Przemysław Koprowski
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Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 4, 1311–1319

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GRADED QUATERNION SYMBOL EQUIVALENCE OF FUNCTION FIELDS

Przemysław Koprowski, Katowice

(Received November 19, 2005)

Abstract. We present criteria for a pair of maps to constitute a quaternion-symbol equivalence (or a Hilbert-symbol equivalence if we deal with global function fields) expressed in terms of vanishing of the Clifford invariant. In principle, we prove that a local condition of a quaternion-symbol equivalence can be transcribed from the Brauer group to the Brauer-Wall group.

Keywords: Brauer group, Brauer-Wall group, Witt equivalence

MSC 2000: 11E81, 11E10, 14H05, 14P05, 16K50

1. Introduction

Two fields K and L are called *Witt equivalent* if their Witt rings of symmetric bilinear forms are isomorphic. The pursuit for criteria for Witt equivalence of global fields gave rise to the notion of a *reciprocity equivalence* (see [7], [8], [9]), later renamed to a *Hilbert-symbol equivalence* (cf. [10], [11], [1]). This in turn became a research subject by itself (see e.g. [1], [11]), especially once it was generalized to higher dimensional forms. Meanwhile a similar approach, utilizing a notion of the so called quaternion-symbol equivalence, was effectively used for investigating Witt equivalence of algebraic function fields of one variable (see [4], [3], [5]).

In this paper we present new criteria for a pair of maps to constitute a Hilbert-symbol equivalence of global function fields (see Proposition 3.6) and formally real algebraic function fields over a fixed real closed field of constants (cf. Proposition 4.4). To this end we introduce (see Definition 2.1) the notion of a *graded quaternion-symbol equivalence*, which is a variation of Hilbert-symbol and quaternion-symbol equivalences.

Recall (see [7], [4]) that a central condition of Hilbert-symbol and quaternion-symbol equivalences was expressed in terms of splitting of local quaternion algebras. Here we investigate the splitting in a narrower sense, by moving the condition from the Brauer group to the Brauer-Wall group. It is slightly surprising that this condition turns out to be equivalent to the previous one, since earlier we effectively controlled the splitting of 2-fold Pfister forms, while now we control only binary hyperbolic forms. Anyway, in Proposition 3.6 we show that for two global function fields the pair of maps (t, T) is a Hilbert-symbol equivalence if and only if it is a graded quaternion-symbol equivalence if and only if it induces isomorphisms of subgroups of local Brauer-Wall groups generated by graded quaternion algebras (it trivially induces isomorphisms of subgroups of local Brauer groups as well, since in this case any such subgroup consists of only two elements). For real function fields we give the analogue of this result in Proposition 4.4. Namely, we show there that for two formally real algebraic function fields over a common real closed field of constants the pair of maps (t, T) is a quaternion-symbol equivalence if and only if it is a graded quaternion-symbol equivalence if and only if it induces isomorphisms of subgroups of local Brauer groups generated by quaternion algebras if and only if it induces isomorphisms of subgroups of local Brauer-Wall groups generated by graded quaternion algebras.

2. Notation and terminology

Throughout this paper letters K, L always denote either global function fields of characteristics different from 2 (Section 3) or formally real function fields over a fixed real closed field (Section 4). We denote by $\Omega(K)$, $\Omega(L)$ the sets of all the points of K, L that are trivial on their fields of constants. In the case that K, L are real we further denote by γ^K, γ^L the subsets of all points having real residue fields. Following [2] we call such points *real*.

For any point $\mathfrak{p} \in \Omega(K)$ we denote by $K_\mathfrak{p}$ the completion of K at $\mathfrak{p}$. Further, $\theta_\mathfrak{p}\colon \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$ is the epimorphism of square class groups induced by the canonical injection $K \hookrightarrow K_\mathfrak{p}$. Similarly $\Theta_\mathfrak{p}\colon WK \to WK_\mathfrak{p}$ is the epimorphism of the Witt rings of K and $K_\mathfrak{p}$.

Further, we denote by $\mathrm{Br}(K)$ and $\mathrm{BW}(K)$ the Brauer and Brauer-Wall groups of K, respectively. Following [6] we denote by $\left(\frac{f,g}{K}\right)$ a quaternion algebra as well as its class in $\mathrm{Br}(K)$, while by $\left\langle\frac{f,g}{K}\right\rangle$ we denote the same quaternion algebra but this time augmented with a $\mathbb{Z}_2$-gradation. We also use the same symbol to denote its class in $\mathrm{BW}(K)$. Finally, $\mathrm{GQ}(K)$ is the subgroup of $\mathrm{BW}(K)$ generated by the classes of all graded quaternion algebras.

In order to simplify the wording of the rest of the paper let us introduce the following definition analogous to the definition of Hilbert-symbol equivalence and quaternion-symbol equivalence.

Definition 2.1. Let K, L be two function fields and let $A \subseteq \Omega(K)$, $B \subseteq \Omega(L)$ be fixed sets of points of K and L. A pair of maps (t, T) in which $t\colon\ \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$ is an isomorphism of square-class groups and $T\colon\ A \to B$ is a bijection is called a *graded quaternion-symbol equivalence* of the fields K, L with respect to the sets A and B if the following two conditions are satisfied:

- t preserves minus one, i.e. $t(-1) = -1$;
- the pair (t, T) preserves the vanishing of local Clifford invariants in the sense that
$$\left\langle \frac{f, g}{K_{\mathfrak{p}}} \right\rangle = 1 \in \mathrm{BW}(K_{\mathfrak{p}}) \iff \left\langle \frac{tf, tg}{L_{T\mathfrak{p}}} \right\rangle = 1 \in \mathrm{BW}(L_{T\mathfrak{p}})$$
for all square classes $f, g \in \dot{K}/\dot{K}^2$ and all points $\mathfrak{p} \in A$.

In general one cannot expect that the graded quaternion-symbol equivalence would coincide with the quaternion-symbol equivalence. The easiest example—although over a power series field rather than a function field—is as follows. Take k to be the field $\mathbb{R}(X)$ of rational functions over the reals and define $K = k((T))$ to be the power series field over k. Then there is only one point of K trivial on k, denote it by $\mathfrak{p}$. The square class group $\dot{K}/\dot{K}^2$ of K can be expressed as

$$\dot{K}/\dot{K}^2 \cong \dot{k}/\dot{k}^2 \times \mathbb{Z}_2.$$

We treat the square class group $\dot{k}/\dot{k}^2$ as an $\mathbb{F}_2$-vector space. The set $\{-1, X, X^2+1\}$ is linearly independent so it can be extended to the basis $\mathcal{B}$ of $\dot{k}/\dot{k}^2$. Define the automorphism t to exchange the basis vectors X and X^2+1 and fix all other elements of $\mathcal{B}$. Extend f to the whole $\dot{K}/\dot{K}^2$. Then the pair (t, T), with T being the only permutation of $\Omega(K) = \{\mathfrak{p}\}$, is a graded quaternion-symbol equivalence of K with itself but it is not a quaternion-symbol equivalence, since $\left(\frac{X, X}{K}\right) \neq 1$ but $\left(\frac{tX, tX}{K}\right) = 1$.

In this paper we are interested exclusively in global and real function fields, hence we always assume that if K, L are global function fields, then $A = \Omega(K)$, $B = \Omega(L)$. Otherwise, if K, L are formally real algebraic function fields over a real closed field then $A = \gamma^K$, $B = \gamma^L$. Therefore, in what follows we omit the phrase 'with respect to the sets A and B'.

Respecting the precedence of Hilbert-symbol equivalence studies, we begin by analysing the global case. In this whole section K, L are two global function fields. For reader's convenience we summarize here some known facts concerning the Hilbert-symbol equivalence of global function fields that we use in the sequel.

(HS1) There exists a Hilbert-symbol equivalence of K and L iff K, L are Witt equivalent iff they have equal levels (i.e. $s(K) = s(L)$)—see [8, Theorem 1.1] and [9, Theorem 1.3].

(HS2) The pair (t, T) is a Hilbert-symbol equivalence iff for every $\mathfrak{p} \in \Omega(K)$ the isomorphism t induces local Harrison isomorphisms $\dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^2 \to \dot{L}_{T\mathfrak{p}}/\dot{L}_{T\mathfrak{p}}^2$—see [8, Proposition 1.4]

(HS3) If the pair (t, T) is a Hilbert-symbol equivalence, then for every point $\mathfrak{p}$ the isomorphism t factors through $\dot{K}_{\mathfrak{p}}^2$, hence it induces isomorphisms of local groups of square classes $\dot{K}_{\mathfrak{p}}/\dot{K}_{\mathfrak{p}}^2 \to \dot{L}_{T\mathfrak{p}}/\dot{L}_{T\mathfrak{p}}^2$. Moreover, it induces isomorphisms of 2-torsion subgroups of local Brauer groups $\mathrm{Br}_2(K_{\mathfrak{p}}) \to \mathrm{Br}_2(L_{T\mathfrak{p}})$ sending $\left(\frac{f,g}{K_{\mathfrak{p}}}\right)$ to $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$.

(HS4) A Hilbert-symbol equivalence is tame (i.e. $\operatorname{ord}_{\mathfrak{p}} f \equiv \operatorname{ord}_{T\mathfrak{p}} tf \pmod 2$) at every point $\mathfrak{p}$ for which -1 is not a square—see [1, Lemma 1.18].

Our aim is to prove that a pair (t, T) is a Hilbert-symbol equivalence if and only if it is a graded quaternion-symbol equivalence. But let us first note an immediate "existentional" consequence of the above facts.

Observation 3.1. *A graded quaternion-symbol equivalence (t, T) preserves local levels, i.e. $s(K_{\mathfrak{p}}) = s(L_{T\mathfrak{p}})$ for every $\mathfrak{p}$.*

Indeed, a level of a (non-dyadic) $\mathfrak{p}$-adic local field is either 1 or 2. Suppose that $s(K_{\mathfrak{p}}) = 2$, this means that -1 is not a square in $K_{\mathfrak{p}}$. Take a quadratic form $\langle -1, -1\rangle$, the local Clifford invariant $\left\langle\frac{-1,-1}{K_{\mathfrak{p}}}\right\rangle \neq 1$ does not vanish. Thus, $\left\langle\frac{t(-1),t(-1)}{L_{T\mathfrak{p}}}\right\rangle = \left\langle\frac{-1,-1}{L_{T\mathfrak{p}}}\right\rangle \neq 1$. Consequently, -1 is not a square in $L_{T\mathfrak{p}}$. Therefore, $s(L_{T\mathfrak{p}}) = 2$. Now, (HS1) implies

Corollary 3.2. *If there exists a graded quaternion-symbol equivalence, then there exists a Hilbert-symbol equivalence.*

We now proceed toward our main result. First we show that any Hilbert-symbol equivalence is graded.

Lemma 3.3. *If (t,T) is a Hilbert-symbol equivalence of K and L, then it is a graded quaternion-symbol equivalence.*

Proof. Let (t,T) be a Hilbert-symbol equivalence. It is well known (see e.g. the first part of the proof of [8, Proposition 1.3]) that it preserves -1. Hence we need to show only the second condition. Take two square classes $f,g\in\dot K/\dot K^2$ and a point $\mathfrak{p}\in\Omega(K)$. Assume that the local Clifford invariant at $\mathfrak{p}$ vanishes: $\left\langle\frac{f,g}{K_\mathfrak{p}}\right\rangle=1\in BW(K_\mathfrak{p})$. This is possible only if the form $\Theta_\mathfrak{p}\langle f,g\rangle$ is hyperbolic, which means that its class in the Witt ring of $K_\mathfrak{p}$ is zero. Now (HS2) implies that t induces a local Harrison isomorphism, hence also a local Witt isomorphism $WK_\mathfrak{p}\to WL_{T\mathfrak{p}}$. Consequently $\Theta_{T\mathfrak{p}}\langle tf,tg\rangle=0\in WL_{T\mathfrak{p}}$ and so $\left\langle\frac{tf,tg}{L_{T\mathfrak{p}}}\right\rangle=1\in\mathrm{BW}(L_{T\mathfrak{p}})$. $\square$

We may now consider the converse implication. First we need the following simple observation. Notice that it holds over arbitrary K,L—not only global ones. We will use it again in the next section.

Observation 3.4. *A graded quaternion-symbol equivalence preserves local squares in the sense that*

$$f\in K_\mathfrak{p}^2\iff tf\in L_{T\mathfrak{p}}^2.$$

Indeed, by the very definition we see

$$f\in K_\mathfrak{p}^2\iff\left\langle\frac{-1,f}{K_\mathfrak{p}}\right\rangle=1\iff\left\langle\frac{t(-1),tf}{L_{T\mathfrak{p}}}\right\rangle=\left\langle\frac{-1,tf}{L_{T\mathfrak{p}}}\right\rangle=1\iff tf\in L_{T\mathfrak{p}}^2.$$

Lemma 3.5. *If (t,T) is a graded quaternion-symbol equivalence of K and L, then it is a Hilbert-symbol equivalence.*

Proof. Assume that (t,T) is a graded quaternion-symbol equivalence. Take two square classes $f,g\in\dot K/\dot K^2$ and a point $\mathfrak{p}\in\Omega(K)$. Consider two cases, first suppose that -1 is not a square in $K_\mathfrak{p}$. Observation 3.1 implies that -1 is not a square in $L_{T\mathfrak{p}}$, either. By the previous observation, t maps local squares onto local squares. Since it preserves -1, so it maps local minus squares onto local minus squares. The group of square classes of $K_\mathfrak{p}$—and $L_{T\mathfrak{p}}$ alike—equals $\{\pm1,\pm p\}$ with p being a fixed uniformizer (see [6, Theorem VI.2.2]). In particular, it is isomorphic to $(\mathbb{Z}_2)^2$. Thus (t,T) preserves the parity of a valuation:

$$\bigwedge_{h\in\dot K/\dot K^2}\operatorname{ord}_\mathfrak{p}h\equiv\operatorname{ord}_\mathfrak{p}th\pmod 2.$$

Now [6, Theorem VI.2.2] asserts that $\left(\frac{-1,\pm p}{K_\mathfrak{p}}\right)=\left(\frac{p,p}{K_\mathfrak{p}}\right)=\left(\frac{-p,-p}{K_\mathfrak{p}}\right)$ is the only non-split quaternion algebra over $K_\mathfrak{p}$, hence $\left(\frac{-1,\pm tp}{L_{T\mathfrak{p}}}\right)=\left(\frac{tp,tp}{L_{T\mathfrak{p}}}\right)=\left(\frac{-tp,-tp}{L_{T\mathfrak{p}}}\right)$ is the only non-split quaternion algebra over $L_{T\mathfrak{p}}$.

Conversely, assume that -1 is a square in $K_\mathfrak{p}$, hence by 3.1 also in $L_{T\mathfrak{p}}$. Now the Hilbert-symbol $\left(\frac{f,g}{K_\mathfrak{p}}\right)$ vanishes if and only if at least one of the following conditions is met:

$$f \in K_\mathfrak{p}^2 \quad \text{or} \quad g \in K_\mathfrak{p}^2 \quad \text{or} \quad fg \in K_\mathfrak{p}^2.$$

In every case, the previous observation implies that $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right) = 1$ as well. □

We are now ready to present our first main result.

Proposition 3.6. *Let K, L be two global function fields of characteristics $\neq 2$, let $t\colon \dot K/\dot K^2 \to \dot L/\dot L^2$ be an isomorphism of their square class groups and $T\colon \Omega(K) \to \Omega(L)$ a bijection of their sets of points. The following three conditions are equivalent:*

(1) *The pair (t,T) preserves Hilbert-symbols in the sense that $\left(\frac{f,g}{K_\mathfrak{p}}\right) = 1$ iff $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right) = 1$ for any $f, g \in \dot K/\dot K^2$ and $\mathfrak{p} \in \Omega(K)$—i.e. (t,T) is a Hilbert-symbol equivalence.*

(2) *The pair (t,T) preserves -1 and local Clifford invariants in the sense that $\left\langle\frac{f,g}{K_\mathfrak{p}}\right\rangle = 1$ iff $\left\langle\frac{tf,tg}{L_{T\mathfrak{p}}}\right\rangle = 1$ for any $f, g \in \dot K/\dot K^2$ and $\mathfrak{p} \in \Omega(K)$—i.e. (t,T) is a graded quaternion-symbol equivalence.*

(3) *The pair (t,T) preserves -1 and induces isomorphisms of subgroups of local Brauer-Wall groups generated by graded quaternion algebras given by $\left\langle\frac{f,g}{K_\mathfrak{p}}\right\rangle \mapsto \left\langle\frac{tf,tg}{L_{T\mathfrak{p}}}\right\rangle$.*

Proof. The equivalence (1) ⇔ (2) follows from Lemmas 3.3 and 3.5. The implication (3) ⇒ (2) is trivial. Hence all we need to do is to show (1) ⇒ (3). Assume that the pair (t,T) is a Hilbert-symbol equivalence. Write the elements of Brauer-Wall group in a 'triple notation' (see [6, Ch. V, sec. 3]), so that $\left\langle\frac{f,g}{K_\mathfrak{p}}\right\rangle$ becomes $\left(\left(\frac{f,g}{K_\mathfrak{p}}\right), 0, -fg\right)$. Recall that $GQ(K_\mathfrak{p})$ is the subgroup of $\mathrm{BW}(K_\mathfrak{p})$ spanned by graded quaternion algebras. It is straightforward to check that it has order 8. Depending on whether -1 is a square or not, this group is isomorphic to either $(\mathbb{Z}_2)^3$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$. In the later case, elements of order four are precisely $\left(\left(\frac{-1,\pm p}{K_\mathfrak{p}}\right), 0, \pm p\right)$. Denote by $X = \left(\frac{u,p}{K_\mathfrak{p}}\right)$ the unique non-split quaternion algebra. The group $\mathrm{GQ}(K_\mathfrak{p})$ consists of elements

$$\begin{gathered} 1 = (1,0,1),\ A := (1,0,u),\ B := (1,0,p),\ C := (1,0,up), \\ D := (X,0,p),\ E := (X,0,up),\ F := (X,0,1),\ G := (X,0,u). \end{gathered}$$

The relations between binary forms and their Clifford invariants are summarized in Table 1. Let $\Upsilon\colon \mathrm{GQ}(K_\mathfrak{p}) \to \mathrm{GQ}(L_{T\mathfrak{p}})$ be the mapping induced by t, namely $\Upsilon\left(\left(\frac{f,g}{K_\mathfrak{p}}\right), 0, -fg\right) = \left(\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right), 0, -tftg\right)$. Now, [6, Theorem V.3.9] provides us with the rule of multiplication in GQ in triple notation. In principle,

$$(3.7)\quad \left(\left(\frac{a,b}{K_\mathfrak{p}}\right), 0, -ab\right) \cdot \left(\left(\frac{f,g}{K_\mathfrak{p}}\right), 0, -fg\right) = \left(\left(\frac{a,b}{K_\mathfrak{p}}\right)\left(\frac{f,g}{K_\mathfrak{p}}\right)\left(-ab, -fgK_\mathfrak{p}\right), 0, abfg\right).$$

By (HS3) we obtain that t induces isomorphisms $\dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^2 \to \dot{L}_{T\mathfrak{p}}/\dot{L}_{T\mathfrak{p}}^2$ and $\mathrm{Br}_2(K_\mathfrak{p}) \to \mathrm{Br}_2(L_{T\mathfrak{p}})$. Hence both the multiplications employed in (3.7) are carried over from $K_\mathfrak{p}$ to $L_{T\mathfrak{p}}$. Thus, Υ is an isomorphism. $\square$

$\left\langle\frac{a,b}{K_\mathfrak{p}}\right\rangle$	1	-1	p	$-p$
1	A	1	C	B
-1	1	A	D	E
p	C	D	G	1
$-p$	B	E	1	G

$\left\langle\frac{a,b}{K_\mathfrak{p}}\right\rangle$	1	u	p	up
1	1	A	B	C
u	A	1	E	D
p	B	E	1	G
up	C	D	G	1

Table 1. The group $\mathrm{GQ}(K_\mathfrak{p})$ when -1 is not a square (left), and when it is a square (right)

We may rephrase the above result as follows.

Corollary 3.8. *Let $t\colon \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$ be an isomorphism of square class groups such that $t(-1) = -1$ and let $T\colon \Omega(K) \to \Omega(L)$ be a bijection. The following conditions are equivalent:*

- *The mapping $\left(\frac{f,g}{K_\mathfrak{p}}\right) \mapsto \left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$ is an isomorphism $\mathrm{Br}_2(K_\mathfrak{p}) \to \mathrm{Br}_2(L_{T\mathfrak{p}})$ for every point $\mathfrak{p} \in \Omega(K)$.*
- *The mapping $\left\langle\frac{f,g}{K_\mathfrak{p}}\right\rangle \mapsto \left\langle\frac{tf,tg}{L_{T\mathfrak{p}}}\right\rangle$ is an isomorphism $\mathrm{GQ}(K_\mathfrak{p}) \to \mathrm{GQ}(L_{T\mathfrak{p}})$ of subgroups of Brauer-Wall groups for every point $\mathfrak{p} \in \Omega(K)$.*

4. Real function fields

Now we turn our attention to the case of real function fields. Thus, in this section $\Bbbk$ is a fixed real closed field and K, L are two formally real algebraic function fields over $\Bbbk$. Recall that here a graded quaternion-symbol equivalence is implicitly meant to be taken with respect to the pair (γ^K, γ^L). We need the following basic lemma.

Lemma 4.1. *Let $t\colon \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$ be an isomorphism of the square class groups of K and L such that $t(-1) = -1$ and let $T\colon \gamma^K \to \gamma^L$ be a bijection. Assume that the pair (t, T) preserves local squares in the sense that*

$$f \in K_\mathfrak{p}^2 \iff tf \in L_{T\mathfrak{p}}^2$$

for every square class $f \in \dot{K}/\dot{K}^2$ and every point $\mathfrak{p} \in \gamma^K$. Then:

1. *The pair (t, T) is "tame" in the sense that it preserves parity of valuation: $\mathrm{ord}_\mathfrak{p} f \equiv \mathrm{ord}_{T\mathfrak{p}} tf \pmod 2$ for every $f \in \dot{K}/\dot{K}^2$ and $\mathfrak{p} \in \gamma^K$.*
2. *For every point $\mathfrak{p} \in \gamma^K$ the mapping $\left(\frac{f,g}{K_\mathfrak{p}}\right) \mapsto \left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$ is an isomorphism from $\mathrm{Br}_2(K_\mathfrak{p})$ onto $\mathrm{Br}_2(L_{T\mathfrak{p}})$.*

P r o o f. Observe that the group of square classes of a local field with real closed residue field (e.g. of $K_{\mathfrak{p}}$ and $L_{T\mathfrak{p}}$) consists of four elements, namely $\pm 1, \pm p$ with p being a fixed uniformizer. Hence it is isomorphic to $(\mathbb{Z}_2)^2$. Since the local squares are mapped onto local squares and -1 is mapped onto -1, so there is no degree of freedom left except to map uniformizers onto uniformizers. This proves (1).

To show (2) observe that $\mathrm{Br}_2(K_{\mathfrak{p}})$ (as well as $\mathrm{Br}_2(L_{T\mathfrak{p}})$) is again isomorphic to $(\mathbb{Z}_2)^2$. Precisely, the class of $\left(\frac{1,f}{K_{\mathfrak{p}}}\right) \cong \left(\frac{f,-f}{K_{\mathfrak{p}}}\right) \cong M_{2,2}(K_{\mathfrak{p}})$ is the unit element and three non-unit elements are given by the classes of $\left(\frac{-1,f}{K_{\mathfrak{p}}}\right) \cong \left(\frac{f,f}{K_{\mathfrak{p}}}\right)$ for any $f \notin K_{\mathfrak{p}}^2$ (see [6, Corollary III.2.6 and Proposition VI.1.9]). The part already proved implies that the mapping $\left(\frac{f,g}{K_{\mathfrak{p}}}\right) \mapsto \left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$ is well defined and preserves the unit element. Hence it is an isomorphism, as any permutation of $(\mathbb{Z}_2)^2$ constant on the unit is an isomorphism. □

Now, any quaternion-symbol equivalence (preserving -1) factors through local squares, hence the above lemma proves the following simple result (which can be found also in [5, Theorem 3.1]):

Corollary 4.2. *If (t,T) is a quaternion-symbol equivalence of K, L with respect to (γ^K, γ^L) and such that $t(-1) = -1$, then it is tame in the sense that $\operatorname{ord}_{\mathfrak{p}} f \equiv \operatorname{ord}_{T\mathfrak{p}} tf \pmod 2$ for any $f \in \dot{K}/\dot{K}^2$ and $\mathfrak{p} \in \gamma^K$.*

This also implies that any quaternion-symbol equivalence is graded.

Corollary 4.3. *If (t,T) is a quaternion-symbol equivalence of K and L with respect to (γ^K, γ^L) and such that $t(-1) = -1$, then it is a graded quaternion-symbol equivalence.*

Indeed, take $f, g \in \dot{K}/\dot{K}^2$ and a point $\mathfrak{p} \in \gamma^K$. Assume that $\left\langle\frac{f,g}{K_{\mathfrak{p}}}\right\rangle = 1$; this means that $\theta_{\mathfrak{p}} f = -\theta_{\mathfrak{p}} g$, so $-fg \in K_{\mathfrak{p}}^2$. Now t factors through local squares, hence $-tftg \in L_{T\mathfrak{p}}^2$. Consequently $\left\langle\frac{tf,tg}{L_{T\mathfrak{p}}}\right\rangle = 1$.

Now from 3.4 we know that any graded quaternion-symbol equivalence preserves local squares, hence the second assertion of Lemma 4.1 implies that it is a quaternion-symbol equivalence. All in all, we have just proved

Proposition 4.4. *Let K, L be two formally real algebraic function fields over a fixed real closed field $\Bbbk$. Let $t\colon \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$ be an isomorphism of their square class groups such that $t(-1) = -1$ and let $T\colon \gamma^K \to \gamma^L$ be a bijection of their set of points. The following three conditions are equivalent:*

(1) *The pair (t,T) preserves quaternion-symbols in the sense that $\left(\frac{f,g}{K_{\mathfrak{p}}}\right) = 1$ iff $\left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right) = 1$ for any $f, g \in \dot{K}/\dot{K}^2$ and $\mathfrak{p} \in \gamma^K$—i.e. (t,T) is a quaternion-symbol equivalence.*

(2) *The pair (t, T) preserves local Clifford invariants in the sense that $\left\langle\frac{f,g}{K_\mathfrak{p}}\right\rangle = 1$ iff $\left\langle\frac{tf,tg}{L_{T\mathfrak{p}}}\right\rangle = 1$ for any $f, g \in \dot{K}/\dot{K}^2$ and $\mathfrak{p} \in \gamma^K$—i.e. (t, T) is a graded quaternion-symbol equivalence.*

(3) *The pair (t, T) induces isomorphisms of subgroups of local Brauer-Wall groups generated by graded quaternion algebras given by $\left\langle\frac{f,g}{K_\mathfrak{p}}\right\rangle \mapsto \left\langle\frac{tf,tg}{L_{T\mathfrak{p}}}\right\rangle$.*

The proof of implication (1) ⇒ (2) is fully analogous to the one for global function fields. Again, like in the global case, we may rephrase the above proposition.

Corollary 4.5. *Let $t\colon \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$ be an isomorphism of square class groups such that $t(-1) = -1$ and let $T\colon \gamma^K \to \gamma^L$ be a bijection. The following conditions are equivalent:*

- *The mapping $\left(\frac{f,g}{K_\mathfrak{p}}\right) \mapsto \left(\frac{tf,tg}{L_{T\mathfrak{p}}}\right)$ is an isomorphism $\mathrm{Br}_2(K_\mathfrak{p}) \to \mathrm{Br}_2(L_{T\mathfrak{p}})$ for every point $\mathfrak{p} \in \gamma^K$.*
- *The mapping $\left\langle\frac{f,g}{K_\mathfrak{p}}\right\rangle \mapsto \left\langle\frac{tf,tg}{L_{T\mathfrak{p}}}\right\rangle$ is an isomorphism $\mathrm{GQ}(K_\mathfrak{p}) \to \mathrm{GQ}(L_{T\mathfrak{p}})$ of subgroups of Brauer-Wall groups for every point $\mathfrak{p} \in \gamma^K$.*

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Author's address: Przemysław Koprowski, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, PL-40-007 Katowice, Poland, e-mail: pkoprowski@member.ams.org.