is a *GMV*-algebra. By Theorem 3.1, Proposition 3.3 and [26] we have

Theorem 3.4. *Let us consider an interior Rl-monoid $(\mathscr{M}, f)$. Further, consider a mapping $\tilde{f}\colon \mathscr{M}/D(\mathscr{M}) \to \mathscr{M}/D(\mathscr{M})$ such that for each element $x \in M$,*

$$\tilde{f}(x/D(\mathscr{M})) := f(x)/D(\mathscr{M}).$$

Then $\tilde{f}$ is an mi-operator on the GMV-algebra $\mathscr{M}/D(\mathscr{M})$.

Let us consider an Rl-monoid $\mathscr{M}$ and the set $R(\mathscr{M}) = \{x \in M\colon x^{\sim -} = x = x^{-\sim}\}$. It is known (see [8]) that if an Rl-monoid $\mathscr{M}$ is good then $\mathscr{R}(\mathscr{M}) = (R(\mathscr{M}); \oplus_R, -_R, \sim_R, 0, 1)$, where "$\oplus_R$" is introduced on $\mathscr{R}(\mathscr{M})$ in the same way as on the whole $\mathscr{M}$, and "$-_R$", "$\sim_R$" are restrictions of unary operations "$-$", "$\sim$" of negations from $\mathscr{M}$ to $\mathscr{R}(\mathscr{M})$, is a GMV-algebra.

Theorem 3.5. *Let us introduce a mapping $\hat{f}\colon R(\mathscr{M}) \to R(\mathscr{M})$ on a good normal interior Rl-monoid $(\mathscr{M}; f)$ by*

$$\hat{f}(x) := (f(x))^{-\sim}$$

for each $x \in R(\mathscr{M})$. Then $\hat{f}$ is an mi-operator on the GMV-algebra $\mathscr{R}(\mathscr{M})$.

P r o o f. Since

$$\hat{f}(x)^{-\sim} = ((f(x))^{-\sim})^{-\sim} = (f(x))^{-\sim} \text{ for each } x \in R(\mathscr{M}),$$

it is clear that $\hat{f}$ is a self-mapping of $\mathscr{R}(\mathscr{M})$. Let us check the conditions from the definition of an mi-operator on a GMV-algebra (see [26]) for $\hat{f}$ and $\mathscr{R}(\mathscr{M})$. For arbitrary $x, y \in R(\mathscr{M})$ we have

1: $\hat{f}(x \odot y) = (f(x \odot y))^{-\sim} = (f(x) \odot f(y))^{-\sim} = (f(x))^{-\sim} \odot (f(y))^{-\sim} = \hat{f}(x) \odot \hat{f}(y)$;

2: $\hat{f}(x) = (f(x))^{-\sim} \leqslant x^{-\sim} = x$;

3: $\hat{f}(\hat{f}(x)) = \hat{f}(f(x)^{-\sim}) = (f((f(x))^{-\sim}))^{-\sim} \geqslant (f(f(x)))^{-\sim} = (f(x))^{-\sim} = \hat{f}(x)$.

Conversely, $(f(x))^{-\sim} = \hat{f}(x) \leqslant x$, so $(f((f(x))^{-\sim}))^{-\sim} \leqslant (f(x))^{-\sim}$ or equivalently $\hat{f}(\hat{f}(x)) \leqslant \hat{f}(x)$;

4: $\hat{f}(1) = (f(1))^{-\sim} = 1^{-\sim} = 1$. □

It was proved that for every good normal Rl-monoid $\mathscr{M}$ the GMV-algebras $\mathscr{R}(\mathscr{M})$ and $\mathscr{M}/D(\mathscr{M})$ are isomorphic ([23], Th. 10), where the mappings $\varphi\colon \mathscr{R}(\mathscr{M}) \to \mathscr{M}/D(\mathscr{M})$ and $\psi\colon \mathscr{M}/D(\mathscr{M}) \to \mathscr{R}(\mathscr{M})$ such that

$$\varphi(x) := x/D(\mathscr{M}),$$
$$\psi(y/D(\mathscr{M})) := y^{-\sim}$$

are mutually inverse isomorphisms.

Let $\mathscr{M}$ be an *Rl*-monoid. Let us denote by $I(\mathscr{M}) = \{a \in M\colon\ a \odot a = a\}$ the set of all *idempotent elements* in M and by $B(\mathscr{M})$ the set of elements from $\mathscr{M}$ which have a complement in the lattice $(M, \vee, \wedge)$. It is known that if $\mathscr{M}$ is a GMV-algebra then $I(\mathscr{M}) = B(\mathscr{M})$—see [10], Prop. 4.2.

Theorem 3.6. *For each Rl-monoid $\mathscr{M}$ we have $B(I(\mathscr{M})) = B(\mathscr{M})$.*

Proof. Clearly $0, 1 \in I(\mathscr{M})$, so 0 is the least and 1 the greatest element in the lattice $(I(\mathscr{M}); \vee, \wedge)$. Let $x \in B(I(\mathscr{M}))$. Then there exists an element $y \in I(\mathscr{M})$ such that $x \vee y = 1$ and $x \wedge y = 0$ in the lattice $(I(\mathscr{M}); \vee, \wedge)$. Since $I(\mathscr{M}) \subseteq M$ and since the operations "$\vee$", "$\wedge$" in $(I(\mathscr{M}); \vee, \wedge)$ are restrictions of the "same" operations in $(M; \vee, \wedge)$, y is also a complement of x in the lattice $(M; \vee, \wedge)$. So $x \in B(\mathscr{M})$.

The converse inclusion is proved in [17], Lemma 15. □

Lemma 3.7. *If $\mathscr{M}$ is an Rl-monoid, $a \in I(\mathscr{M})$ and $x \in \mathscr{M}$ then $a \wedge x = a \odot y$.*

Proof. See [17], Lemma 6. □

Lemma 3.8. *If $\mathscr{M}$ is a normal good Rl-monoid and $a \in I(\mathscr{M})$ then $a^{-\sim} \in I(\mathscr{M})$ and $a^{\sim} \oplus a^{\sim} = a^{\sim}$.*

Proof. For an arbitrary element $a \in I(\mathscr{M})$ we have

$$a^{-\sim} \odot a^{-\sim} = (a \odot a)^{-\sim} = a^{-\sim}$$

thanks to normality of $\mathscr{M}$. Moreover,

$$a^{\sim} \oplus a^{\sim} = (a^{\sim-} \odot a^{\sim-})^{\sim} = (a \odot a)^{\sim-\sim} = a^{\sim}.$$

□

Theorem 3.9. *If $\mathscr{M}$ is an Rl-monoid then $I(\mathscr{M})$ is a subalgebra of the reduct $(M; \odot, \vee, \wedge, 0, 1)$ of the Rl-monoid $\mathscr{M}$.*

Proof. Closedness of $I(\mathscr{M})$ with respect to the operation "$\wedge$" follows from Lemma 3.7. It is enough to check that $I(\mathscr{M})$ is closed with respect to the operation "$\vee$". Let $a, b \in I(\mathscr{M})$. Since "$\odot$" is distributive over the lattice operations join and meet, we conclude

$$(a \vee b) \odot (a \vee b) = (a \odot a) \vee (a \odot b) \vee (b \odot a) \vee (b \odot b) = a \vee b \vee (a \odot b) \vee (b \odot a) = a \vee b.$$

□

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Author's address: Filip Švrček, Department of Algebra and Geometry, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: `filipsvrcek@seznam.cz`.