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ON A CLASS OF NONLINEAR PROBLEMS INVOLVING A  
 $p(x)$ -LAPLACE TYPE OPERATOR

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*Abstract.* We study the boundary value problem  $-\operatorname{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) = f(x, u)$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^N$ . Our attention is focused on two cases when  $f(x, u) = \pm(-\lambda|u|^{m(x)-2}u + |u|^{q(x)-2}u)$ , where  $m(x) = \max\{p_1(x), p_2(x)\}$  for any  $x \in \overline{\Omega}$  or  $m(x) < q(x) < N \cdot m(x)/(N - m(x))$  for any  $x \in \overline{\Omega}$ . In the former case we show the existence of infinitely many weak solutions for any  $\lambda > 0$ . In the latter we prove that if  $\lambda$  is large enough then there exists a nontrivial weak solution. Our approach relies on the variable exponent theory of generalized Lebesgue-Sobolev spaces, combined with a  $\mathbb{Z}_2$ -symmetric version for even functionals of the Mountain Pass Theorem and some adequate variational methods.

*Keywords:*  $p(x)$ -Laplace operator, generalized Lebesgue-Sobolev space, critical point, weak solution, electrorheological fluid

*MSC 2000:* 35D05, 35J60, 35J70, 58E05, 68T40, 76A02

## 1. INTRODUCTION AND PRELIMINARY RESULTS

In the recent years increasing attention has been paid to the study of differential and partial differential equations involving variable exponent conditions. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics or calculus of variations. In particular, we mention that Ruzicka developed in [26] a model of electrorheological fluid for which the essential part of the dissipative energy is given by

$$\int |\mathbf{Df}|^{p(x)} dx,$$

where  $\mathbf{Df}$  represents the symmetric part of the gradient. The same type of energy also appears in the papers of Zhikov [32], Marcellini [16] and Acerbi-Mingione [1].

For more information on modelling physical phenomena by equations involving  $p(x)$ -growth conditions we refer to [1], [5], [13], [24], [26], [30]. The appearance of such physical models was facilitated by the development of variable Lebesgue and Sobolev spaces,  $L^{p(x)}$  and  $W^{1,p(x)}$ , where  $p(x)$  is a real-valued function. Variable exponent Lebesgue spaces appeared for the first time in literature as early as 1931 in an article by W. Orlicz [23]. The spaces  $L^{p(x)}$  are special cases of Orlicz spaces  $L^\varphi$  originated by Nakano [22] and developed by Musielak and Orlicz [20], [21], where  $f \in L^\varphi$  if and only if  $\int \varphi(x, |f(x)|) dx < \infty$  for a suitable  $\varphi$ . Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context we refer to the studies of Tsenov [29], Sharapudinov [27] and Zhikov [32], [33].

This paper is motivated by the phenomenon that can be modelled by the equations

$$(1) \quad \begin{cases} -\operatorname{div}(|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u = f(x, u) & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ) is a bounded domain with smooth boundary and  $1 < p_i(x)$ ,  $p_i(x) \in C(\overline{\Omega})$  for  $i \in \{1, 2\}$ . Our goal will be to obtain nontrivial weak solutions for problem (1) in the generalized Sobolev space  $W^{1,m(x)}(\Omega)$ , where  $m(x) = \max\{p_1(x), p_2(x)\}$  for any  $x \in \overline{\Omega}$ , for some particular nonlinearities of the type  $f(x, u)$ . Problems of type (1) have been intensively studied in the past decades. We refer to [2], [10], [11], [31], [18], [17], [19] for some interesting results. We point out the presence in problem (1) of the operator  $\Delta_{p_1(x)}u + \Delta_{p_2(x)}u$ , where for a real-valued function  $p(x)$  we define the  $p(x)$ -Laplace operator by  $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ . This is a natural extension of the  $p$ -Laplace operator, with  $p$  a positive constant. However, such generalizations are not trivial since the  $p(x)$ -Laplace operator possesses a more complicated structure than the  $p$ -Laplace operator, for example it is inhomogeneous.

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces  $L^{p(x)}(\Omega)$  and  $W_0^{1,p(x)}(\Omega)$ , where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ . We refer the reader to the book of J. Musielak [20] and the papers of O. Kovacic and J. Rákosník [14], H. G. Leopold [15], D. Edmunds et al. [6], [7], [8] and X. L. Fan et al. [9], [12].

Set

$$C_+(\overline{\Omega}) = \{h; h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any  $h \in C_+(\overline{\Omega})$  we define

$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).$$

For any  $p(x) \in C_+(\overline{\Omega})$ , we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u; u \text{ is a measurable real-valued function such that} \right. \\ \left. \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$\|u\|_{p(x)} = \inf \left\{ \mu > 0; \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces [14, Theorem 2.5], the Hölder inequality holds [14, Theorem 2.1], they are reflexive if and only if  $1 < p^- \leq p^+ < \infty$  [14, Corollary 2.7] and continuous functions are dense if  $p^+ < \infty$  [14, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [14, Theorem 2.8]: if  $0 < |\Omega| < \infty$  and  $r_1, r_2$  are variable exponents so that  $r_1(x) \leq r_2(x)$  almost everywhere in  $\Omega$  then there exists a continuous embedding  $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$ , whose norm does not exceed  $|\Omega| + 1$ .

We denote by  $L^{p'(x)}(\Omega)$  the conjugate space of  $L^{p(x)}(\Omega)$ , where  $1/p(x) + 1/p'(x) = 1$ . For any  $u \in L^{p(x)}(\Omega)$  and  $v \in L^{p'(x)}(\Omega)$  the Hölder type inequality

$$(2) \quad \left| \int_{\Omega} uv dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the  $L^{p(x)}(\Omega)$  space, which is the mapping  $\varrho_{p(x)}: L^{p(x)}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\varrho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} dx.$$

If  $(u_n)$ ,  $u \in L^{p(x)}(\Omega)$  and  $p^+ < \infty$  then the following relations hold true:

$$(3) \quad \|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \varrho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+},$$

$$(4) \quad \|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \varrho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-},$$

$$(5) \quad \|u_n - u\|_{p(x)} \rightarrow 0 \Leftrightarrow \varrho_{p(x)}(u_n - u) \rightarrow 0.$$

Spaces with  $p^+ = \infty$  have been studied by Edmunds, Lang and Nekvinda [6].

Next, we define  $W_0^{1,p(x)}(\Omega)$  as the closure of  $C_0^\infty(\Omega)$  under the norm

$$\|u\|_{p(x)} = \|\nabla u\|_{p(x)}.$$

The space  $(W_0^{1,p(x)}(\Omega), \|\cdot\|_{p(x)})$  is a separable and reflexive Banach space. We note that if  $q \in C_+(\overline{\Omega})$  and  $q(x) < p^*(x)$  for all  $x \in \overline{\Omega}$  then the embedding  $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$  is compact and continuous, where  $p^*(x) = Np(x)/(N - p(x))$  if  $p(x) < N$  or  $p^*(x) = +\infty$  if  $p(x) \geq N$  [14, Theorems 3.9 and 3.3] (see also [9, Theorems 1.3 and 1.1]).

**Remark 1.** If  $p_1(x), p_2(x) \in C_+(\overline{\Omega})$  then it is clear that  $m(x) \in C_+(\overline{\Omega})$  where  $m(x) = \max\{p_1(x), p_2(x)\}$  for any  $x \in \overline{\Omega}$ . On the other hand, since  $p_1(x), p_2(x) \leq m(x)$  for any  $x \in \overline{\Omega}$ , it follows by Theorem 2.8 in [14] that  $W_0^{1,m(x)}(\Omega)$  is continuously embedded in  $W_0^{1,p_i(x)}(\Omega)$  for  $i \in \{1, 2\}$ .

## 2. MAIN RESULTS

In this paper we study problem (1) in the particular cases when

$$f(x, t) = \pm(-\lambda|t|^{m(x)-2}t + |t|^{q(x)-2}t)$$

where  $m(x) = \max\{p_1(x), p_2(x)\}$  for any  $x \in \overline{\Omega}$  and  $q(x) \in C_+(\Omega)$  with  $m(x) < q(x) < N \cdot m(x)/(N - m(x))$  for any  $x \in \overline{\Omega}$  and  $\lambda > 0$ .

First, we consider the problem

$$(6) \quad \begin{cases} -\operatorname{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) \\ \quad = -\lambda|u|^{m(x)-2}u + |u|^{q(x)-2}u & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

We say that  $u \in W_0^{1,m(x)}(\Omega)$  is a *weak solution* of problem (6) if

$$\int_{\Omega} (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v \, dx + \lambda \int_{\Omega} |u|^{m(x)-2}uv \, dx - \int_{\Omega} |u|^{q(x)-2}uv \, dx = 0$$

for all  $v \in W_0^{1,m(x)}(\Omega)$ .

We will prove

**Theorem 1.** *For every  $\lambda > 0$  problem (6) has infinitely many weak solutions provided  $2 \leq p_i^-$  for  $i \in \{1, 2\}$ ,  $m^+ < q^-$  and  $q^+ < N \cdot m^-/(N - m^-)$ .*

Next, we study the problem

$$(7) \quad \begin{cases} -\operatorname{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) \\ \quad = \lambda|u|^{m(x)-2}u - |u|^{q(x)-2}u & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

We say that  $u \in W_0^{1,m(x)}(\Omega)$  is a *weak solution* of problem (7) if

$$\int_{\Omega} (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v \, dx - \lambda \int_{\Omega} |u|^{m(x)-2} uv \, dx + \int_{\Omega} |u|^{q(x)-2} uv \, dx = 0$$

for all  $v \in W_0^{1,m(x)}(\Omega)$ .

We will prove

**Theorem 2.** *There exists  $\lambda^* > 0$  such that for any  $\lambda \geq \lambda^*$  problem (7) has a nontrivial weak solution provided  $m^+ < q^-$  and  $q^+ < N \cdot m^- / (N - m^-)$ .*

A careful analysis of the proofs shows that Theorems 1 and 2 still remain valid for more general classes of differential operators. For example, we can replace the  $p(x)$ -Laplace type operator  $\operatorname{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u)$  by the generalized mean curvature operator  $\operatorname{div}(((1 + |\nabla u|^2)^{(p(x)-2)/2} \nabla u))$  (see Example (ii) on page 2629 in [18]).

We remark that in the particular case corresponding to  $p_1(x) = p_2(x) = m(x) = 2$  and  $q(x) = q$ ,  $q$  being a constant, problem (6) becomes

$$(8) \quad \begin{cases} -\Delta u = -\lambda u + |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem has been studied by Ambrosetti and Rabinowitz [3] provided  $2 < q < 2^* = 2N/(N-2)$ . Using the Mountain Pass Theorem combined with the observation that the operator  $-\Delta + \lambda I$  ( $\lambda > 0$ ) is coercive in  $H_0^1(\Omega)$ , Ambrosetti and Rabinowitz showed that problem (8) has a positive solution for any  $\lambda > 0$ . The result we establish in Theorem 1 establishes the existence of *infinitely many solutions* (not necessarily positive) for a related class of boundary value problems, but involving another differential operator in the class of variable exponent Sobolev spaces.

Finally, we point out the strong difference between the result of Theorem 1 and Theorem 2. While for problem (6) we find infinitely many solutions, for problem (7) we find only the existence of at least one nontrivial solution, for  $\lambda > 0$  sufficiently large. This fact is connected with the method applied in order to find solutions for either of the quoted problems. For problem (6) we apply a  $\mathbb{Z}_2$ -symmetric version (for even functionals) of the Mountain Pass Theorem. The application of this theorem is intimately linked with the fact that under the assumptions of Theorem 1, the nonlinear term  $f_1(x, t) := -\lambda|t|^{m(x)-2}t + |t|^{q(x)-2}t$  satisfies the Ambrosetti-Rabinowitz condition  $0 \leq q(x) \int_0^t f_1(x, s) \, ds \leq t f_1(x, t)$  for all  $t \geq 0$  with  $q(x) > m(x)$  for all  $x \in \bar{\Omega}$ . This condition fails if  $f_2(x, t) := \lambda|t|^{m(x)-2}t - |t|^{q(x)-2}t$  but, in that case, we show that the corresponding energy functional is coercive and lower semicontinuous.

The result of Theorem 2 is also in keeping with that of Theorem 2.1 in [18], where a problem of the type

$$(9) \quad \begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda(u^{\gamma-1} - u^{\beta-1}) & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \\ u \geq 0 & \text{for } x \in \Omega. \end{cases}$$

with  $1 < \beta < \gamma < \inf_{x \in \overline{\Omega}} p(x)$  is studied. By Theorem 2.1 in [18] we find at least two nontrivial solutions for problem (9) for  $\lambda > 0$  large enough. Even if the problems (7) and (9) seems to have a similar nonlinear term on the right-hand side, the existence of a second solution for problem (7) cannot be stated this time, since it should be obtained by applying the Mountain Pass Theorem which cannot be used for the nonlinear term of problem (7) in accord with the information already pointed out above.

### 3. PROOF OF THEOREM 1

The key argument in the proof of Theorem 1 is the following  $\mathbb{Z}_2$ -symmetric version (for even functionals) of the Mountain Pass Theorem (see Theorem 9.12 in [25]):

**Mountain Pass Theorem.** *Let  $X$  be an infinite dimensional real Banach space and let  $I \in C^1(X, \mathbb{R})$  be even, satisfying the Palais-Smale condition (i.e., any sequence  $\{x_n\} \subset X$  such that  $\{I(x_n)\}$  is bounded and  $I'(x_n) \rightarrow 0$  in  $X^*$  has a convergent subsequence) and  $I(0) = 0$ . Suppose that*

(I1) *there exist two constants  $\varrho, a > 0$  such that  $I(x) \geq a$  if  $\|x\| = \varrho$ ,*

(I2) *for each finite dimensional subspace  $X_1 \subset X$ , the set  $\{x \in X_1; I(x) \geq 0\}$  is bounded.*

*Then  $I$  has an unbounded sequence of critical values.*

Let  $E$  denote the generalized Sobolev space  $W_0^{1,m(x)}(\Omega)$  and let  $\lambda > 0$  be arbitrary but fixed.

The energy functional corresponding to problem (6) is defined as  $J_\lambda: E \rightarrow \mathbb{R}$ ,

$$J_\lambda(u) = \int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx \\ + \lambda \int_{\Omega} \frac{1}{m(x)} |u|^{m(x)} dx - \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx.$$

A simple calculation based on Remark 1, relations (3) and (4) and the compact embedding of  $E$  into  $L^{s(x)}(\Omega)$  for all  $s \in C_+(\overline{\Omega})$  with  $s(x) < m^*(x)$  on  $\overline{\Omega}$  shows that

$J_\lambda$  is well-defined on  $E$  and  $J_\lambda \in C^1(E, \mathbb{R})$  with the derivative given by

$$\begin{aligned} \langle J'_\lambda(u), v \rangle &= \int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v \, dx \\ &\quad + \lambda \int_\Omega |u|^{m(x)-2} uv \, dx - \int_\Omega |u|^{q(x)-2} uv \, dx \end{aligned}$$

for any  $u, v \in E$ . Thus the weak solutions of (6) are exactly the critical points of  $J_\lambda$ .

We show now that the Mountain Pass Theorem can be applied in this case.

**Lemma 1.** *There exist  $\eta > 0$  and  $\alpha > 0$  such that  $J_\lambda(u) \geq \alpha > 0$  for any  $u \in E$  with  $\|u\|_{m(x)} = \eta$ .*

**Proof.** We first point out that since  $m(x) = \max\{p_1(x), p_2(x)\}$  for any  $x \in \overline{\Omega}$ , we have

$$(10) \quad |\nabla u(x)|^{p_1(x)} + |\nabla u(x)|^{p_2(x)} \geq |\nabla u(x)|^{m(x)}, \quad \forall x \in \overline{\Omega}.$$

On the other hand, we have

$$(11) \quad |u(x)|^{q^-} + |u(x)|^{q^+} \geq |u(x)|^{q(x)}, \quad \forall x \in \overline{\Omega}.$$

Using (10) and (11) we deduce that

$$\begin{aligned} (12) \quad J_\lambda(u) &\geq \frac{1}{\max\{p_1^+, p_2^+\}} \cdot \int_\Omega |\nabla u|^{m(x)} \, dx - \frac{1}{q^-} \cdot \left( \int_\Omega |u|^{q^-} \, dx + \int_\Omega |u|^{q^+} \, dx \right) \\ &\geq \frac{1}{m^+} \cdot \int_\Omega |\nabla u|^{m(x)} \, dx - \frac{1}{q^-} \cdot \left( \int_\Omega |u|^{q^-} \, dx + \int_\Omega |u|^{q^+} \, dx \right) \end{aligned}$$

for any  $u \in E$ .

Since  $m^+ < q^- \leq q^+ < m^*(x)$  for any  $x \in \overline{\Omega}$  and  $E$  is continuously embedded in  $L^{q^-}(\Omega)$  and in  $L^{q^+}(\Omega)$ , it follows that there exist two positive constants  $C_1$  and  $C_2$  such that

$$(13) \quad \|u\|_{m(x)} \geq C_1 \cdot |u|_{q^+}, \quad \|u\|_{m(x)} \geq C_2 \cdot |u|_{q^-}, \quad \forall u \in E.$$

Next, we focus our attention on the case when  $u \in E$  with  $\|u\|_{m(x)} < 1$ . For such a  $u$  by relation (4) we obtain

$$(14) \quad \int_\Omega |\nabla u|^{m(x)} \, dx \geq \|u\|_{m(x)}^{m^+}.$$



Relations (12), (13) and (14) imply

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{m^+} \cdot \|u\|_{m(x)}^{m^+} - \frac{1}{q^-} \cdot \left[ \left( \frac{1}{C_1} \cdot \|u\|_{m(x)} \right)^{q^+} + \left( \frac{1}{C_2} \cdot \|u\|_{m(x)} \right)^{q^-} \right] \\ &= (\beta - \gamma \cdot \|u\|_{m(x)}^{q^+ - m^+} - \delta \cdot \|u\|_{m(x)}^{q^- - m^+}) \cdot \|u\|_{m(x)}^{m^+} \end{aligned}$$

for any  $u \in E$  with  $\|u\|_{m(x)} < 1$ , where  $\beta$ ,  $\gamma$  and  $\delta$  are positive constants.

We remark that the function  $g: [0, 1] \rightarrow \mathbb{R}$  defined by

$$g(t) = \beta - \gamma \cdot t^{q^+ - m^+} - \delta \cdot t^{q^- - m^+}$$

is positive in a neighborhood of the origin. We conclude that Lemma 1 holds true.  $\square$

**Lemma 2.** *If  $E_1 \subset E$  is a finite dimensional subspace, the set  $S = \{u \in E_1; J_\lambda(u) \geq 0\}$  is bounded in  $E$ .*

*Proof.* In order to prove Lemma 2, we first show that

$$(15) \quad \int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx \leq K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}), \quad \forall u \in E$$

where  $K_1$  is a positive constant.

Indeed, using relations (3) and (4) we obtain

$$(16) \quad \int_{\Omega} |\nabla u|^{p_1(x)} dx \leq |\nabla u|_{p_1(x)}^{p_1^-} + |\nabla u|_{p_1(x)}^{p_1^+} = \|u\|_{p_1(x)}^{p_1^-} + \|u\|_{p_1(x)}^{p_1^+}, \quad \forall u \in E.$$

On the other hand, Remark 1 implies that there exists a positive constant  $K_0$  such that

$$(17) \quad \|u\|_{p_1(x)} \leq K_0 \cdot \|u\|_{m(x)}, \quad \forall u \in E.$$

Inequalities (16) and (17) yield

$$\int_{\Omega} |\nabla u|^{p_1(x)} dx \leq (K_0 \cdot \|u\|_{m(x)})^{p_1^-} + (K_0 \cdot \|u\|_{m(x)})^{p_1^+}, \quad \forall u \in E$$

and thus (15) holds true.

By similar arguments we conclude that there exists a positive constant  $K_2$  such that

$$(18) \quad \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx \leq K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}), \quad \forall u \in E.$$

Using again (3) and (4) we arrive at

$$\int_{\Omega} |u|^{m(x)} dx \leq |u|_{m(x)}^{m^-} + |u|_{m(x)}^{m^+}, \quad \forall u \in E.$$

The fact that  $E$  is continuously embedded in  $L^{m(x)}(\Omega)$  ensures the existence of a positive constant  $\overline{K}$  such that

$$|u|_{m(x)} \leq \overline{K} \cdot \|u\|_{m(x)}, \quad \forall u \in E.$$

The last two inequalities show that for each  $\lambda > 0$  there exists a positive constant  $K_3(\lambda)$  such that

$$(19) \quad \lambda \cdot \int_{\Omega} \frac{1}{m(x)} |u|^{m(x)} dx \leq K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}), \quad \forall u \in E.$$

By inequalities (15), (18) and (19) we get

$$\begin{aligned} J_{\lambda}(u) &\leq K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &\quad + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+} \int_{\Omega} |u|^{q(x)} dx \end{aligned}$$

for all  $u \in E$ .

Let  $u \in E$  be arbitrary but fixed. We define

$$\Omega_{<} = \{x \in \Omega; |u(x)| < 1\}, \quad \Omega_{\geq} = \Omega \setminus \Omega_{<}.$$

Then we have

$$\begin{aligned} J_{\lambda}(u) &\leq K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &\quad + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+} \int_{\Omega} |u|^{q(x)} dx \\ &\leq K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &\quad + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+} \int_{\Omega_{\geq}} |u|^{q(x)} dx \\ &\leq K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &\quad + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+} \int_{\Omega_{\geq}} |u|^{q^-} dx \\ &\leq K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &\quad + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+} \int_{\Omega} |u|^{q^-} dx + \frac{1}{q^+} \int_{\Omega_{<}} |u|^{q^-} dx. \end{aligned}$$

But there exists a positive constant  $K_4$  such that

$$\frac{1}{q^+} \int_{\Omega_{<}} |u|^{q^-} \leq K_4, \quad \forall u \in E.$$

Thus we deduce that

$$\begin{aligned} J_\lambda(u) &\leq K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &\quad + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+} \int_{\Omega} |u|^{q^-} dx + K_4, \quad \forall u \in E. \end{aligned}$$

The functional  $|\cdot|_{q^-} : E \rightarrow \mathbb{R}$  defined by

$$|u|_{q^-} = \left( \int_{\Omega} |u|^{q^-} dx \right)^{1/q^-}$$

is a norm in  $E$ . In the finite dimensional subspace  $E_1$  the norms  $|\cdot|_{q^-}$  and  $\|\cdot\|_{m(x)}$  are equivalent, so there exists a positive constant  $K = K(E_1)$  such that

$$\|u\|_{m(x)} \leq K \cdot |u|_{q^-}, \quad \forall u \in E_1.$$

As a consequence we have that there exists a positive constant  $K_5$  such that

$$\begin{aligned} J_\lambda(u) &\leq K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &\quad + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - K_5 \cdot \|u\|_{m(x)}^{q^-} + K_4, \quad \forall u \in E_1. \end{aligned}$$

Hence

$$\begin{aligned} K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) \\ - K_5 \cdot \|u\|_{m(x)}^{q^-} + K_4 \geq 0, \quad \forall u \in S \end{aligned}$$

and since  $q^- > m^+$  we conclude that  $S$  is bounded in  $E$ .

The proof of Lemma 2 is complete. □

**Lemma 3.** *If  $\{u_n\} \subset E$  is a sequence which satisfies the conditions*

$$(20) \quad |J_\lambda(u_n)| < M,$$

$$(21) \quad J'_\lambda(u_n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

where  $M$  is a positive constant, then  $\{u_n\}$  possesses a convergent subsequence.

**Proof.** First, we show that  $\{u_n\}$  is bounded in  $E$ . Assume the contrary. Then, passing if necessary to a subsequence, still denoted by  $\{u_n\}$ , we may assume that  $\|u_n\|_{m(x)} \rightarrow \infty$  as  $n \rightarrow \infty$ . Thus, we may assume that  $\|u_n\|_{m(x)} > 1$  for any integer  $n$ .

By (21) we deduce that there exists  $N_1 > 0$  such that for any  $n > N_1$  we have

$$\|J'_\lambda(u_n)\| \leq 1.$$

On the other hand, for any  $n > N_1$  fixed, the application

$$E \ni v \rightarrow \langle J'_\lambda(u_n), v \rangle$$

is linear and continuous. The above information yields that

$$|\langle J'_\lambda(u_n), v \rangle| \leq \|J'_\lambda(u_n)\| \cdot \|v\|_{m(x)} \leq \|v\|_{m(x)}, \quad \forall v \in E, \quad n > N_1.$$

Setting  $v = u_n$  we have

$$\begin{aligned} -\|u_n\|_{m(x)} &\leq \int_{\Omega} |\nabla u_n|^{p_1(x)} dx + \int_{\Omega} |\nabla u_n|^{p_2(x)} dx \\ &\quad + \lambda \int_{\Omega} |u_n|^{m(x)} dx - \int_{\Omega} |u_n|^{q(x)} dx \leq \|u_n\|_{m(x)} \end{aligned}$$

for all  $n > N_1$ . We obtain

$$(22) \quad \begin{aligned} -\|u_n\|_{m(x)} - \int_{\Omega} |\nabla u_n|^{p_1(x)} dx - \int_{\Omega} |\nabla u_n|^{p_2(x)} dx \\ - \lambda \int_{\Omega} |u_n|^{m(x)} dx \leq - \int_{\Omega} |u_n|^{q(x)} dx \end{aligned}$$

for any  $n > N_1$ .

Provided that  $\|u_n\|_{m(x)} > 1$  relations (20), (22) and (3) imply

$$\begin{aligned} M > J_\lambda(u_n) &\geq \left(\frac{1}{m^+} - \frac{1}{q^-}\right) \cdot \int_{\Omega} (|\nabla u_n|^{p_1(x)} + |\nabla u_n|^{p_2(x)}) dx \\ &\quad + \lambda \cdot \left(\frac{1}{m^+} - \frac{1}{q^-}\right) \cdot \int_{\Omega} |u_n|^{m(x)} dx - \frac{1}{q^-} \cdot \|u_n\|_{m(x)} \\ &\geq \left(\frac{1}{m^+} - \frac{1}{q^-}\right) \cdot \int_{\Omega} |\nabla u_n|^{m(x)} dx - \frac{1}{q^-} \cdot \|u_n\|_{m(x)} \\ &\geq \left(\frac{1}{m^+} - \frac{1}{q^-}\right) \cdot \|u_n\|_{m(x)}^{m^-} - \frac{1}{q^-} \cdot \|u_n\|_{m(x)}. \end{aligned}$$

Letting  $n \rightarrow \infty$  we obtain a contradiction. It follows that  $\{u_n\}$  is bounded in  $E$ .

Since  $\{u_n\}$  is bounded in  $E$  we deduce that there exists a subsequence, again denoted by  $\{u_n\}$ , and  $u_0 \in E$  such that  $\{u_n\}$  converges weakly to  $u_0$  in  $E$ . Since by Remark 1 and Theorem 1.3 in [9]  $E$  is compactly embedded in  $L^{m(x)}(\Omega)$  and in  $L^{q(x)}(\Omega)$  it follows that  $\{u_n\}$  converges strongly to  $u_0$  in  $L^{m(x)}(\Omega)$  and  $L^{q(x)}(\Omega)$ . The above information and relation (21) imply

$$\langle J'_\lambda(u_n) - J'_\lambda(u_0), u_n - u_0 \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

On the other hand, we have

$$\begin{aligned} & \int_{\Omega} (|\nabla u_n|^{p_1(x)-2} \nabla u_n + |\nabla u_n|^{p_2(x)-2} \nabla u_n - |\nabla u_0|^{p_1(x)-2} \nabla u_0 \\ & \quad - |\nabla u_0|^{p_2(x)-2} \nabla u_0) \cdot (\nabla u_n - \nabla u_0) \, dx \\ &= \langle J'_\lambda(u_n) - J'_\lambda(u_0), u_n - u_0 \rangle - \lambda \cdot \int_{\Omega} (|u_n|^{m(x)-2} u_n - |u_0|^{m(x)-2} u_0)(u_n - u_0) \, dx \\ & \quad + \int_{\Omega} (|u_n|^{q(x)-2} u_n - |u_0|^{q(x)-2} u_0)(u_n - u_0) \, dx. \end{aligned}$$

Using the fact that  $\{u_n\}$  converges strongly to  $u_0$  in  $L^{q(x)}(\Omega)$  and inequality (2) we have

$$\begin{aligned} & \left| \int_{\Omega} (|u_n|^{q(x)-2} u_n - |u_0|^{q(x)-2} u_0)(u_n - u_0) \, dx \right| \\ & \leq \left| \int_{\Omega} |u_n|^{q(x)-2} u_n (u_n - u_0) \, dx \right| + \left| \int_{\Omega} |u_0|^{q(x)-2} u_0 (u_n - u_0) \, dx \right| \\ & \leq C_3 \cdot \|u_n\|^{q(x)-1} \Big|_{\frac{q(x)}{q(x)-1}} \cdot \|u_n - u_0\|_{q(x)} + C_4 \cdot \|u_0\|^{q(x)-1} \Big|_{\frac{q(x)}{q(x)-1}} \cdot \|u_n - u_0\|_{q(x)} \end{aligned}$$

where  $C_3$  and  $C_4$  are two positive constants. Since  $\|u_n - u_0\|_{q(x)} \rightarrow 0$  as  $n \rightarrow \infty$  we deduce that

$$(24) \quad \lim_{n \rightarrow \infty} \int_{\Omega} (|u_n|^{q(x)-2} u_n - |u_0|^{q(x)-2} u_0)(u_n - u_0) \, dx = 0.$$

By similar arguments we deduce that

$$(25) \quad \lim_{n \rightarrow \infty} \int_{\Omega} (|u_n|^{m(x)-2} u_n - |u_0|^{m(x)-2} u_0)(u_n - u_0) \, dx = 0.$$

By (23), (24) and (25) we get

$$(26) \quad \lim_{n \rightarrow \infty} \int_{\Omega} (|\nabla u_n|^{p_1(x)-2} \nabla u_n + |\nabla u_n|^{p_2(x)-2} \nabla u_n - |\nabla u_0|^{p_1(x)-2} \nabla u_0 \\ - |\nabla u_0|^{p_2(x)-2} \nabla u_0) \cdot (\nabla u_n - \nabla u_0) \, dx = 0.$$

It is known that

$$(27) \quad (|\xi|^{r-2}\xi - |\psi|^{r-2}\psi) \cdot (\xi - \psi) \geq \left(\frac{1}{2}\right)^r |\xi - \psi|^r, \quad \forall r \geq 2, \quad \xi, \psi \in \mathbb{R}^N.$$

Relations (26) and (27) yield

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n - \nabla u_0|^{p_1(x)} dx + \int_{\Omega} |\nabla u_n - \nabla u_0|^{p_2(x)} dx = 0$$

and using relation (10) we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n - \nabla u_0|^{m(x)} dx = 0.$$

This fact and relation (5) imply  $\|u_n - u_0\|_{m(x)} \rightarrow 0$  as  $n \rightarrow \infty$ . The proof of Lemma 3 is complete.  $\square$

**Proof of Theorem 1 completed.** It is clear that the functional  $J_\lambda$  is even and verifies  $J_\lambda(0) = 0$ . Lemma 3 implies that  $J_\lambda$  satisfies the Palais-Smale condition. On the other hand, Lemmas 1 and 2 show that conditions (I1) and (I2) are satisfied. The Mountain Pass Theorem can be applied to the functional  $J_\lambda$ . We conclude that equation (6) has infinitely many weak solutions in  $E$ . The proof of Theorem 1 is complete.

#### 4. PROOF OF THEOREM 2

Let  $E$  denote the generalized Sobolev space  $W_0^{1,m(x)}(\Omega)$  and let  $\lambda > 0$  be arbitrary but fixed.

We start by introducing the energy functional corresponding to problem (7) as  $I_\lambda: E \rightarrow \mathbb{R}$ ,

$$\begin{aligned} I_\lambda(u) &= \int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx \\ &\quad - \lambda \int_{\Omega} \frac{1}{m(x)} |u|^{m(x)} dx + \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx. \end{aligned}$$

The same arguments as those used in the case of the functional  $J_\lambda$  show that  $I_\lambda$  is well-defined on  $E$  and  $I_\lambda \in C^1(E, \mathbb{R})$  with the derivative given by

$$\begin{aligned} \langle I'_\lambda(u), v \rangle &= \int_{\Omega} (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v dx \\ &\quad - \lambda \int_{\Omega} |u|^{m(x)-2} uv dx + \int_{\Omega} |u|^{q(x)-2} uv dx \end{aligned}$$

for any  $u, v \in E$ . We obtain that the weak solutions of (7) are the critical points of  $I_\lambda$ .

This time our idea is to show that  $I_\lambda$  possesses a nontrivial global minimum point in  $E$ . With this end in view we start by proving two auxiliary results.

**Lemma 4.** *The functional  $I_\lambda$  is coercive on  $E$ .*

**Proof.** In order to prove Lemma 4 we first show that for any  $a, b > 0$  and  $0 < k < l$  the following inequality holds:

$$(28) \quad a \cdot t^k - b \cdot t^l \leq a \cdot \left(\frac{a}{b}\right)^{k/(l-k)}, \quad \forall t \geq 0.$$

Indeed, since the function

$$[0, \infty) \ni t \rightarrow t^\theta$$

is increasing for any  $\theta > 0$  it follows that

$$a - b \cdot t^{l-k} < 0, \quad \forall t > \left(\frac{a}{b}\right)^{1/(l-k)}$$

and

$$t^k \cdot (a - b \cdot t^{l-k}) \leq a \cdot t^k < a \cdot \left(\frac{a}{b}\right)^{k/(l-k)}, \quad \forall t \in \left[0, \left(\frac{a}{b}\right)^{1/(l-k)}\right].$$

The above two inequalities show that (28) holds true.

Using (28) we deduce that for any  $x \in \Omega$  and  $u \in E$  we have

$$\begin{aligned} \frac{\lambda}{m^-} |u(x)|^{m(x)} - \frac{1}{q^+} |u(x)|^{q(x)} &\leq \frac{\lambda}{m^-} \left[ \frac{\lambda \cdot q^+}{m^-} \right]^{m(x)/(q(x)-m(x))} \\ &\leq \frac{\lambda}{m^-} \left[ \left( \frac{\lambda \cdot q^+}{m^-} \right)^{m^+/(q^- - m^+)} + \left( \frac{\lambda \cdot q^+}{m^-} \right)^{m^-/(q^+ - m^-)} \right] = \mathcal{C}, \end{aligned}$$

where  $\mathcal{C}$  is a positive constant independent of  $u$  and  $x$ . Integrating the above inequality over  $\Omega$  we obtain

$$(29) \quad \frac{\lambda}{m^-} \int_{\Omega} |u|^{m(x)} dx - \frac{1}{q^+} \int_{\Omega} |u|^{q(x)} dx \leq \mathcal{D}$$

where  $\mathcal{D}$  is a positive constant independent of  $u$ .

Using inequalities (10) and (29) we obtain for any  $u \in E$  with  $\|u\|_{m(x)} > 1$  that

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{m^+} \int_{\Omega} |\nabla u|^{m(x)} dx - \frac{\lambda}{m^-} \int_{\Omega} |u|^{m(x)} dx + \frac{1}{q^+} \int_{\Omega} |u|^{q(x)} dx \\ &\geq \frac{1}{m^+} \|u\|_{m(x)}^{m^-} - \left( \frac{\lambda}{m^-} \int_{\Omega} |u|^{m(x)} dx - \frac{1}{q^+} \int_{\Omega} |u|^{q(x)} dx \right) \\ &\geq \frac{1}{m^+} \|u\|_{m(x)}^{m^-} - \mathcal{D}. \end{aligned}$$

Thus  $I_\lambda$  is coercive and the proof of Lemma 4 is complete.  $\square$

**Lemma 5.** *The functional  $I_\lambda$  is weakly lower semicontinuous.*

**Proof.** First we prove that the functionals  $\Lambda_i: E \rightarrow \mathbb{R}$ ,

$$\Lambda_i(u) = \int_{\Omega} \frac{1}{p_i(x)} |\nabla u|^{p_i(x)} dx, \quad \forall i \in \{1, 2\}$$

are convex. Indeed, since the function

$$[0, \infty) \ni t \rightarrow t^\theta$$

is convex for any  $\theta > 1$ , we deduce that for each  $x \in \Omega$  fixed it the inequality

$$\left| \frac{\xi + \psi}{2} \right|^{p_i(x)} \leq \left| \frac{|\xi| + |\psi|}{2} \right|^{p_i(x)} \leq \frac{1}{2} |\xi|^{p_i(x)} + \frac{1}{2} |\psi|^{p_i(x)}, \quad \forall \xi, \psi \in \mathbb{R}^N, \quad i \in \{1, 2\}$$

holds. Using the above inequality we deduce that

$$\left| \frac{\nabla u + \nabla v}{2} \right|^{p_i(x)} \leq \frac{1}{2} |\nabla u|^{p_i(x)} + \frac{1}{2} |\nabla v|^{p_i(x)}, \quad \forall u, v \in E, \quad x \in \Omega, \quad i \in \{1, 2\}.$$

Multiplying with  $1/p_i(x)$  and integrating over  $\Omega$  we obtain

$$\Lambda_i\left(\frac{u+v}{2}\right) \leq \frac{1}{2} \Lambda_i(u) + \frac{1}{2} \Lambda_i(v), \quad \forall u, v \in E, \quad i \in \{1, 2\}.$$

Thus  $\Lambda_1$  and  $\Lambda_2$  are convex. It follows that  $\Lambda_1 + \Lambda_2$  is convex.

Next, we show that the functional  $\Lambda_1 + \Lambda_2$  is weakly lower semicontinuous on  $E$ . Taking into account that  $\Lambda_1 + \Lambda_2$  is convex, by Corollary III.8 in [4] it is enough to show that  $\Lambda_1 + \Lambda_2$  is strongly lower semicontinuous on  $E$ . We fix  $u \in E$  and  $\varepsilon > 0$ . Let  $v \in E$  be arbitrary. Since  $\Lambda_1 + \Lambda_2$  is convex and inequality (2) holds true we have

$$\begin{aligned} \Lambda_1(v) + \Lambda_2(v) &\geq \Lambda_1(u) + \Lambda_2(u) + \langle \Lambda'_1(u) + \Lambda'_2(u), v - u \rangle \\ &\geq \Lambda_1(u) + \Lambda_2(u) - \int_{\Omega} |\nabla u|^{p_1(x)-1} |\nabla(v-u)| dx \\ &\quad - \int_{\Omega} |\nabla u|^{p_2(x)-1} |\nabla(v-u)| dx \\ &\geq \Lambda_1(u) + \Lambda_2(u) - D_1 \cdot \|\nabla u\|^{p_1(x)-1} \Big|_{\frac{p_1(x)}{p_1(x)-1}} \cdot \|\nabla(u-v)\|_{p_1(x)} \\ &\quad - D_2 \cdot \|\nabla u\|^{p_2(x)-1} \Big|_{\frac{p_2(x)}{p_2(x)-1}} \cdot \|\nabla(u-v)\|_{p_2(x)} \\ &\geq \Lambda_1(u) + \Lambda_2(u) - D_3 \cdot \|u-v\|_{m(x)} \\ &\geq \Lambda_1(u) + \Lambda_2(u) - \varepsilon \end{aligned}$$



for all  $v \in E$  with  $\|u - v\|_{m(x)} < \varepsilon / [\|\nabla u\|^{p_1(x)-1} |_{\frac{p_1(x)}{p_1(x)-1}} + \|\nabla u\|^{p_2(x)-1} |_{\frac{p_2(x)}{p_2(x)-1}}]$ . We have denoted by  $D_1, D_2$  and  $D_3$  three positive constants. It follows that  $\Lambda_1 + \Lambda_2$  is strongly lower semicontinuous and since it is convex we obtain that  $\Lambda_1 + \Lambda_2$  is weakly lower semicontinuous.

Finally, we remark that if  $\{u_n\} \subset E$  is a sequence which converges weakly to  $u$  in  $E$  then  $\{u_n\}$  converges strongly to  $u$  in  $L^{m(x)}(\Omega)$  and  $L^{q(x)}(\Omega)$ . Thus,  $I_\lambda$  is weakly lower semicontinuous. The proof of Lemma 5 is complete.  $\square$

**Proof of Theorem 2.** By Lemmas 4 and 5 we deduce that  $I_\lambda$  is coercive and weakly lower semicontinuous on  $E$ . Then Theorem 1.2 in [28] implies that there exists a global minimizer  $u_\lambda \in E$  of  $I_\lambda$  and thus a weak solution of problem (7).

We show that  $u_\lambda$  is not trivial for  $\lambda$  large enough. Indeed, letting  $t_0 > 1$  be a fixed real and  $\Omega_1$  an open subset of  $\Omega$  with  $|\Omega_1| > 0$  we deduce that there exists  $u_0 \in C_0^\infty(\Omega) \subset E$  such that  $u_0(x) = t_0$  for any  $x \in \overline{\Omega}_1$  and  $0 \leq u_0(x) \leq t_0$  in  $\Omega \setminus \Omega_1$ . We have

$$\begin{aligned} I_\lambda(u_0) &= \int_\Omega \frac{1}{p_1(x)} |\nabla u_0|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\nabla u_0|^{p_2(x)} dx \\ &\quad - \lambda \int_\Omega \frac{1}{m(x)} |u_0|^{m(x)} dx + \int_\Omega \frac{1}{q(x)} |u_0|^{q(x)} dx \\ &\leq L - \frac{\lambda}{m^+} \int_{\Omega_1} |u_0|^{m(x)} dx \leq L - \frac{\lambda}{m^+} \cdot t_0^{m^-} \cdot |\Omega_1| \end{aligned}$$

where  $L$  is a positive constant. Thus, there exists  $\lambda^* > 0$  such that  $I_\lambda(u_0) < 0$  for any  $\lambda \in [\lambda^*, \infty)$ . It follows that  $I_\lambda(u_\lambda) < 0$  for any  $\lambda \geq \lambda^*$  and thus  $u_\lambda$  is a nontrivial weak solution of problem (7) for  $\lambda$  large enough. The proof of Theorem 2 is complete.  $\square$

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