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CURVES WITH FINITE TURN

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Abstract. In this paper we study the notions of finite turn of a curve and finite turn of tangents of a curve. We generalize the theory (previously developed by Alexandrov, Pogorelov, and Reshetnyak) of angular turn in Euclidean spaces to curves with values in arbitrary Banach spaces. In particular, we manage to prove the equality of angular turn and angular turn of tangents in Hilbert spaces. One of the implications was only proved in the finite dimensional context previously, and equivalence of finiteness of turn with finiteness of turn of tangents in arbitrary Banach spaces. We also develop an auxiliary theory of one-sidedly smooth curves with values in Banach spaces. We use analytic language and methods to provide analogues of angular theorems. In some cases our approach yields stronger results (for example Corollary 5.12 concerning the permanent properties of curves with finite turn) than those that were proved previously with geometric methods in Euclidean spaces.

Keywords: curve with finite turn, tangent of a curve, curve with finite convexity, delta-convex curve, d.c. curve

MSC 2000: 14H50, 46T20

1. Introduction

This paper is concerned with a generalization of the notion of curves with finite angular turn with values in Euclidean spaces to the notion of curves with finite turn with values in arbitrary Banach spaces. The theory of curves with finite angular turn was developed by several authors; see e.g. [1], [4], [6]. Theorem 4.11 that shows that finite turn is equivalent to finite turn of tangents was proved by Pogorelov (see [6]) in $\mathbb{R}^3$ and by Alexandrov and Reshetnyak [1] in $\mathbb{R}^n$. Gronychová [4] proved the (easier) implication that finite angular turn implies finite angular turn of tangents in the case of an arbitrary Hilbert space. We managed to prove the converse implication. We do not know whether the turn equals the turn of tangents in an arbitrary Banach

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space; we prove that they are equivalent with constant 2. Our notions of curves with finite turn and with finite turn of tangents generalize the angular notions. We also develop an auxiliary theory of one-sidedly smooth curves, which generalizes the theory of such curves with values in Euclidean spaces from [1]. We extend some results from [1], [4], [6]. Theorem 4.10 shows that the notions of curves with finite turn and curves with finite turn of tangents coincide. The last part of the paper deals with the relation between delta-convex (d.c.) curves and curves with finite turn (of tangents). We prove that under some natural assumptions, delta-convexity and finiteness of turn are equivalent. We use this equivalence to prove a stability theorem for curves with finite turn; see Theorem 5.11 and Corollaries 5.12 and 5.14.

Curves with finite turn were investigated by several authors; see [1], [4], [6] and others. Let X be a Banach space. By S_X we denote the set unit sphere of X. A *path* is a continuous function $\varphi\colon [a,b] \to X$, where $a < b$. A *curve* Φ (*corresponding to the path* φ) is a set

$$\begin{aligned}\Phi = \{\psi\colon [c,d] \to X\colon & \text{ there exists a continuous, strictly monotone,} \\ & \text{and onto } \omega\colon [c,d] \to [a,b] \text{ such that } \psi = \varphi \circ \omega\}.\end{aligned}$$

We call any $\mu \in \Phi$ a parametrization of Φ. A curve Φ is uniquely determined by any $\psi \in \Phi$, so we can without any confusion refer to a curve ψ (where $\psi\colon [c,d] \to X$ and $\psi \in \Phi$) when in fact we mean the curve Φ. *In the sequel, we shall assume that all curves are locally non-constant; i.e. one of the parametrizations (equivalently all parametrizations) are not constant on any open interval contained in the domain.* In the proofs, we can always assume that X is separable (we can always work with $\overline{\operatorname{span}}\{\varphi([a,b])\}$ instead of X). Let us define the *length of* φ as

$$s(\varphi) = s(\varphi, [a,b]) = \bigvee_a^b \varphi,$$

where $\bigvee_a^b \varphi$ is the variation of φ on the interval $[a,b]$. We will say that a curve φ is *rectifiable* provided $s(\varphi) < \infty$. It is well known (see e.g. Theorem 2.1.4 of [1]) that a (locally non-constant) rectifiable curve ξ has a unique arc-length parametrization ψ, which is characterized by (1.1) (i.e. there exists a continuous monotone function $\omega\colon [0, s(\xi)] \to [a,b]$ such that $\psi := \xi \circ \omega$ satisfies (1.1)). We say that a curve $\psi\colon [0, s(\psi)] \to X$ is *parametrized by the arc-length* provided

$$s(\psi, [r,t]) = t - r \quad \text{for } r < t,\ \ r,t \in [0, s(\psi)]. \tag{1.1}$$

Let D be a finite partition of $[a,b]$ (i.e. $D = \{a = x_0 < x_1 < \ldots < x_n = b\}$). For a partition D, denote $\nu(D) = \max\limits_{0 \leqslant i \leqslant n-1} (x_{i+1} - x_i)$. Define

$$\mathrm{P}(\varphi, D) = \sum_{i=1}^{n-1} \left\| \frac{\varphi(x_{i+1}) - \varphi(x_i)}{\|\varphi(x_{i+1}) - \varphi(x_i)\|} - \frac{\varphi(x_i) - \varphi(x_{i-1})}{\|\varphi(x_i) - \varphi(x_{i-1})\|} \right\|,$$

if the quantity on the right-hand side makes sense, otherwise take $\mathrm{P}(\varphi, D) = 0$. This quantity corresponds to the turn of a polygon inscribed to φ. We shall say that a curve φ has *finite turn* provided $\mathrm{P}_a^b \varphi = \sup\limits_D \mathrm{P}(\varphi, D) < \infty$, where the supremum is taken over all partitions D of $[a,b]$. We call the quantity $\mathrm{P}_a^b \varphi$ *turn of* φ *(on* $[a,b]$*)*.

Let H be a Hilbert space. Then we can define the *angle* between two non-zero vectors $x, y \in H$ as

$$\angle(x,y) = \arccos \frac{\langle x, y \rangle}{\|x\| \|y\|}.$$

Note that $\angle(x,y) = 2 \arcsin(\frac{1}{2}\|x - y\|)$ for all $x, y \in S_H$. Suppose that $\varphi\colon [a,b] \to H$ is a curve. For a partition $D = \{x_i\}_{i=0}^n$ of $[a,b]$ define

$$\angle\mathrm{P}(\varphi, D) = \sum_{i=1}^{n-1} \angle\left(\frac{\varphi(x_{i+1}) - \varphi(x_i)}{\|\varphi(x_{i+1}) - \varphi(x_i)\|}, \frac{\varphi(x_i) - \varphi(x_{i-1})}{\|\varphi(x_i) - \varphi(x_{i-1})\|} \right),$$

if the quantity on the right-hand side makes sense, otherwise take $\angle\mathrm{P}(\varphi, D) = 0$. This quantity corresponds to the variation of angles of lines of a polygon inscribed to φ. We shall say that a curve φ has *finite angular turn* provided $\angle\mathrm{P}_a^b \varphi = \sup\limits_D \angle\mathrm{P}(\varphi, D) < \infty$, where the supremum is taken over all partitions D of $[a,b]$. We call the quantity $\angle\mathrm{P}_a^b \varphi$ the *angular turn of* φ *(on* $[a,b]$*)*.

We define the *right tangent* τ_+ of φ at $x \in [a,b)$ as

$$\tau_+(x) = \tau_+(\varphi, x) = \lim_{t \searrow 0} \frac{\varphi(x+t) - \varphi(x)}{\|\varphi(x+t) - \varphi(x)\|}, \tag{1.2}$$

and the *left tangent* τ_- as

$$\tau_-(x) = \tau_-(\varphi, x) = \lim_{t \searrow 0} \frac{\varphi(x) - \varphi(x-t)}{\|\varphi(x) - \varphi(x-t)\|}, \tag{1.3}$$

provided the limits exist. We shall say that a curve φ has *a finite turn of tangents*, if the tangent $\tau_+(x)$ exists for all $x \in [a,b)$, the tangent $\tau_-(b)$ exists, and $\mathrm{T}_a^b \varphi = \sup\limits_D \mathrm{T}(\varphi, D) < \infty$, where the supremum is taken over all partitions of $[a,b]$, and for a partition $D = \{x_i\}_{i=0}^n$ we define

$$\mathrm{T}(\varphi, D) = \sum_{i=0}^{n-2} \|\tau_+(x_{i+1}) - \tau_+(x_i)\| + \|\tau_+(x_{n-1}) - \tau_-(b)\|. \tag{1.4}$$

The quantity $\mathrm{T}_a^b\varphi$ is called the *turn of tangents of* φ (*on* $[a,b]$). It is easy to see (see proof of Lemma 4.4) that for one-sidedly smooth curves we have $\mathrm{T}_a^b\varphi = \overline{\mathrm{T}}_a^b\varphi$, where $\overline{\mathrm{T}}_a^b\varphi$ is defined as $\mathrm{T}_a^b\varphi$, but instead of $\mathrm{T}_a^b(\varphi, D)$ we take

$$\overline{\mathrm{T}}(\varphi, D) = \sum_{i=0}^{n-2} \|\tau_+(x_{i+1}) - \tau_+(x_i)\|. \tag{1.5}$$

We can also define $\mathrm{L}_a^b\varphi = \sup_D \mathrm{L}(\varphi, D)$, where the supremum is taken over all partitions of $[a,b]$, and for a partition $D = \{x_i\}_{i=0}^n$ we define

$$\mathrm{L}(\varphi, D) = \sum_{i=1}^{n-1} \|\tau_-(x_{i+1}) - \tau_-(x_i)\| + \|\tau_-(x_1) - \tau_+(a)\|.$$

Let H be a Hilbert space and $\varphi\colon\ [a,b] \to H$ a curve. We shall say that the curve φ has *finite angular turn of tangents*, if the tangent $\tau_+(x)$ exists for all $x \in [a,b)$, the tangent $\tau_-(b)$ exists, and $\angle\mathrm{T}_a^b\varphi = \sup_D \angle\mathrm{T}(\varphi, D) < \infty$, where the supremum is taken over all partitions of $[a,b]$, and for a partition $D = \{x_i\}_{i=0}^n$ we define

$$\angle\mathrm{T}(\varphi, D) = \sum_{i=0}^{n-2} \angle(\tau_+(x_{i+1}), \tau_+(x_i)) + \angle(\tau_+(x_{n-1}), \tau_-(b)).$$

The quantity $\angle\mathrm{T}_a^b\varphi$ is called the *angular turn of tangents of* φ (*on* $[a,b]$).

Remark 1.1. Note that for any $\|a\| = \|b\| = 1$, $a \neq b$ we have that

$$\|a-b\| \leqslant \angle(a,b) \leqslant \frac{1}{2}\pi\|a-b\|.$$

Thus we have the following inequalities: $\mathrm{T}_a^b\varphi \leqslant \angle\mathrm{T}_a^b\varphi \leqslant \frac{1}{2}\pi\mathrm{T}_a^b\varphi$.

For a function $f\colon\ [a,b] \to X$, we shall denote by f'_+ (and f'_- respectively) the *right* and *left directional derivative*, i.e.

$$f'_+(x) = \lim_{t\searrow 0} \frac{f(x+t) - f(x)}{t}$$

and $f'_-(x) = \lim_{t\searrow 0}(f(x) - f(x-t))/t$ provided the limit exists.

2. Preliminaries

For integration of Banach space-valued functions we shall use the Bochner integral (for the definition and some facts about this integral see [2]). We shall need the following version of the Fundamental Theorem of Calculus:

Lemma 2.1. *Let X be a Banach space, and let $\varphi\colon [a,b] \to X$ be an absolutely continuous function such that φ' exists almost everywhere in (a,b) and $\int_a^b \|\varphi'(x)\| \,\mathrm{d}x < \infty$. Then*

$$\varphi(d) - \varphi(c) = \int_c^d \varphi'(x) \,\mathrm{d}x \tag{2.1}$$

for all $a \leqslant c < d \leqslant b$.

Proof. Define $g(x) = \varphi'(x)$ for x, where $\varphi'(x)$ exists, and $g(x) = 0$ elsewhere. For any $x^* \in X^*$ and for all $x \in (a,b)$, where $\varphi'(x)$ exists, we see that

$$\langle x^*, g(x)\rangle = (\langle x^*, \varphi(\cdot)\rangle)'(x) = \langle x^*, \varphi'(x)\rangle.$$

Because φ is absolutely continuous, we get that $\langle x^*, g\rangle$ is a measurable function. Application of Proposition 5.1 from [2] (we can assume that X is separable, as it can be replaced by $\overline{\operatorname{span}}\{\varphi([a,b])\}$ if necessary) yields that g is measurable. Because

$$\int_a^b \|g(x)\| \,\mathrm{d}x = \int_a^b \|\varphi'(x)\| \,\mathrm{d}x < \infty,$$

Proposition 5.2 from [2] implies that g is Bochner integrable. Now for any $x^* \in X^*$ we see that

$$\begin{aligned}\langle x^*, \varphi(d)\rangle - \langle x^*, \varphi(c)\rangle &= \int_c^d \langle x^*, \varphi'(y)\rangle \,\mathrm{d}y \\ &= \int_c^d \langle x^*, g(y)\rangle \,\mathrm{d}y = \left\langle x^*, \int_c^d g(y) \,\mathrm{d}y \right\rangle,\end{aligned}$$

where the first equality is an application of the Fundamental Theorem of Calculus to absolutely continuous functions. As $g = \varphi'$ a.e., we see that equality (2.1) holds. □

We want to generalize the angle in the Hilbert space to an arbitrary Banach space X. We shall use the following quantity instead of an angle: If $x, y \in X$ are two non-zero vectors, then we shall take $\|x/\|x\| - y/\|y\|\|$. This quantity has the following remarkable property (which also holds for the angle in a Hilbert space):

Lemma 2.2. *Let X be a Banach space and $0 \neq u, v \in X$. If $u \notin \operatorname{span}(\{v\})$, then*

$$\left\| \frac{u}{\|u\|} - \frac{u+v}{\|u+v\|} \right\| \leqslant \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\|.$$

P r o o f. The following proof is due to N. Kalton [5]. The statement also follows from Lemma 4F from [8]. Suppose $\|u\| = \|v\| = 1$ and $z = tv+(1-t)u$; let $\xi = z/\|z\|$. Then

$$\|\xi - u\| \leqslant \|\xi - z\| + \|z - u\|.$$

Now

$$\|z - u\| = t\|v - u\|.$$

On the other hand,

$$\|\xi - z\| = 1 - \|z\|.$$

But

$$\|z\| \geqslant \|v\| - \|(1-t)(v-u)\| = 1 - (1-t)\|v - u\|.$$

Thus

$$\|\xi - z\| \leqslant (1-t)\|v - u\|$$

and so

$$\|\xi - u\| \leqslant \|v - u\|.$$

□

Lemma 2.3. *Let X be a Banach space. Suppose that $\|x - y\| = \varepsilon < \frac{1}{2}$ for some $x, y \in S_X$. Then $\|\lambda x - y\| \geqslant \frac{1}{2}\varepsilon$ for any $\lambda \in \mathbb{R}$ (i.e. $\operatorname{dist}(y, \operatorname{span}\{x\}) \geqslant \frac{1}{2}\varepsilon$).*

P r o o f. Take $x^* \in X^*$ with $x^*(y) = \|x^*\| = 1$. Then $x^*(x) \geqslant \frac{1}{2}$ and so for any $\lambda \leqslant 0$ we get $\|\lambda x - y\| \geqslant x^*(y) - \lambda x^*(x) \geqslant 1 - \frac{1}{2}\lambda \geqslant 1$. For $0 \leqslant \lambda < 1 - \frac{1}{2}\varepsilon$ we get that $\|\lambda x - y\| \geqslant 1 - \lambda \geqslant \frac{1}{2}\varepsilon$ and for $\lambda > 1 + \frac{1}{2}\varepsilon$ we get that $\|\lambda x - y\| \geqslant \lambda - 1 \geqslant \frac{1}{2}\varepsilon$. Now for $\lambda \in [1 - \frac{1}{2}\varepsilon, 1 + \frac{1}{2}\varepsilon]$ we obtain $\|\lambda x - y\| \geqslant \|x - y\| - |1 - \lambda| \geqslant \frac{1}{2}\varepsilon$. □

We will need the following lemma:

Lemma 2.4. *Let X be a Banach space and $\varphi\colon [a, b] \to X$ a curve. Then the following holds:*

(i) *Suppose that there exists a countable $M \subset [a, b]$ such that the right tangent $\tau_+(\varphi, x)$ exists for all $x \in [a, b] \setminus M$. If $\varphi(s) = \varphi(t)$, and $s < t$, then there exist $u, \xi \in [s, t)$ such that $\|\tau_+(\xi) - \tau_+(u)\| \geqslant 1$.*

(ii) *Suppose that the right tangent $\tau_+(\varphi, x)$ exists for all $x \in [a, b)$, $\tau_-(\varphi, b)$ exists, and $\omega\colon [c, d] \to [a, b]$ is continuous, onto, and strictly monotone. Then for $\psi = \varphi \circ \omega$ we have that $\mathrm{T}_a^b \varphi = \mathrm{T}_c^d \psi$.*

P r o o f. Ad (i): Without any loss of generality, assume that $\varphi(s) = \varphi(t) = 0$. Because M is at most countable, choose $\xi \in [s, t) \setminus M$ and $x^* \in S_{X^*}$ such that $x^*(\tau_+(\xi)) = 1$. There are two cases: either $x^*(\varphi(\xi)) \geqslant 0$ or $x^*(\varphi(\xi)) < 0$.

In the first case it easily follows that $m := \sup\{x^*(\varphi(r))\colon\ r \in [\xi, t]\} > 0$. Find $0 < h < m$ such that $h \notin x^*(\varphi(M))$ and define $u := \sup\{v \in [\xi, t]\colon\ x^*(\varphi(v)) \geqslant h\}$. Then $\xi < u < t$ and $x^*(\varphi(u)) = h$ (and thus $u \notin M$). For any $v \in (u, t]$, we see that $h \geqslant x^*(\varphi(v))$ and so $x^*(\tau_+(u)) \leqslant 0$. Thus

$$\|\tau_+(\xi) - \tau_+(u)\| \geqslant x^*(\tau_+(\xi) - \tau_+(u)) \geqslant 1. \tag{2.2}$$

In the case when $x^*(\varphi(\xi)) < 0$, find $h \in (x^*(\varphi(\xi)), 0)$ such that $h \notin x^*(\varphi(M))$ and take $u := \sup\{v \in [s, \xi]\colon\ x^*(\varphi(v)) \geqslant h\}$. Then $s < u < \xi$, $x^*(\tau_+(u)) \leqslant 0$, and we obtain (2.2).

Ad (ii): Without any loss of generality suppose that ω is increasing. Observe that $\tau_+(\varphi, x) = \tau_+(\psi, t)$ (or $\tau_-(\varphi, x) = \tau_-(\psi, t)$), where $x = \omega(t)$. The rest follows easily from the definition of the turn of tangents. □

We have the following generalized "mean-value theorem" for tangents:

Proposition 2.5. *Let X be a Banach space (or a Hilbert space) and $\varphi\colon\ [a, b] \to X$ a curve. Suppose that there exists $w \in S_X$, a countable $M \subset [a, b]$ and $0 < \varepsilon < \frac{1}{4}$ such that*

$$\|\tau_+(x) - w\| \leqslant \varepsilon \quad \text{for } x \in [a, b) \setminus M. \tag{2.3}$$

Then for any $a \leqslant c < d \leqslant b$ we have

$$\left\| \frac{\varphi(d) - \varphi(c)}{\|\varphi(d) - \varphi(c)\|} - w \right\| \leqslant 2\varepsilon \tag{2.4}$$

(or

$$\left\| \frac{\varphi(d) - \varphi(c)}{\|\varphi(d) - \varphi(c)\|} - w \right\| \leqslant \varepsilon$$

if X is the Hilbert space).

Proof. Let us first treat the case when X is a Banach space. Take $x^* \in X^*$ with $x^*(w) = \|x^*\| = 1$. Note that φ is one-to-one on $[a, b]$. To see this, suppose that $\varphi(c) = \varphi(d)$ for $a \leqslant c < d \leqslant b$. Lemma 2.4 (i) yields that $\|\tau_+(\xi) - w\| \geqslant \frac{1}{2}$ or $\|\tau_+(u) - w\| \geqslant \frac{1}{2}$, and this is a contradiction with (2.3).

Suppose that there exist c, d such that $a \leqslant c < d \leqslant b$ and

$$\left\| \frac{\varphi(d) - \varphi(c)}{\|\varphi(d) - \varphi(c)\|} - w \right\| > 2\varepsilon' > 2\varepsilon. \tag{2.5}$$

By continuity, we can assume that $c, d \notin M$. We can also assume that the left-hand expression in (2.5) is less than $\frac{1}{2}$ (otherwise shift d toward c). By adding a suitable vector to φ, we can assume that $\varphi(d)+\varphi(c)=0$. Using Lemma 2.3 with $x=(\varphi(d)-\varphi(c))/(\|\varphi(d)-\varphi(c)\|)=\varphi(d)/\|\varphi(d)\|$ and $y=w$ we obtain $\operatorname{dist}(w,\operatorname{span}\{x\})>\varepsilon'$. By the Hahn-Banach theorem we obtain $x^*\in X^*$, $\|x^*\|\leqslant 1$ with $x^*(x)=0$ and $x^*(w)>\varepsilon'$. For any $\varepsilon_1>0$ there exists $z^*\in X^*$ with $\|z^*\|<\varepsilon_1$, $z^*(w)=0$ and $z^*(x)<0$. Now take $w^*=x^*+z^*$. Then $\|w^*\|\leqslant 1+\varepsilon_1$, $w^*(w)>\varepsilon'$ and $w^*(\varphi(d))<0$. Take $h\in(w^*(\varphi(d)),0)$ such that $h\notin w^*(\varphi(M))$. Let $t_0=\sup\{t\in[c,d]\colon\ w^*(\varphi(t))\geqslant h\}$. Then $c<t_0<d$, $w^*(\tau_+(t_0))\leqslant 0$ and $t_0\notin M$ (because $h=w^*(\varphi(t_0))$). Thus

$$\|w-\tau_+(t_0)\|\geqslant(1+\varepsilon_1)^{-1}w^*(w-\tau_+(t_0))>(1+\varepsilon_1)^{-1}\varepsilon',$$

and this is a contradiction with our assumptions for small ε_1, as the right-hand expression is strictly bigger than ε.

Now suppose that X is a Hilbert space. By the first part of the proof we see that φ is one-to-one. Let $w\in S_X$ and define

$$C_w^\varepsilon=\left\{x\in X\colon\ \left\|\frac{x}{\|x\|}-w\right\|\leqslant\varepsilon\right\}.$$

Then C_w^ε is a convex cone and $C_w^\varepsilon\cup\{0\}$ is a closed convex set. Suppose that (2.4) is not true for some $c<d$, thus

$$\left\|\frac{\varphi(d)-\varphi(c)}{\|\varphi(d)-\varphi(c)\|}-w\right\|>\varepsilon'>\varepsilon.$$

Without any loss of generality we can assume that $\varphi(c)=0$. By the Hahn-Banach theorem there exists $x^*\in X^*$ such that $x^*(z)<x^*(\varphi(d))/\|\varphi(d)\|$ for any $z\in C_w^{\varepsilon'}\cup\{0\}$. We easily see that $x^*(\varphi(d))>0$, $x^*(z)\leqslant 0$ for $z\in C_w^{\varepsilon'}$ (because $C_w^{\varepsilon'}$ is a cone), and

$$x^*(y)<0\quad\text{for } y\in S_X\cap C_w^\varepsilon.$$

Take $0<\varepsilon_1<x^*(\varphi(d))$ such that $\varepsilon_1\notin x^*(\varphi(M))$. Define $t_0:=\sup\{t\in[c,d]\colon x^*(\varphi(t))<\varepsilon_1\}$. Then $x^*(\varphi(t_0))=\varepsilon_1$ (thus $t_0\notin M$), and $x^*(\varphi(t))\geqslant\varepsilon_1$ for $t\in[t_0,d]$. From this we obtain that $x^*(\tau_+(t_0))\geqslant 0$, and we have contradicted (2.6). □

Following [1], Chapter 3, we shall consider the notion of a one-sidedly smooth curve. We shall say that a curve $\varphi\colon [a,b] \to X$ has a *right tangent in the strong sense at* x provided there exists a right tangent $\tau_+(x)$ at x and for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $x \leqslant s < t < x + \delta$ we have

$$\left\| \tau_+(x) - \frac{\varphi(t) - \varphi(s)}{\|\varphi(t) - \varphi(s)\|} \right\| < \varepsilon. \tag{3.1}$$

In an analogous way we can define the notion of a left tangent in the strong sense. We say that a curve $\varphi\colon [a,b] \to X$ is *one-sidedly smooth* provided there exist left tangents in the strong sense at all $x \in (a,b]$, and right tangents in the strong sense at all $x \in [a,b)$.

Here are some basic properties of one-sidedly smooth curves (part (ii) is a generalization of Theorem 3.3.2 from [1]):

Lemma 3.1. *Let $\varphi\colon [a,b] \to X$ be a one-sidedly smooth curve. Then*

(i) *if $\omega\colon [c,d] \to [a,b]$ is continuous and strictly monotone, then $\overline{\varphi} = \varphi \circ \omega\colon [c,d] \to X$ is a one-sidedly smooth curve;*

(ii) *for any $\varepsilon > 0$ the set $\{x \in (a,b)\colon \|\tau_+(x) - \tau_-(x)\| > \varepsilon\}$ is finite (i.e. we have $\tau_+(x) = \tau_-(x)$ except for a countable set $S \subset (a,b)$).*

Proof. For part (i), we can suppose without any loss of generality that ω is increasing. Note that $\tau_\pm(\overline{\varphi}, s) = \tau_\pm(\varphi, \omega(s))$ for any $s \in (c,d)$ and the rest follows easily.

For part (ii), suppose that the set $A := \{x \in (a,b)\colon \|\tau_+(x) - \tau_-(x)\| > \varepsilon\}$ is infinite for some $\varepsilon > 0$. Then A has a limit point x in $[a,b]$. Without any loss of generality suppose that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset A$ with $x_n \searrow x$. Select $\delta > 0$ such that (3.1) holds for $x \leqslant s < t < x + \delta$ with $\frac{1}{4}\varepsilon$. Take $n \in \mathbb{N}$ such that $x_n < x + \delta$. We obtain that for some $u, v > 0$ with $x < x_n - u < x_n < x_n + v < x + \delta$ we have

$$\left\| \frac{\varphi(x_n + v) - \varphi(x_n)}{\|\varphi(x_n + v) - \varphi(x_n)\|} - \frac{\varphi(x_n) - \varphi(x_n - u)}{\|\varphi(x_n) - \varphi(x_n - u)\|} \right\| > \frac{1}{2}\varepsilon.$$

Thus either

$$\left\| \frac{\varphi(x_n + v) - \varphi(x_n)}{\|\varphi(x_n + v) - \varphi(x_n)\|} - \tau_+(x) \right\| > \frac{1}{4}\varepsilon$$

or

$$\left\| \tau_+(x) - \frac{\varphi(x_n) - \varphi(x_n - u)}{\|\varphi(x_n) - \varphi(x_n - u)\|} \right\| > \frac{1}{4}\varepsilon.$$

One of these possibilities must occur and thus we obtain a contradiction with (3.1). □

We have the following generalization of Lemma 3.11 from [1]:

Lemma 3.2. *Suppose that* $\varphi\colon [a,b] \to X$ *has a right tangent in the strong sense at* $x \in [a,b)$. *Then* $\varphi|_{[x,x+\delta]}$ *is rectifiable for some* $\delta > 0$ *and*

$$\frac{s(\varphi,[x,y])}{\|\varphi(y)-\varphi(x)\|} \to 1$$

when $y \searrow x$.

Remark 3.3. An analogous statement holds if we replace the right tangent in the strong sense with the notion of the left tangent in the strong sense, and make obvious modifications of the statement.

Proof. Denote $\tau = \tau_+(x)$ and select $x^* \in X^*$ with $x^*(\tau) = \|x^*\| = 1$. For any $0 < \varepsilon < 1$ take $\delta > 0$ such that for $x \leqslant s < t \leqslant x+\delta$ we get

$$\left\| \frac{\varphi(t)-\varphi(s)}{\|\varphi(t)-\varphi(s)\|} - \tau \right\| < \varepsilon.$$

Select $y \in (x, x+\delta]$ and let $D = \{x_i\}_{i=0}^n$ be a partition of $[x,y]$. Now estimate

$$x^*(\varphi(x_{i+1})-\varphi(x_i)) \leqslant \|\varphi(x_{i+1})-\varphi(x_i)\| \leqslant \frac{1}{1-\varepsilon}\, x^*(\varphi(x_{i+1})-\varphi(x_i))$$

for $x_i, x_{i+1} \in D$. Adding these inequalities up, we obtain by a telescoping argument that (the first inequality is trivial):

$$1 \leqslant \frac{s(\varphi,[x,y])}{\|\varphi(y)-\varphi(x)\|} \leqslant \frac{1}{1-\varepsilon}.$$

Complete the proof by sending $\varepsilon \to 0$. □

Corollary 3.4. *Any one-sidedly smooth curve is rectifiable.*

We can prove the following generalization of Theorem 3.3.1 from [1]:

Theorem 3.5. *Let* $\varphi\colon [a,b] \to X$ *be a curve.*

(i) *Suppose that there exists a countable set* $M \subset [a,b]$ *such that*

$$\lim_{\substack{y \searrow x \\ y \notin M}} \tau_+(y) =: w$$

exists. Then $\tau_+(x) = w$ *exists as a right tangent in the strong sense.*

(ii) *Suppose that there exists a countable set* $N \subset [a, b]$ *such that*

$$\lim_{\substack{y \nearrow x \\ y \notin N}} \tau_+(y) =: w$$

exists. Then $\tau_-(x) = w$ *exists as a left tangent in the strong sense.*

Proof. We shall only prove (i) as (ii) is analogous. Take $0 < \varepsilon < 1/4$. Then there exists a $\delta > 0$ such that for $y \in [x, x+\delta] \setminus M$ we have $\|w - \tau_+(y)\| \leqslant \varepsilon$. Proposition 2.5 now implies that for any $x \leqslant c < d \leqslant x + \delta$ we have

$$\Big\| \frac{\varphi(d) - \varphi(c)}{\|\varphi(d) - \varphi(c)\|} - w \Big\| \leqslant 2\varepsilon,$$

and thus $w = \tau_+(x)$ is a right tangent in the strong sense. □

The next lemma shows some properties of the arc-length parametrization of a curve that is one-sidedly smooth. It is a generalization of Theorem 3.3.3 from [1].

Proposition 3.6. *Let* X *be a Banach space, and let* $\varphi\colon [a, b] \to X$ *be a one-sidedly smooth curve. Let* $F\colon [0, l] \to X$ *(where* $l = s(\varphi)$*) be the arc-length parametrization of* φ*. Then*

(i) $F'_+(x) = \tau_+(F, x)$ *for* $x \in [0, l)$*, and* $F'_-(x) = \tau_-(F, x)$ *for* $x \in (0, l]$*,*
(ii) F' *exists except for a countable set of points in* $[0, l]$*,*
(iii) F'_+ *is right continuous at all* $x \in [0, l)$ *(*F'_- *is left continuous at all* $x \in (0, l]$*),*
(iv) $\mathrm{T}_0^l F = \mathrm{T}_a^b \varphi$*.*

Proof. Corollary 3.4 implies that φ is rectifiable, and thus we obtain the existence of the arc-length parametrization of φ; call it F. Note that $l(t) = s(\varphi, [a, t])$ is an increasing function on $[a, b]$ and $\varphi(l^{-1}(r)) = F(r)$ for $r \in [0, l]$. Thus by Lemma 2.4 (ii) our condition (iv) holds. Note that $\tau_\pm(F, t) = \tau_\pm(\varphi, l^{-1}(t))$ for $t \in (0, l)$, and $\tau_\pm(F, t)$ is a right (left) tangent in the strong sense (and similarly for $t = 0, l$ considering the corresponding unilateral tangents). By Lemma 3.1 (i) and by Lemma 3.2 we see that for all $x \in [0, l)$ we have

$$\lim_{t \searrow 0} \frac{\|F(x+t) - F(x)\|}{t} = 1. \tag{3.2}$$

Thus

$$\begin{aligned} F'_+(x) &= \lim_{t \searrow 0} \frac{F(x+t) - F(x)}{t} \\ &= \lim_{t \searrow 0} \frac{F(x+t) - F(x)}{\|F(x+t) - F(x)\|} \, \frac{\|F(x+t) - F(x)\|}{t} = \tau_+(F, x), \end{aligned} \tag{3.3}$$

where the last equality follows by (3.2). Similarly for $F'_-(x)$ and $x \in (0, l]$. This concludes the proof of condition (i). Condition (iii) follows from our condition (i) and from the definition of a strict unilateral tangent.

To prove (ii), note that because F is one-sidedly smooth, Lemma 3.1 (ii) implies the equality $\tau_+(F, x) = \tau_-(F, x)$ except for a countable set $S \subset [0, l]$, and thus (3.3) implies that $F'(x)$ exists for all $x \in [0, l] \setminus S$. □

4. Finite turn

We shall need the following lemma:

Lemma 4.1. *Suppose that X is a normed linear space with two norms $\|\cdot\|$ and $|\cdot|$ such that*

$$C_1\|x\| \leqslant |x| \leqslant C_2\|x\| \quad \text{for all } x \in X \tag{4.1}$$

and for some $C_1, C_2 > 0$. Then for $x, y \in X$ with $\|x\| = \|y\| = 1$ we have that

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right| \leqslant 2C_1^{-1}C_2\|x - y\|.$$

P r o o f. Take $x, y \in X$ with $\|x\| = \|y\| = 1$. Then

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right| \leqslant |y|^{-1}\big||x| - |y|\big| + |y|^{-1}|x - y| \leqslant 2C_1^{-1}C_2\|x - y\|.$$

□

Remark 4.2. It is well known that for any two norms $\|\cdot\|$ and $|\cdot|$ on a finite-dimensional space X there exist some $C_1, C_2 > 0$ such that (4.1) holds.

Now we can prove

Proposition 4.3. *Let X be a Banach space, let $\|\cdot\|$ and $|\cdot|$ be two norms on X satisfying (4.1), and let $\varphi\colon [a, b] \to (X, \|\cdot\|)$ be a curve with finite turn of tangents. Then φ also has finite turn of tangents if we consider $\varphi\colon [a, b] \to (X, |\cdot|)$; more precisely, $|\cdot| - \mathrm{T}_a^b\varphi \leqslant 2C_1^{-1}C_2\mathrm{T}_a^b\varphi$, where $|\cdot| - \mathrm{T}_a^b\varphi$ is defined as $\mathrm{T}_a^b\varphi$, but we replace $\|\cdot\|$ by $|\cdot|$ (also in (1.2) and in (1.3)).*

P r o o f. Suppose that $\tau_+(x) = \lim\limits_{t \searrow x} \varphi(t) - \varphi(x)/\|\varphi(t) - \varphi(x)\|$. Then by Lemma 4.1 applied to

$$\frac{\|\varphi(t) - \varphi(x)\|^{-1}(\varphi(t) - \varphi(x))}{\|\varphi(t) - \varphi(x)\|^{-1}|\varphi(t) - \varphi(x)|}$$

and $\tau_+(x)/|\tau_+(x)|$ for $t > x$ we see that

$$\tau'_+(x) = \lim_{t \searrow x} \frac{\varphi(t) - \varphi(x)}{|\varphi(t) - \varphi(x)|} = \frac{\tau_+(x)}{|\tau_+(x)|},$$

and similarly for $\tau'_-(b)$. Now an application of Lemma 4.1 to the definition of $|\cdot|-\mathrm{T}_a^b\varphi$ yields the conclusion of the proposition. □

As a corollary, we get that finiteness of the turn of tangents of a curve does not depend on the equivalent norm.

Let us summarize the basic properties of curves with finite turn of tangents. A similar lemma holds for curves with finite angular turn of tangents and is presented in [4].

Lemma 4.4. *Let X be a Banach space, $\varphi\colon [a,b] \to X$, and let φ have finite turn of tangents. Then*

(I) *for each $x \in (a,b]$ the left tangent $\tau_-(x)$ exists as a left tangent in the strong sense, and $\mathrm{L}_a^b\varphi = \mathrm{T}_a^b\varphi < \infty$ (and thus φ has right tangents in the strong sense at all $x \in [a,b)$);*

(ii) *φ is one-sidedly smooth;*

(iii) *$\mathrm{T}_x^{x+t}\varphi \to 0$ as $t \searrow 0$ and $\mathrm{T}_s^x\varphi \to 0$ as $s < x$, $s \nearrow x$ for any $x \in [a,b)$ (and $x \in (a,b]$, respectively);*

(iv) *for each $x \in [a,b)$ there exists $\varepsilon > 0$ such that φ is one-to-one on $[x, x+\varepsilon)$;*

(v) *for any $\varepsilon > 0$ there are only finitely many $x \in (a,b)$ such that $\|\tau_+(x) - \tau_-(x)\| \geqslant \varepsilon$.*

Proof. Ad (i): We can assume that $x = 0$. We claim that for any $\varepsilon_1 > 0$ there is a $\delta_1 > 0$ such that if for any $0 < \delta' < \delta'' < \delta_1$ we consider a partition $D = \{-\delta'' = x_0 < \ldots < x_n = -\delta'\}$ of $[-\delta'', -\delta']$, then

$$\sum_{i=0}^{n-1} \|\tau_+(x_{i+1}) - \tau_+(x_i)\| < \varepsilon_1. \tag{4.2}$$

If not, then we easily obtain a contradiction with the fact that φ has finite turn of tangents. If we define $W_n = \{\tau_+(y)\colon -1/n < y < 0\}$, then $\bigcap_n \overline{W_n} = \{w\}$ for some $w \in S_X$. By Theorem 3.5 (ii) we see that w is the left tangent in the strong sense at x.

The fact that $\mathrm{L}_a^b\varphi = \mathrm{T}_a^b\varphi$ follows by an easy approximation argument.

Ad (ii): This follows immediately from part (i).

Ad (iii): If this is not true, we easily get a contradiction with the fact that φ has finite turn of tangents. To see this, note that by part (i) of our lemma, we have that $\tau_+(x) = \lim_{y \searrow x} \tau_+(y)$ and that $\mathrm{T}_c^d\varphi \to 0$ as $d \to x$ for any $x < c < d$.

Ad (iv): For $x \in [a,b)$ take $\varepsilon > 0$ such that $\mathrm{T}_x^{x+\varepsilon}\varphi < 1$. Then φ is one-to-one on $[x, x+\varepsilon)$, otherwise we get a contradiction by Lemma 2.4 (i).

Ad (v): This follows by part (ii) of Lemma 3.1 and by part (ii) of the current lemma. $\square$

Lemma 4.5. *Let $\varphi\colon [a,b] \to X$ be a curve with finite turn of tangents. Then there exists an arc-length parametrization $F\colon [0,l] \to X$ of φ and it satisfies:*

(i) *$F'_+(s) = \tau_+(F,s)$ for any $s \in [0,l)$, and $F'_-(s) = \tau_-(F,s)$ for any $s \in (0,l]$,*

(ii) *$F'(x)$ exists except for a countable set of points $x \in (0,l)$,*

(iii) *$\mathrm{T}_0^l F = \mathrm{T}_a^b \varphi$,*

(iv) *for any $0 \leqslant p < q \leqslant l$ we have*

$$\left| \frac{\|F(q) - F(p)\|}{q-p} - 1 \right| \leqslant \mathrm{T}_p^q F. \tag{4.3}$$

P r o o f. By part (ii) of Lemma 4.4 we see that a curve with finite turn of tangents is one-sidedly smooth. Thus by Proposition 3.6 there exists the arc-length parametrization F of φ. Part (iv) of Lemma 3.6 implies that $\mathrm{T}_0^l F = \mathrm{T}_a^b \varphi$. It is easy to see that F is 1-Lipschitz (because of being an arc-length parametrization). A simple computation now yields (using Lemma 2.1) that

$$\begin{aligned}
0 &\leqslant q - p - \|F(q) - F(p)\| \\
&= \int_p^q \left(\|F'(t)\| - \frac{1}{q-p} \int_p^q \|F'(s)\| \,\mathrm{d}s \right) \mathrm{d}t \\
&\leqslant \int_p^q \left\| F'(t) - \frac{1}{q-p} \int_p^q F'(s) \,\mathrm{d}s \right\| \mathrm{d}t \\
&\leqslant \frac{1}{q-p} \int_p^q \left\| \int_p^q (F'(t) - F'(s)) \,\mathrm{d}s \right\| \mathrm{d}t \\
&\leqslant \frac{1}{q-p} \int_p^q \int_p^q \|F'(t) - F'(s)\| \,\mathrm{d}s \,\mathrm{d}t \leqslant (q-p)\mathrm{T}_p^q F.
\end{aligned}$$

This implies the condition (4.3). All the other properties are consequences of the fact that F is one-sidedly smooth and follow from Proposition 3.6. $\square$

Definition 4.6. Let X be a normed linear space. We say that $A\colon X \setminus \{0\} \times X \setminus \{0\} \to (0,\infty)$ is an *angular form* provided it satisfies for $u, v, w \in X \setminus \{0\}$:

(i) $A(u,u) = 0$, $A(u,v) = A(v,u)$,

(ii) $A(u,v) \leqslant A(u,w) + A(w,v)$,

(iii) $A(u, u+v) \leqslant A(u,v)$ provided $u + v \neq 0$.

It is easily seen that the angle $\angle(\cdot,\cdot)$ in a Hilbert space is an angular form. Lemma 2.2 together with the triangle inequality imply that in any normed linear space the quantity $A_1(u,v) = \big\|u/\|u\| - v/\|v\|\big\|$ is also an angular form. For a normed linear space X with an angular form A and a curve $\varphi\colon [a,b] \to X$ we can define the *general angular turn* (or *A-turn*) as $\text{A-P}_a^b\varphi := \sup_D \text{A-P}(\varphi, D)$, where the supremum is taken over all partitions $D = \{x_i\}_{i=0}^n$ of $[a,b]$ and

$$\text{A-P}(\varphi, D) := \sum_{i=1}^{n-1} A(\varphi(x_{i+1}) - \varphi(x_i), \varphi(x_i) - \varphi(x_{i-1})) \tag{4.4}$$

provided the right hand side is defined, and $\text{A-P}(\varphi, D) = 0$ otherwise.

Lemma 4.7. *Let X be a normed linear space with an angular form A, $\varphi\colon [a,b] \to X$ a curve, and D, D' partitions of $[a,b]$ such that $D \subset D'$. Then*

$$\text{A-P}(\varphi, D) \leqslant \text{A-P}(\varphi, D')$$

provided $\text{A-P}(\varphi, D')$ *is defined by* (4.4).

Proof. We use the proof of Lemma 3.17 from [4]. Let $D = \{a = x_0 < \ldots < x_n = n\}$. It is enough to prove the statement for $D' = D \cup \{t\}$ and $t \notin D$; the rest follows by induction. Suppose that $t \in (x_0, x_1)$. Apply the property (ii) from Definition 4.6 with $u = \varphi(x_2) - \varphi(x_1)$, $v = (\varphi(x_1) - \varphi(t) + \varphi(t) - \varphi(x_0))$, $w = \varphi(x_1) - \varphi(t)$, and the property (iii) with $u = \varphi(x_1) - \varphi(t)$, $v = \varphi(t) - \varphi(x_0)$, to obtain

$$\begin{aligned} A(\varphi(x_2) - \varphi(x_1), \varphi(x_1) - \varphi(x_0)) &\leqslant A(\varphi(x_2) - \varphi(x_1), \varphi(x_1) - \varphi(t)) \\ &\quad + A(\varphi(x_1) - \varphi(t), \varphi(t) - \varphi(x_0)). \end{aligned}$$

Thus $\text{A-P}_a^b(\varphi, D) \leqslant \text{A-P}_a^b(\varphi, D')$.

For $t \in (x_{n-1}, x_n)$, the proof is analogous. Thus suppose that $t \in (x_i, x_{i+1})$, where $1 \leqslant i \leqslant n-2$. From the properties of A (see Definition 4.6) we obtain for $u, v, z, w_1, w_2 \in X \setminus \{0\}$:

$$A(u,v) + A(v,z) \leqslant A(u,w_1) + A(w_1,v) + A(v,w_2) + A(w_2,z). \tag{4.5}$$

Apply (4.5) with $u = \varphi(x_{i+2}) - \varphi(x_{i+1})$, $v = (\varphi(x_{i+1}) - \varphi(t) + \varphi(t) - \varphi(x_i))$, $z = \varphi(x_i) - \varphi(x_{i-1})$, $w_1 = \varphi(x_{i+1}) - \varphi(t)$, and $w_2 = \varphi(t) - \varphi(x_i)$ to get

$$\begin{aligned} &A(\varphi(x_{i+2}) - \varphi(x_{i+1}), \varphi(x_{i+1}) - \varphi(x_i)) + A(\varphi(x_{i+1}) - \varphi(x_i), \varphi(x_i) - \varphi(x_{i-1})) \\ &\quad \leqslant A(\varphi(x_{i+2}) - \varphi(x_{i+1}), \varphi(x_{i+1}) - \varphi(t)) + A(\varphi(x_{i+1}) - \varphi(t), \varphi(t) - \varphi(x_i)) \\ &\qquad + A(\varphi(t) - \varphi(x_i), \varphi(x_i) - \varphi(x_{i-1})). \end{aligned}$$

Thus $\text{A-P}(\varphi, D) \leqslant \text{A-P}(\varphi, D')$. □

An analogue of the next lemma, which holds for curves with finite angular turn in a Hilbert space, is given in [4].

Lemma 4.8. *Let X be a Banach space, $\varphi\colon [a,b] \to X$, and let φ have a finite turn. Then*

(i) $\mathrm{P}_x^{x+t}\varphi \to 0$ *as* $t \searrow 0$ *for any* $x \in [a,b)$, *and the function* φ *is one-to-one on* $[x, x+\varepsilon)$ *for each* $x \in [a,b)$ *and some* $\varepsilon > 0$ *(and also on* $(x-\varepsilon, x]$ *for all* $x \in (a,b]$ *and some* $\varepsilon > 0$*);*

(ii) *for each* $x \in [a,b)$ *the right tangent* $\tau_+(x)$ *exists as the right tangent in the strong sense, and for each* $x \in (a,b]$ *the left tangent* $\tau_-(x)$ *exists as the left tangent in the strong sense;*

(iii) *if* $\omega\colon [c,d] \to [a,b]$ *is continuous, onto, and strictly monotone, then for* $\xi = \varphi \circ \omega$ *we have* $\mathrm{P}_a^b\varphi = \mathrm{P}_c^d\xi$.

P r o o f. The proof of this lemma for the case of angular turn in a Hilbert space is found in [4].

To prove (i), assume that $\mathrm{P}_x^{x+t}\varphi > \delta > 0$ for some $\delta > 0$ and all $0 < t < b - x$. We claim that if this is the case, then there exist sequences $\alpha_j, \beta_j \searrow x$ with $\beta_{j+1} < \alpha_j < \beta_j$ such that $\mathrm{P}_{\beta_j}^{\alpha_j}\varphi > \frac{1}{4}\delta$. To see this, fix $0 < t < b - x$, and find $D = \{x_i\}_{i=0}^n$, a partition of $[x, x+t]$ such that $\mathrm{P}(\varphi, D) > \frac{1}{2}\delta$. Then (by Lemma 4.7) either

$$\left\| \frac{\varphi(x_2) - \varphi(x_1)}{\|\varphi(x_2) - \varphi(x_1)\|} - \frac{\varphi(x_1) - \varphi(x_0)}{\|\varphi(x_1) - \varphi(x_0)\|} \right\| > \frac{1}{4}\delta$$

or $\mathrm{P}_{x_0}^{x_1}\varphi > \frac{1}{4}\delta$. In the former case, by continuity of φ, there is $y \in (x, x_1)$ such that

$$\left\| \frac{\varphi(x_2) - \varphi(x_1)}{\|\varphi(x_2) - \varphi(x_1)\|} - \frac{\varphi(x_1) - \varphi(y)}{\|\varphi(x_1) - \varphi(y)\|} \right\| > \frac{1}{4}\delta.$$

Thus $\mathrm{P}_y^{x_2} > \frac{1}{4}\delta$. Choose $\alpha_1 = y$, $\beta_1 = x_2$, and proceed with $t = \frac{1}{2}(x + \alpha_1)$. In the latter case, take $\alpha_1 = x_2$, $\beta_1 = x_n$, and proceed with $t = \frac{1}{2}(x + \alpha_1)$. Now continue in the obvious fashion. However, the existence of the sequence $(\alpha_i, \beta_i)_{i\in\mathbb{N}}$ with $\mathrm{P}_{\alpha_i}^{\beta_i}\varphi > \frac{1}{4}\delta$ easily contradicts the assumption that φ has a finite turn.

To prove (ii), without any loss of generality assume that $x = 0$ and $\varphi(x) = 0$. By (i), take $\varepsilon > 0$ and find $\delta > 0$ such that $P_0^\delta\varphi < \varepsilon$. For any $0 < y < z < w < \delta$ we get

$$\begin{aligned}
\left\| \frac{\varphi(y)}{\|\varphi(y)\|} - \frac{\varphi(z)}{\|\varphi(z)\|} \right\| &\leqslant \left\| \frac{\varphi(y)}{\|\varphi(y)\|} - \frac{\varphi(z) - \varphi(y)}{\|\varphi(z) - \varphi(y)\|} \right\| \\
&\quad + \left\| \frac{\varphi(z) - \varphi(y)}{\|\varphi(z) - \varphi(y)\|} - \frac{\varphi(w) - \varphi(z)}{\|\varphi(w) - \varphi(z)\|} \right\| \\
&\quad + \left\| \frac{\varphi(w) - \varphi(z)}{\|\varphi(w) - \varphi(z)\|} - \frac{\varphi(z)}{\|\varphi(z)\|} \right\| \leqslant 2\mathrm{P}_0^\delta\varphi < 2\varepsilon.
\end{aligned}$$

Thus $\tau_+(x)$ exists by completeness of X. To show that $\tau_+(x)$ is the tangent in the strong sense, notice that for $0 < t < s < \delta$ we have

$$\begin{aligned}(4.6)\qquad \left\| \frac{\varphi(s)-\varphi(t)}{\|\varphi(s)-\varphi(t)\|} - \tau_+(x) \right\| &\leqslant \left\| \frac{\varphi(s)-\varphi(t)}{\|\varphi(s)-\varphi(t)\|} - \frac{\varphi(t)}{\|\varphi(t)\|} \right\| \\ &+ \left\| \frac{\varphi(t)}{\|\varphi(t)\|} - \tau_+(x) \right\| \leqslant 3\mathrm{P}_0^{\delta}\varphi < 3\varepsilon.\end{aligned}$$

The proof for left tangents is analogous.

Finally, part (iii) is an easy consequence of the fact that ω is a homeomorphism of $[c,d]$ onto $[a,b]$. □

Let us compare the finite turn (of tangents) and the finite angular turn (of tangents) in the case of Hilbert space-valued curves.

Proposition 4.9. *Let H be a Hilbert space and $\varphi\colon [a,b] \to H$ a curve.*

(i) *If $\tau_+(x)$ exists for all $x \in [a,b)$, $\tau_-(x)$ exists for all $x \in (a,b]$, and $\tau_+(x) = \tau_-(x)$ for all $x \in (a,b)$, then $\mathrm{T}_a^b\varphi = \angle\mathrm{T}_a^b\varphi$, and $\mathrm{P}_a^b\varphi = \angle\mathrm{P}_a^b\varphi$.*

(ii) *If $\tau_+(x)$ exists for all $x \in [a,b)$, $\tau_-(b)$ exists, and $\mathrm{T}_a^b\varphi < \infty$, then*

$$\begin{aligned}(4.7)\qquad \mathrm{T}_a^b\varphi - \sum_{\substack{x\in(a,b):\\ \tau_+(x)\neq\tau_-(x)}} \|\tau_+(x)-\tau_-(x)\| \\ = \angle\mathrm{T}_a^b\varphi - \sum_{\substack{x\in(a,b):\\ \tau_+(x)\neq\tau_-(x)}} 2\arcsin\left(\frac{\|\tau_+(x)-\tau_-(x)\|}{2}\right).\end{aligned}$$

(iii) *If $\tau_+(x)$ exists for all $x \in [a,b)$, $\tau_-(b)$ exists, and $\mathrm{P}_a^b\varphi < \infty$, then*

$$\begin{aligned}\mathrm{P}_a^b\varphi - \sum_{\substack{x\in(a,b):\\ \tau_+(x)\neq\tau_-(x)}} \|\tau_+(x)-\tau_-(x)\| \\ = \angle\mathrm{P}_a^b\varphi - \sum_{\substack{x\in(a,b):\\ \tau_+(x)\neq\tau_-(x)}} 2\arcsin\left(\frac{\|\tau_+(x)-\tau_-(x)\|}{2}\right).\end{aligned}$$

Proof. By Remark 1.1, we easily see that $\mathrm{P}_a^b\varphi < \infty$ if and only if $\angle\mathrm{P}_a^b\varphi < \infty$, and $\mathrm{T}_a^b\varphi < \infty$ if and only if $\angle\mathrm{T}_a^b\varphi < \infty$.

To prove (i), note that by Remark 1.1 we always have $\mathrm{T}_a^b\varphi \leqslant \angle\mathrm{T}_a^b\varphi$. To see the other inequality, suppose that $\mathrm{T}_a^b\varphi < \infty$, take a partition D of $[a,b]$, and define $\tau_+(b) := \tau_-(b)$. Pick $\varepsilon > 0$ and find a refinement $D' = \{x_i\}_{i=0}^n$ of D such that $\|\tau_+(x) - \tau_+(y)\| \leqslant \varepsilon$ for $x, y \in [x_i, x_{i+1}]$, and $x_i, x_{i+1} \in D'$. To see that such a refinement exists, we use a simple compactness argument based on Lemma 4.4 (v) and take a minimal (with respect to inclusion) finite subcover.

Because $\angle(x,y) = 2\arcsin(\frac{1}{2}\|x-y\|)$ for $\|x\| = \|y\| = 1$, we obtain that there exists a non-decreasing function $f\colon [0,2] \to \mathbb{R}$ with $\lim_{t\searrow 0} f(t) = 1$ such that if $x, y \in S_X$, then $\angle(x,y) \leqslant f(\|x-y\|)\|x-y\|$. Thus

$$\angle\mathrm{T}(\varphi, D) \leqslant \angle\mathrm{T}(\varphi, D') \leqslant f(\varepsilon)\mathrm{T}(\varphi, D') \leqslant f(\varepsilon)\mathrm{T}_a^b\varphi.$$

To complete the proof, send $\varepsilon \to 0$. Because D was arbitrary, we are done. The equality $\mathrm{P}_a^b\varphi = \angle\mathrm{P}_a^b\varphi$ is proved similarly.

We shall only prove (ii), as (iii) is analogous. Suppose that D is a partition of $[a,b]$ and $\varepsilon > 0$. Find a finite $M \subset \{x \in (a,b)\colon\ \tau_+(x) \neq \tau_-(x)\}$ such that

$$\sum_{\substack{x\in(a,b):\\ \tau_+(x)\neq\tau_-(x)}} \angle(\tau_+(x), \tau_-(x)) < \sum_{x\in M} \angle(\tau_+(x), \tau_-(x)) + \varepsilon. \tag{4.8}$$

Find a partition $D' \supset D$ such that $M \subset D' = \{x_i\}_{i=0}^n$, and for $x_{i+1} \in M$ we have

$$\big|\|\tau_+(x_i) - \tau_+(x_{i+1})\| - \|\tau_+(x_{i+1}) - \tau_-(x_{i+1})\|\big| \leqslant \frac{\varepsilon}{m} \tag{4.9}$$

and

$$\big|\angle(\tau_+(x_i), \tau_+(x_{i+1})) - \angle(\tau_+(x_{i+1}), \tau_-(x_{i+1}))\big| \leqslant \frac{\varepsilon}{m}, \tag{4.10}$$

where $m = \#(M)$. This can be achieved by a simple compactness argument which uses Lemma 4.4. Now we are ready to estimate (using (4.8), (4.9), and (4.10)):

$$\begin{aligned}
&T(\varphi, D) - \sum_{\substack{x\in(a,b):\\ \tau_+(x)\neq\tau_-(x)}} \|\tau_+(x) - \tau_-(x)\| \\
&\quad\leqslant T(\varphi, D') - \sum_{x\in M} \|\tau_+(x) - \tau_-(x)\| \\
&\quad\leqslant T(\varphi, D') - \sum_{x_{i+1}\in M} \|\tau_+(x_i) - \tau_+(x_{i+1})\| + \varepsilon \\
&\quad\leqslant \sum_{x_{i+1}\in D'\setminus M} \angle(\tau_+(x_i), \tau_+(x_{i+1})) + \varepsilon \\
&\quad\leqslant \angle\mathrm{T}(\varphi, D') - \sum_{x_{i+1}\in M} \angle(\tau_+(x_i), \tau_+(x_{i+1})) + \varepsilon \\
&\quad\leqslant \angle\mathrm{T}(\varphi, D') - \sum_{x\in M} \angle(\tau_+(x), \tau_-(x)) + 2\varepsilon \\
&\quad\leqslant \angle\mathrm{T}_a^b\varphi - \sum_{\substack{x\in(a,b):\\ \tau_+(x)\neq\tau_-(x)}} 2\arcsin\Big(\frac{\|\tau_+(x) - \tau_-(x)\|}{2}\Big) + 3\varepsilon.
\end{aligned}$$

To obtain the desired inequality in (4.7), send $\varepsilon \to 0$, and then take supremum over all partitions D of $[a, b]$. The proof of the reverse inequality follows similar lines. □

Now we can prove the main theorem:

Theorem 4.10. *Let X be a Banach space, and let $\varphi\colon [a, b] \to X$ be a curve. Then φ has finite turn if and only if φ has finite turn of tangents.*

If X is a Hilbert space, then $\mathrm{P}_a^b\varphi = \mathrm{T}_a^b\varphi$.

Proof. Suppose that φ has finite turn. We shall follow the proof of Theorem 3.32 from [4]. Lemma 4.8 (ii) implies the existence of $\tau_+(x)$ for $x \in [a, b)$ and of $\tau_-(b)$. Choose an arbitrary partition $D = \{x_i\}_{i=0}^n$ of $[a, b]$. Take $\delta > 0$ such that $0 < \delta < \frac{1}{2} \min\limits_{0\leqslant i\leqslant n-1} (x_{i+1} - x_i)$. Then we have $x_i < x_i + \delta < x_{i+1}$ for $i = 0, \ldots, n-2$, and $x_{n-1} < x_{n-1} + \delta < x_n - \delta < x_n$. Denote

$$T(\delta) = \sum_{i=0}^{n-2} \left\| \frac{\varphi(x_{i+1}+\delta) - \varphi(x_{i+1})}{\|\varphi(x_{i+1}+\delta) - \varphi(x_{i+1})\|} - \frac{\varphi(x_i+\delta) - \varphi(x_i)}{\|\varphi(x_i+\delta) - \varphi(x_i)\|} \right\| + \left\| \frac{\varphi(x_n) - \varphi(x_n-\delta)}{\|\varphi(x_n) - \varphi(x_n-\delta)\|} - \frac{\varphi(x_{n-1}+\delta) - \varphi(x_{n-1})}{\|\varphi(x_{n-1}+\delta) - \varphi(x_{n-1})\|} \right\|.$$

Then

$$\begin{aligned} T(\delta) \leqslant \sum_{i=0}^{n-2} & \left\| \frac{\varphi(x_{i+1}+\delta) - \varphi(x_{i+1})}{\|\varphi(x_{i+1}+\delta) - \varphi(x_{i+1})\|} - \frac{\varphi(x_{i+1}) - \varphi(x_i+\delta)}{\|\varphi(x_{i+1}) - \varphi(x_i+\delta)\|} \right\| \\ &+ \left\| \frac{\varphi(x_{i+1}) - \varphi(x_i+\delta)}{\|\varphi(x_{i+1}) - \varphi(x_i+\delta)\|} - \frac{\varphi(x_i+\delta) - \varphi(x_i)}{\|\varphi(x_i+\delta) - \varphi(x_i)\|} \right\| \\ &+ \left\| \frac{\varphi(x_n) - \varphi(x_n-\delta)}{\|\varphi(x_n) - \varphi(x_n-\delta)\|} - \frac{\varphi(x_n-\delta) - \varphi(x_{n-1}+\delta)}{\|\varphi(x_n-\delta) - \varphi(x_{n-1}+\delta)\|} \right\| \\ &+ \left\| \frac{\varphi(x_n) - \varphi(x_n-\delta)}{\|\varphi(x_n) - \varphi(x_n-\delta)\|} - \frac{\varphi(x_n-\delta) - \varphi(x_{n-1})}{\|\varphi(x_n-\delta) - \varphi(x_{n-1})\|} \right\| \leqslant \mathrm{P}_a^b\varphi. \end{aligned}$$

Thus also

$$\mathrm{T}(\varphi, D) = \sum_{i=0}^{n-2} \|\tau_+(x_{i+1}) - \tau_+(x_i)\| + \|\tau_-(b) - \tau_+(x_{n-1})\| = \lim_{\delta\searrow 0} \mathrm{T}(\delta) \leqslant \mathrm{P}_a^b\varphi.$$

As we have chosen an arbitrary partition D of $[a, b]$, we obtain $\mathrm{T}_a^b\varphi \leqslant \mathrm{P}_a^b\varphi < \infty$.

Suppose that φ has finite turn of tangents. First note that for any $a, b \in X$ with $\|a\| = 1$ and $b \neq 0$ we get that

$$\left\| a - \frac{b}{\|b\|} \right\| \leqslant \frac{2}{\|b\|} \|a - b\|. \tag{4.11}$$

If X is a Hilbert space, then if $x, y \in X$ are such that $\|x\| = \|y\| = 1$, $\varepsilon \in [0,1)$, $1-\varepsilon < \xi, \eta \leqslant 1$, then

$$\|x-y\| \leqslant \frac{1}{1-\varepsilon}\,\|\xi x - \eta y\|. \tag{4.12}$$

Take the parametrization F of φ from Lemma 4.5. Define $l = s(F)$, so F is defined on $[0,l]$. Take a partition $D_0 = \{y_i\}_{i=0}^m$ of $[0,l]$ such that φ is one-to-one on $[y_i, y_{i+1}]$ for $i = 0,\dots,m-1$. This can be achieved by compactness, Lemma 4.4 (iv), and symmetrical rôles of right and left tangents.

To prove our theorem, take any partition D of $[0,l]$. If $\mathrm{P}(F,D) = 0$, then there is nothing to prove, otherwise pick $\varepsilon > 0$. Define $D' = D \cup D_0$. Further, find a refinement $D'' = \{x_i\}_{i=0}^n$ of D' such that $\nu(D'') < \varepsilon$ and $\mathrm{T}_{x_i}^{x_{i+1}} F \leqslant \varepsilon$ for any $i = 0,\dots,n-1$. This can be achieved by a simple compactness argument using Lemma 4.4 and taking the (minimal with respect to inclusion) finite subcover. Now for any $i = 1,\dots,n-1$, estimate

$$\begin{aligned}
&\left\|\frac{F(x_{i+1})-F(x_i)}{\|F(x_{i+1})-F(x_i)\|} - \frac{F(x_i)-F(x_{i-1})}{\|F(x_i)-F(x_{i-1})\|}\right\| \\
&\qquad\leqslant 2\frac{x_i - x_{i-1}}{\|F(x_i)-F(x_{i-1})\|}\left\|\frac{F(x_{i+1})-F(x_i)}{x_{i+1}-x_i} - \frac{F(x_i)-F(x_{i-1})}{x_i - x_{i-1}}\right\| \\
&\qquad\leqslant 2\Big(1+\frac{\varepsilon}{1-\varepsilon}\Big)\left\|\frac{F(x_{i+1})-F(x_i)}{x_{i+1}-x_i} - \frac{F(x_i)-F(x_{i-1})}{x_i - x_{i-1}}\right\|,
\end{aligned} \tag{4.13}$$

where we use (4.11) with $a = (F(x_{i+1})-F(x_i))/\|F(x_{i+1})-F(x_i)\|$,

$$b = \frac{F(x_i)-F(x_{i-1})}{x_i - x_{i-1}}\,\frac{x_{i+1}-x_i}{\|F(x_{i+1})-F(x_i)\|},$$

and (4.3) with $\mathrm{T}_{x_i}^{x_{i+1}} F \leqslant \varepsilon$.

The last term from (4.13) can be estimated in the following way (using Lemma 2.1):

$$\begin{aligned}
&\left\|\frac{F(x_{i+1})-F(x_i)}{x_{i+1}-x_i} - \frac{F(x_i)-F(x_{i-1})}{x_i - x_{i-1}}\right\| \\
&\qquad= \left\|(x_{i+1}-x_i)^{-1}\int_{x_i}^{x_{i+1}} \tau_+(z)\,\mathrm{d}z - (x_i - x_{i-1})^{-1}\int_{x_{i-1}}^{x_i} \tau_+(z)\,\mathrm{d}z\right\| \\
&\qquad= \left\|\int_0^1 (\tau_+(\lambda(x_{i+1}-x_i)+x_i) - \tau_+(\lambda(x_i - x_{i-1})+x_{i-1}))\,\mathrm{d}\lambda\right\|.
\end{aligned} \tag{4.14}$$

To complete the proof, note that

$$\begin{aligned}(4.15)\quad \sum_{i=1}^{n-1}\Big\|\int_0^1 &(\tau_+(\lambda(x_{i+1}-x_i)+x_i)-\tau_+(\lambda(x_i-x_{i-1})+x_{i-1}))\,\mathrm{d}\lambda\Big\| \\ &\leqslant \int_0^1\sum_{i=1}^{n-1}\|\tau_+(\lambda(x_{i+1}-x_i)+x_i)-\tau_+(\lambda(x_i-x_{i-1})+x_{i-1})\|\,\mathrm{d}\lambda \\ &\leqslant \mathrm{T}_0^l F,\end{aligned}$$

where the last inequality follows from the fact that for any $\lambda\in[0,1)$ we have that

$$D(\lambda)=\{\lambda(x_1-x_0)+x_0,\ldots,\lambda(x_n-x_{n-1})+x_{n-1}\}$$

is a partition of the interval $[\lambda(x_1-x_0)+x_0,\lambda(x_n-x_{n-1})+x_{n-1}]\subset[0,l]$. Thus (summing over $i=1,\ldots,n-1$ in (4.13) and putting the estimates (4.13), (4.14) and (4.15) together) we get that

$$\mathrm{P}(F,D)\leqslant \mathrm{P}(F,D'')\leqslant 2\Big(1+\frac{\varepsilon}{1-\varepsilon}\Big)\mathrm{T}_0^l F,$$

where the first inequality follows from Lemma 4.7 with $A(u,v)=\big\|u/\|u\|-v/\|v\|\big\|$. We have obtained (send $\varepsilon\to 0$) that $\mathrm{P}_0^l F\leqslant 2\mathrm{T}_0^l F$ and thus $\mathrm{P}_0^l F<\infty$. Finally, if X is a Hilbert space, then we use (4.12) instead of (4.11), and we obtain $\mathrm{T}_0^l F=\mathrm{P}_0^l F$ and thus $\mathrm{T}_a^b\varphi=\mathrm{P}_a^b\varphi$. □

Using similar ideas as in the proof of Theorem 4.10 we can prove the following theorem, which generalizes Theorem 5.2.1 from [1] and Theorem III.1.10 from [6]. We shall take an alternative approach and use Proposition 4.9 instead.

Theorem 4.11. *Let H be a Hilbert space, and let $\varphi\colon[a,b]\to H$ be a curve. Then φ has finite angular turn if and only if φ has finite angular turn of tangents, and $\angle\mathrm{T}_a^b\varphi=\angle\mathrm{P}_a^b\varphi$.*

P r o o f. First, note that Theorem 4.10 implies that $\mathrm{T}_a^b\varphi=\mathrm{P}_a^b\varphi$. From this equality it follows that

$$\mathrm{T}_a^b\varphi-\sum_{\substack{x\in(a,b):\\ \tau_+(x)\neq\tau_-(x)}}\|\tau_+(x)-\tau_-(x)\|=\mathrm{P}_a^b\varphi-\sum_{\substack{x\in(a,b):\\ \tau_+(x)\neq\tau_-(x)}}\|\tau_+(x)-\tau_-(x)\|.$$

Thus an application of Proposition 4.9 (parts (ii) and (iii)) implies the equality $\angle\mathrm{T}_a^b\varphi=\angle\mathrm{P}_a^b\varphi$. □

The following definition comes from [10]:

Definition 5.1 ([10, Definition 1.1])**.** Let X, Y be normed linear spaces, let $A \subset X$ be an open convex set. A mapping $F\colon A \to Y$ is called *d.c.* (*on* A), if there exists a continuous function $f\colon A \to \mathbb{R}$ such that $y^* \circ F + f$ is a continuous convex function on A for each $y^* \in Y$, $\|y^*\| = 1$. If this is the case, we say that f is a *control function of* F.

We need to extend the definition from [10] to functions defined on closed intervals.

Definition 5.2. We say that a curve $\varphi\colon [a,b] \to X$ is *d.c.* (*on* $[a,b]$), provided there exists $\varepsilon > 0$ and a d.c. map $\psi\colon (a-\varepsilon, b+\varepsilon) \to X$ such that $\psi|_{[a,b]} = \varphi$.

The notion of turn is closely related to the notion of "convexity", which goes back to de la Vallée Poussin (1908; cf. [7]).

Definition 5.3. Let X be a normed linear space and $f\colon [a,b] \to X$ a mapping. For every partition $D = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a,b]$ we put

$$\mathrm{K}_a^b(f,D) = \sum_{i=1}^{n-1} \left\| \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right\|.$$

We define the *convexity of* f *on* $[a,b]$ as

$$\mathrm{K}_a^b f = \sup \mathrm{K}_a^b(f,D),$$

where the supremum is taken over all partitions D of $[a,b]$.

The following theorem of L. Veselý and L. Zajíček relates the notion of convexity with d.c. curves.

Theorem 5.4 ([10, Theorem 2.3])**.** *Let X be a Banach space and let $f\colon (a,b) \to X$ be a continuous mapping. Then the following conditions are equivalent.*
(i) *f is d.c. on (a,b).*
(ii) *$f'_+(x)$ exists for each $x \in (a,b)$ and f'_+ has locally finite variation on I.*
(iii) *$\mathrm{K}_c^d f < \infty$ for each interval $[c,d] \subset (a,b)$.*

Let us note the consequences of the previous theorem for our definition of delta-convexity.

Lemma 5.5. *Let $\varphi\colon [a,b] \to X$ be continuous. Then the following statements are equivalent:*

(a) *φ is d.c. (according to Definition 5.2);*

(b) $\mathrm{K}_a^b\varphi < \infty$;

(c) *$\varphi'_+(x)$ exists for all $x \in [a,b)$, $\varphi'_-(b)$ exists, and $\bigvee_a^b \varphi'_+ < \infty$, where we take $\varphi'_+(b) := \varphi'_-(b)$.*

P r o o f. (a) $\Longrightarrow$ (b). There exists $\varepsilon > 0$ and a d.c. $\psi\colon (a-\varepsilon, b+\varepsilon) \to X$ such that $\psi|_{[a,b]} = \varphi$. Thus by Theorem 5.4 we obtain that $\mathrm{K}_a^b\varphi < \infty$.

(b) $\Longrightarrow$ (c). Take $\varepsilon > 0$ and extend φ to ψ on $(a-\varepsilon, b+\varepsilon)$ as

$$\psi(x) := \begin{cases} \varphi(x) & \text{for } x \in [a,b], \\ \varphi(a) + (x-a)\varphi'_+(a) & \text{for } x \in (a-\varepsilon, a), \\ \varphi(b) + (x-b)\varphi'_-(b) & \text{for } x \in (b, b+\varepsilon). \end{cases} \tag{5.1}$$

Note that $\mathrm{K}_c^d\psi \leqslant \mathrm{K}_a^b\varphi < \infty$ for $a-\varepsilon < c < d < b+\varepsilon$ (this follows from Lemma 2.2 by L. Veselý [9]). Now apply Theorem 5.4.

(c) $\Longrightarrow$ (a). Take $\varepsilon > 0$ and extend φ to ψ on $(a-\varepsilon, b+\varepsilon)$ as in (5.1). Now note that $\bigvee_c^d \psi'_+ \leqslant \bigvee_a^b \psi'_+ < \infty$ (where we have $\psi'_+(b) = \varphi'_-(b)$) for $a-\varepsilon < c < d < b+\varepsilon$, and thus by Theorem 5.4 we obtain that ψ is d.c. on $(a-\varepsilon, b+\varepsilon)$. □

It is well known that if $f\colon (a,b) \to X$ is locally d.c. (in the sense of [10]) then it is d.c. (in the sense of [10]); see Theorem 1.20 in [10]. We shall use this fact without explicitly mentioning it.

Remark 5.6. Theorems 4.2 and 5.2 from [10] imply the following:

(i) If $\varphi\colon [a,b] \to [c,d]$ is d.c. and bilipschitz, then φ^{-1} is also d.c.

(ii) If X is a Banach space and $f\colon [a,b] \to [c,d]$, $g\colon [c,d] \to X$ are d.c., then $g \circ f$ is also d.c.

(iii) Let X, Y be Banach spaces, $U \subset X$ open. If $F\colon [a,b] \to X$ is d.c. with $F([a,b]) \subset U$ and $G\colon U \to Y$ is d.c., then $G \circ F$ is also d.c.

P r o o f. To prove (i), take $\varepsilon > 0$ and extend φ to $\psi\colon (a-\varepsilon, b+\varepsilon) \to \mathbb{R}$ as in (5.1). Then ψ is bilipschitz, d.c., and onto some open interval containing $[c,d]$. Apply Theorem 5.2 from [10] to ψ, and note that $\psi^{-1}|_{[c,d]} = \varphi^{-1}$.

The parts (ii) and (iii) follow easily by Theorem 4.2 from [10]. □

Using the proof of Theorem 4.10, we can prove

Proposition 5.7. *Let X be a Banach space and $\varphi\colon [a,b]\to X$ a curve.*

(i) *If φ has finite turn, then the arc-length parametrization F of φ satisfies $\mathrm{K}_0^l F = \mathrm{T}_a^b\varphi < \infty$, and F is d.c.*

(ii) *If φ is parametrized by the arc-length and $\mathrm{K}_0^l\varphi < \infty$, then φ has finite turn (of tangents) and $\mathrm{K}_0^l\varphi = \mathrm{T}_0^l\varphi < \infty$, where $l = s(\varphi)$.*

Proof. Note that $\mathrm{T}_a^b\varphi = \mathrm{T}_0^l F$. Thus the inequality $\mathrm{K}_0^l F \leqslant \mathrm{T}_a^b\varphi$ in part (i) follows from (4.14) and (4.15) in the proof of Theorem 4.10. The inequality $\mathrm{K}_0^l F \geqslant \mathrm{T}_a^b\varphi$ can be established in a similar way as the inequality $\mathrm{T}_a^b\varphi \leqslant \mathrm{P}_a^b\varphi$ in the first part of the proof of Theorem 4.10.

Part (ii) follows from the fact that $\varphi'_+(x) = \tau_+(x,\varphi)$ for all $x\in[0,l)$. □

The previous proposition has the following

Corollary 5.8. *Let $\varphi\colon [0,l]\to X$ be a curve parametrized by the arc-length. Then $\mathrm{K}_0^l\varphi = \mathrm{T}_0^l\varphi$.*

Delta-convexity is not equivalent (without any further assumptions) with finiteness of the turn as is shown by the following example.

Example 5.9. Finiteness of the turn (of tangents) of a curve $\varphi\colon [a,b]\to X$ does not necessarily imply that $\mathrm{K}_a^b\varphi < \infty$. To see this, take any $\varphi\colon [0,1]\to\mathbb{R}$ such that φ is continuous, strictly increasing, and φ is not d.c. Then φ has finite turn (of tangents), but φ is not d.c.

Take $f\colon [0,1]\to\mathbb{R}$ such that f is C^2 (meaning that there is a C^2 function $g\colon \mathbb{R}\to\mathbb{R}$ such that $g|_{[0,1]} = f$), there exist sequences $(a_i)_i, (\delta_i)_i$ such that $f|_{(a_i-\delta_i,a_i]}$ is increasing, $f|_{[a_i,a_i+\delta_i)}$ is decreasing for all $i\in\mathbb{N}$, and

$$(a_i-\delta_i, a_i+\delta_i)\cap(a_j-\delta_j, a_j+\delta_j) = \emptyset$$

for $i\neq j$. Then f is d.c. (by Proposition 1.11 from [10]), but f does not have a finite turn of tangents (as $\tau_-(a_i,f) = 1$, $\tau_+(a_i,f) = -1$, and thus $\mathrm{T}_0^1 f = \infty$). Thus delta-convexity does not in general imply finite turn.

Proposition 5.10. *Let us suppose that $\varphi\colon [0,l]\to X$ is absolutely continuous, $\|\varphi'(x)\| = 1$ for almost all x, and φ has finite turn (of tangents). Then φ is d.c.*

Proof. We shall prove first that φ is parametrized by the arc-length. By Lemma 2.1, we see that φ is Lipschitz. Thus $f(t) := s(\varphi,[0,t])$ is Lipschitz for $t\in[0,l]$. By Lemma 4.5, φ has the right tangent in the strong sense at each $x\in[0,l)$, and thus by Lemma 3.2 we obtain that $f'(x) = 1$ for almost all $x\in[0,l]$, as

$$f'_+(x) = \lim_{t\searrow 0}\frac{s(\varphi,[x,x+t])}{\|\varphi(x+t)-\varphi(x)\|}\,\frac{\|\varphi(x+t)-\varphi(x)\|}{t} = 1,$$

provided $\|\varphi'(x)\| = 1$, and similarly (by an argument for left tangents) $f'_-(x) = 1$ for all such x. Thus $f(t) = t$ for all $t \in [0, l]$, and thus φ is parametrized by the arc-length.

By Lemma 4.5 we have that $\varphi'_+(x) = \tau_+(x)$ for all x, and Lemma 5.5 implies the delta-convexity of φ. □

We can now generalize Theorem 5.4.2 from [1]:

Theorem 5.11. *Let $\varphi\colon X \to \mathbb{R}$ be a curve such that $\varphi\colon [a,b] \to X$ is d.c., such that $\|\varphi'_\pm(x)\| > 0$ for all $x \in (a,b)$, and such that*

$$\min(\|\varphi'_+(a)\|, \|\varphi'_-(b)\|) > 0.$$

Then the arc-length parametrization of φ is d.c. and thus φ has finite turn (of tangents).

P r o o f. By Proposition 1.10 from [10] we see that φ is Lipschitz. From Proposition 3.9 from [10] it follows that $\varphi'(x)$ exists except for a countable set of x's. By compactness and Note 3.2 from [10] there is an $\varepsilon > 0$ such that

$$\|\varphi'_+(x)\| > \varepsilon > 0 \quad \text{for all } x \in [a,b). \tag{5.2}$$

Now for $t \in [a,b]$ define $l(t) = \int_a^t \|\varphi'(x)\| \, \mathrm{d}x$. This function is obviously strictly monotone (by (5.2)). There is a $\delta > 0$ and $\widetilde{\varphi}\colon (a-\delta, b+\delta) \to X$ which is d.c. and such that $\widetilde{\varphi}|_{[a,b]} \equiv \varphi$. Thus by Proposition 1.10 from [10] there is $L > 0$ such that $\widetilde{\varphi}|_{[a-\frac{1}{2}\delta, b+\frac{1}{2}\delta]}$ is L-Lipschitz. This implies that $\|\varphi'_+(x)\| \leqslant L$ for $x \in [a,b)$ and thus it follows that l is Lipschitz. Note that l^{-1} is also Lipschitz by (5.2). By Note 3.2 from [10] we see that l'_+ exists everywhere in $[a,b)$, and that $l'_+(x) = \|\varphi'_+(x)\|$. Take a partition $D = \{x_i\}_{i=0}^n$ of $[a,b]$ and estimate

$$\begin{aligned}\sum_{i=0}^{n-1} \|l'_+(x_{i+1}) - l'_+(x_i)\| &= \sum_{i=0}^{n-1} \big| \|\varphi'_+(x_{i+1})\| - \|\varphi'_+(x_i)\| \big| \\ &\leqslant \sum_{i=0}^{n-1} \|\varphi'_+(x_{i+1}) - \varphi'_+(x_i)\| \leqslant \bigvee_a^b \varphi'_+ < \infty,\end{aligned}$$

where the last inequality follows from the fact that ψ is d.c. Thus by Lemma 5.5 we obtain that l is d.c. (as $\bigvee l'_+ < \infty$). Remark 5.6 (i) implies (l is bilipschitz) that l^{-1} is d.c. Define $F(s) = \varphi \circ l^{-1}(s)$. Then F is d.c. (as a composition of two delta-convex mappings) by Remark 5.6 (ii). Put $l = l(b)$. By Lemma 5.5 we obtain that $\bigvee_0^l F'_+(x) < \infty$. It is easy to see that $F'_+(x) = \tau_+(F, x)$ for all $x \in [0, l(b))$ as

$$\begin{aligned} F'_+(x) &= \lim_{t \searrow 0} \frac{F(x+t) - F(x)}{t} \\ &= \lim_{t \searrow 0} \frac{F(x+t) - F(x)}{\|F(x+t) - F(x)\|} \frac{\|F(x+t) - F(x)\|}{t} = \tau_+(F, x)\|F'_+(t)\|. \end{aligned}$$

On the other hand, $\|F'_+(t)\| = \|\psi'_+(l^{-1}(t))(l^{-1})'_+(t)\| = 1$. Thus we obtain $\mathrm{T}_0^l F = \bigvee_0^l F'_+(x) < \infty$.

Let us only remark that F is the arc-length parametrization of φ (see e.g. the first part of the proof of Proposition 5.10). □

The previous theorem has the following corollary, which generalizes[1] Theorem 5.4.3 from [1].

Corollary 5.12. *Let X, Y be Banach spaces. Let $\varphi\colon [a, b] \to X$ be a curve with finite turn, $U \subset X$ open, $\varphi([a, b]) \subset U$, and let $G\colon U \to Y$ be a locally d.c. mapping such that*

$$\|D_+G(\varphi(x), \tau_+(\varphi, x))\| > 0$$

for $x \in [a, b)$ and

$$\|D_-G(\varphi(x), \tau_-(\varphi, x))\| > 0$$

for all $x \in (a, b]$. Then $G \circ \varphi$ has finite turn.

Remark 5.13. By $D_+G(x, y)$ we denote the one-sided y-directional derivative of G at x, i.e. $D_+G(x, y) = f'_+(0)$, where $f(t) = G(x + ty)$.

P r o o f. Let F be the arc-length parametrization of φ. Note that by Remark 5.6 (iii) we obtain that $G \circ F$ is d.c. To apply Theorem 5.11, it is enough to prove that $\|(G \circ F)'_+(x)\| > 0$ for all $x \in [a, b)$ (and $\|(G \circ F)'_-(x)\| > 0$ for $x \in (a, b]$, which follows by an analogous argument). To see this, choose $x \in [a, b)$. Without any loss of generality, we may (and will) assume that $x = 0$, $F(x) = 0$, and $G(F(x)) = 0$. We obtain that $F(t) - tF'_+(0) = \omega(t)$ with $\lim_{t \searrow 0} \omega(t)/t = 0$, because $F'_+(0) = \tau_+(F, 0)$. Now estimate

$$\begin{aligned} \|(G \circ F)'_+(0)\| &= \lim_{t \searrow 0} t^{-1}\|G(F(t))\| \\ &\geqslant \lim_{t \searrow 0} t^{-1}\|G(tF'_+(0))\| - \lim_{t \searrow 0} t^{-1}\|G(tF'_+(0)) - G(tF'_+(0) + \omega(t))\| \\ &\geqslant \|D_+G(x, \tau_+(F, x))\| - L \lim_{t \searrow 0} t^{-1}\|\omega(t)\| \\ &= \|D_+G(x, \tau_+(F, x))\| > 0, \end{aligned}$$

[1] Note that Proposition 1.11 from [10] implies that $C^{1,1}$-mappings between Euclidean spaces are d.c.

where L is the local Lipschitz constant of G at $F(x)$ (see Proposition 1.1.2 from [10]). Now we can apply Theorem 5.11 to obtain that $G \circ F$ has finite turn, and Lemma 4.8 (iii) shows that $G \circ \varphi$ has finite turn. $\square$

The following corollary generalizes Theorem 16 from [6]. If X, Y are Banach spaces, we denote by $X \oplus_2 Y$ their L_2 sum, i.e. $X \times Y$ equipped with the norm $\|(x,y)\|_2 = \sqrt{\|x\|^2 + \|y\|^2}$.

Corollary 5.14. *Let $U \subset X$ be open, let $G\colon U \to Y$ be a locally d.c. mapping, and let $\varphi\colon [a,b] \to X$ be a curve with finite turn such that $\varphi([a,b]) \subset U$. Let $\widetilde{\varphi}\colon [a,b] \to X \oplus_2 Y$ be defined as $\widetilde{\varphi}(x) = (\varphi(x), G(\varphi(x)))$. Then $\widetilde{\varphi}$ has finite turn.*

Proof. Define a mapping $\Phi\colon X \to X \oplus_2 Y$ as $\Phi(x) = (x, G(x))$. Note that $\|D_{\pm}\Phi(x,y)\| > 0$ for all $y \in S_X$ and apply Corollary 5.12 to φ and Φ. $\square$

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