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BANASCHEWSKI'S THEOREM FOR GENERALIZED MV-ALGEBRAS

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Abstract. A generalized MV-algebra $\mathscr{A}$ is called representable if it is a subdirect product of linearly ordered generalized MV-algebras. Let S be the system of all congruence relations ϱ on $\mathscr{A}$ such that the quotient algebra $\mathscr{A}/\varrho$ is representable. In the present paper we prove that the system S has a least element.

Keywords: generalized MV-algebra, representability, congruence relation, unital lattice ordered group

MSC 2000: 06D35, 06F15

1. Introduction

The concept of the generalized MV-algebra was introduced independently by Georgescu and Iorgulescu [6], [7] and by Rachůnek [10] (in [6] and [7], the term "pseudo MV-algebra" was applied).

For the terminology and notation cf. Section 2 below.

Dvurečenskij [4] proved that each generalized MV-algebra is an interval of a unital lattice ordered group. This enables one to search for analogies between the theory of lattice ordered groups and the theory of generalized MV-algebras.

A lattice ordered group is *representable* if it is a subdirect product of linearly ordered groups. The representability of a generalized MV-algebra is defined analogously; this notion was investigated in [7]; cf. also Dvurečenskij and Pulmannová [5], Section 3.4.

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The motivation and the aim of the present paper are as follows.

For a lattice ordered group G let $W(G)$ be the union of all normal prime filters of the positive cone G^+ of G. Put

$$K_0(G) = \{x \in G\colon\ |x| \notin W(G)\}.$$

Banaschewski [1] proved that $K_0(G)$ is an ℓ-ideal of G and that $G/K_0(G)$ is the largest quotient lattice ordered group of G which is representable.

In other words, $K_0(G)$ is the least ℓ-ideal of G having the property that $G/K_0(G)$ is representable.

To each ℓ-ideal of G there corresponds a congruence relation on G, and conversely. Let S_0 be the system of all congruence relations ϱ on G such that G/ϱ is representable. Banaschewski's result yields that the system S_0 possesses a least element.

In [1], Banaschewski remarked that it may be of interest to have a characterization of $W(G)$ and $K_0(G)$ internally in terms of elements of G and that it remains an open question whether $W(G)$ is the set of all elements $a > 0$ of G such that, for some $x_1, \ldots, x_n \in G$, the relation

$$(x_1 + a - x_1) \wedge \ldots \wedge (x_n + a - x_n) = 0$$

is valid.

The author [8] showed that the answer to this question is 'No' and presented the desired characterizations of $W(G)$ and $K_0(G)$ in terms of elements of G.

In the present paper we prove

(∗) Let $\mathscr{A}$ be a generalized MV-algebra and let S be the system of all congruence relations ϱ on $\mathscr{A}$ such that the quotient algebra $\mathscr{A}/\varrho$ is representable. Then the system S has a least element.

In the proof we substantially apply some results of the author's article [9]; these were formulated for MV-algebras, but remain valid for generalized MV-algebras as well.

Further, using the results of [8], we give a constructive description of the least element of S in terms of elements from G, where G is a lattice ordered group with a strong unit u such that $\mathscr{A}$ is the interval $[0, u]$ of G.

A generalized MV-algebra is defined to be an algebraic structure $\mathscr{A} = (A; \oplus, ^-, ^\sim, 0, 1)$ of type (2,1,1,0,0) such that the axioms (A1)–(A8) from [6] are satisfied.

For $x, y \in A$ we put $x \leqslant y$ if $x^- \oplus y = 1$. Then $(A; \leqslant)$ is a distributive lattice with the least element 0 and with the greatest element 1; we put $(A; \leqslant) = \ell(\mathscr{A})$.

The group operation in a lattice ordered group will be denoted by the symbol $+$, though the commutativity of this operation is not assumed (cf. also Birkhoff [2] and Conrad [3]). G^+ denotes the positive cone of a lattice ordered group G. An element $u \in G^+$ is a *strong unit* of G if for each $g \in G$ there exists a positive integer n with $g \leqslant nu$.

Let u be a fixed strong unit of G; then (G, u) is said to be a *unital lattice ordered group.*

For a unital lattice ordered group (G, u) we set $A = [0, u]$ (the interval in G with the end-points 0 and u). Further, for $x, y \in A$ we put

$$x \oplus y = (x + y) \wedge u,$$
$$x^- = u - x, \quad x^\sim = -x + u, \quad 1 = u.$$

Then $(A; \oplus, ^-, ^\sim, 0, 1)$ is a generalized MV-algebra; it will be denoted by $\Gamma(G, u)$.

According to Dvurečenskij [4], for each generalized MV-algebra $\mathscr{A}$ there exists a unital lattice ordered group (G, u) such that $\mathscr{A} = \Gamma(G, u)$. Also, the partial order defined in $\mathscr{A}$ coincides with the partial order on A induced from G.

Let $(\mathscr{A}_i)_{i \in I}$ be an indexed system of generalized MV-algebras. The *direct product* $\prod\limits_{i \in I} \mathscr{A}_i$ is defined in the usual way; its elements are denoted by $(a_i)_{i \in I}$, where $a_i \in A_i$.

A generalized MV-algebra $\mathscr{A}$ is a *subdirect product* of the indexed system $(\mathscr{A}_i)_{i \in I}$ if there exists a one-to-one homomorphism $\varphi\colon \mathscr{A} \to \prod\limits_{i \in I} \mathscr{A}_i$ such that, whenever $i_0 \in I$ and $z \in A_{i_0}$, then there exists $a \in A$ with $\varphi(a) = (a_i)_{i \in I}$, where $a_{i_0} = z$. We also say that φ is a subdirect product decomposition of $\mathscr{A}$.

Let $\operatorname{Con} \mathscr{A}$ be the system of all congruence relations on $\mathscr{A}$. For $\varrho \in \operatorname{Con} \mathscr{A}$, the symbol $\mathscr{A}/\varrho$ has the obvious meaning. If $x \in A$, we put $x(\varrho) = \{y \in A\colon y \varrho x\}$. Let $\varrho_1, \varrho_2 \in \operatorname{Con} \mathscr{A}$; we set $\varrho_1 \leqslant \varrho_2$ if for each $x \in A$, $x(\varrho_1) \subseteq x(\varrho_2)$. Under the relation $\leqslant$, $\operatorname{Con} \mathscr{A}$ is a complete lattice.

Analogous notions are applied for lattice ordered groups.

For a lattice ordered group G let $\mathscr{J}(G)$ be the system of all ℓ-ideals of G. This system is partially ordered by the set-theoretical inclusion. Further, let $\operatorname{Con} G$ be the system of all congruence relations on G. It is well-known that for each $\varrho \in \operatorname{Con} G$, $0(\varrho)$ is an ℓ-ideal of G and the mapping $\operatorname{Con} G \to \mathscr{J}(G)$ defined by $\varrho \to 0(\varrho)$ is an isomorphism of $\operatorname{Con} G$ onto $\mathscr{J}(G)$.

Again, let $\mathscr{A}$ be a generalized MV-algebra. A nonempty subset X of A is a *normal ideal* of $\mathscr{A}$ if it satisfies the following conditions:

(i) X is closed with respect to the operation $\oplus$;
(ii) if $x \in X$ and $x_1 \in A$, $x_1 \leqslant x$, then $x_1 \in X$;
(iii) $a \oplus X = X \oplus a$ for each $a \in A$.

This notion was investigated in [6] and [10]; cf. also [5]. Let $\mathscr{N}\mathscr{J}(\mathscr{A})$ be the system of all normal ideals of $\mathscr{A}$; this system is partially ordered by the set theoretical inclusion. The relation between $\mathscr{N}\mathscr{J}(\mathscr{A})$ and $\operatorname{Con}\mathscr{A}$ is similar to that between $\mathscr{J}(G)$ and $\operatorname{Con} G$, namely: for each $\varrho \in \operatorname{Con}\mathscr{A}$, $0(\varrho)$ belongs to $\mathscr{N}\mathscr{J}(\mathscr{A})$ and the mapping $\operatorname{Con}\mathscr{A} \to \mathscr{N}\mathscr{J}(\mathscr{A})$ defined by $\varrho \to 0(\varrho)$ is an isomorphism of $\operatorname{Con}\mathscr{A}$ onto $\mathscr{N}\mathscr{J}(\mathscr{A})$.

3. Subdirect product decompositions

In the present section we assume that $\mathscr{A}$ is a generalized MV-algebra and (G, u) is a unital lattice ordered group with $\mathscr{A} = \Gamma(G, u)$. Recall that if the operation $\oplus$ in $\mathscr{A}$ is commutative, then $\mathscr{A}$ is an *MV-algebra*.

Proposition 3.1 (Cf. [5]). *For each $Y \in \mathscr{J}(G)$ we put $\psi(Y) = Y \cap A$. Then ψ is an isomorphism of $\mathscr{J}(G)$ onto $\mathscr{N}\mathscr{J}(\mathscr{A})$.*

Let $\varrho^1 \in \operatorname{Con} G$. Then $0(\varrho^1) \in \mathscr{J}(G)$. Put $0(\varrho^1) = Y$; hence $\psi(Y) \in \mathscr{N}\mathscr{J}(\mathscr{A})$. There exists a uniquely determined $\varrho \in \operatorname{Con}\mathscr{A}$ with $0(\varrho) = \psi(Y)$. In view of Section 2 and of 3.1 we have

Lemma 3.2. *The mapping $\chi\colon \operatorname{Con} G \to \operatorname{Con}\mathscr{A}$ defined by $\chi(\varrho^1) = \varrho$ for each $\varrho^1 \in \operatorname{Con} G$ is an isomorphism of $\operatorname{Con} G$ onto $\operatorname{Con}\mathscr{A}$.*

Subdirect product decompositions of MV-algebras were investigated by the author [9].

A straightforward verification shows that the results of Section 1 and Section 2 of [9] remain valid if

(a) the MV-algebra $\mathscr{A}$ is replaced by a generalized MV-algebra;
(b) the symbol $\neg$ is replaced by $^-$;
(c) in the proof of 2.3, the argument concerning the operation $^\sim$ is added (which is analogous to the argument used for the operation $\neg$).

In this sense we will understand the quotations concerning the definitions and results of [9].

In view of the well-known Birkhoff's result on the relation between subdirect product decompositions and congruence relation (cf., e.g., [2], Chapter VI), when considering a subdirect product decompositions of any algebra X we can suppose without loss of generality that the corresponding subdirect factors have the form X/ϱ_i ($i \in I$), where ϱ_i are congruence relations on X such that $\bigwedge_{i\in I} \varrho_i = \mathrm{Id}_X$ (we denote by Id_X the identity on X). Moreover, for each $x \in X$ and each $i \in I$, the component of x in X/ϱ_i is equal to $x(\varrho_i)$. In this situation we say that the subdirect product decomposition under consideration is determined by the system $(\varrho_i)_{i\in I}$.

Let $\varrho^1 \in \mathrm{Con}\, G$. The element $u(\varrho^1)$ is a strong unit of the lattice ordered group G/ϱ^1, hence we can construct the generalized MV-algebra

$$\mathscr{A}_{\varrho^1} = \Gamma(G/\varrho^1, u(\varrho^1)).$$

We define a binary relation ϱ on A as follows: for any $a_1, a_2 \in A$ we put $a_1 \varrho a_2$ iff $a_1 \varrho^1 a_2$. It is easy to verify that ϱ belongs to $\mathrm{Con}\, \mathscr{A}$ and that $\varrho = \chi(\varrho^1)$, where χ is as in 3.2. For each $g(\varrho^1) \in \mathscr{A}_{\varrho^1}$ we put

$$\psi_{\varrho^1}(g(\varrho^1)) = g(\varrho^1) \cap A.$$

In view of the above remark concerning the validity of results of [9] for generalized MV-algebras we have

Proposition 3.3 (Cf. [9], Proposition 2.4). *Let $\varrho^1 \in \mathrm{Con}\, G$ and $\varrho = \chi(\varrho^1)$. Then ψ_{ϱ^1} is an isomorphism of $\mathscr{A}_{\varrho^1}$ onto $\mathscr{A}/\varrho$.*

Theorem 3.4 (Cf. [9], Theorem 2.5). *Let (G, u) and $\mathscr{A}$ be as above. If σ is a subdirect product decomposition of G which is determined by a system $\{\varrho^i\}_{i\in I} \subseteq \mathrm{Con}\, G$, then*

(i) *there exists a subdirect product decomposition $\sigma_1 = \psi^*(\sigma)$ of $\mathscr{A}$ which is determined by the system $\{\chi(\varrho^i)\}_{i\in I}$;*

(ii) *for each $i \in I$, the quotient algebra $\mathscr{A}/\chi(\varrho^i)$ is isomorphic to $\Gamma(G/\varrho^i, u(\varrho^i))$.*

Lemma 3.5. *Let σ_0 be a subdirect product decomposition of $\mathscr{A}$ which is determined by a system $\{\varrho_0^i\}_{i\in I} \subseteq \mathrm{Con}\, \mathscr{A}$. Let χ be as in 3.2. Put $\varrho^i = \chi^{-1}(\varrho_0^i)$ for each $i \in I$. Then the system $\{\varrho^i\}_{i\in I}$ determines a subdirect product decomposition of G.*

P r o o f. From the fact that $\{\varrho_0^i\}_{i\in I}$ determines a subdirect product decomposition of $\mathscr{A}$ we obtain $\bigwedge_{i\in I} \varrho_0^i = \mathrm{Id}\, A$. In view of 3.2, χ^{-1} is an isomorphism of $\mathrm{Con}\, G$ onto $\mathrm{Con}\, \mathscr{A}$, hence $\bigwedge_{i\in I} \varrho^i = \mathrm{Id}\, G$. Then in view of Birkhoff's theorem, $\{\varrho^i\}_{i\in I}$ determines a subdirect product decomposition of G. □

Lemma 3.6. *Let $\varrho^1 \in \operatorname{Con} G$, $\varrho = \chi(\varrho^1)$. Then g/ϱ^1 is linearly ordered if and only if $\mathscr{A}(\varrho)$ is linearly ordered.*

P r o o f. It is well-known that if $\mathscr{A} = \Gamma(G, u)$, then $\mathscr{A}$ is linearly ordered if and only if G is linearly ordered. Now it suffices to apply Proposition 3.3. $\square$

Lemma 3.7. *G is representable if and only if $\mathscr{A}$ is representable.*

P r o o f. Assume that G is representable. Then there exists a system $\{\varrho^i\}_{i\in I} \subseteq \operatorname{Con} G$ such that (i) all G/ϱ^i are linearly ordered, and (ii) this system determines a subdirect product decomposition of G. For each $i \in I$ let $\varrho_0^i = \chi(\varrho^i)$. Then in view of 3.4, the system $\{\varrho_0^i\}_{i\in I}$ determines a subdirect product decomposition of $\mathscr{A}$. Moreover, according to 3.3, all generalized MV-algebras $\mathscr{A}/\varrho_0^i$ are linearly ordered. Hence $\mathscr{A}$ is representable.

Conversely, suppose that $\mathscr{A}$ is representable; thus there exists $\{\varrho_0^i\}_{i\in I} \subseteq \operatorname{Con} \mathscr{A}$ determining a subdirect product decomposition of $\mathscr{A}$ such that all $\mathscr{A}/\varrho_0^i$ are linearly ordered. Let ϱ^i be as in 3.5. In view of 3.5, the system $\{\varrho_0^i\}_{i\in I}$ determines a subdirect product decomposition of G; according to 3.3, all G/ϱ^i are linearly ordered. $\square$

Lemma 3.8. *Let $\varrho^1 \in \operatorname{Con} G$; put $\varrho = \chi(\varrho^1)$. Then G/ϱ^1 is representable if and only if $\mathscr{A}/\varrho$ is representable.*

P r o o f. This is a consequence of 3.3 and 3.7. $\square$

Let S and S_0 be as in Section 1.

Lemma 3.9. *Assume that $\bar{\varrho}$ is the least element of S_0. Then $\chi(\bar{\varrho})$ is the least element of S.*

P r o o f. According to 3.8 and 3.2 we conclude that χ is a bijection of S_0 onto S; moreover, if $\varrho_1, \varrho_2 \in S_0$, then

$$\varrho_1 \leqslant \varrho_2 \Leftrightarrow \chi(\varrho_0) \leqslant \chi(\varrho_2).$$

Let $\varrho \in S$. There exists $\varrho^1 \in S_0$ with $\chi(\varrho^1) = \varrho$. Then $\varrho^1 \geqslant \bar{\varrho}$, whence $\chi(\varrho^1) \geqslant \chi(\bar{\varrho})$. Thus $\varrho \geqslant \chi(\bar{\varrho})$. $\square$

According to [1], the set S_0 has a least element. Then in view of 3.9, the assertion $(*)$ from Section 1 is valid.

Using the results of [8], we can give a constructive description of the least element of S (in terms of elements of G). We proceed as follows.

By induction we define subsets K_n and $\overline{K}_n$ of G by putting $K_1 = \overline{K}_1 = \{0\}$; if $1 < n \in \mathbb{N}$ then let K_n be the set of all $0 \leqslant a \in G$ such that $(x_1+a-x_1)\wedge(x_2+a-x_2) \in$

$\overline{K}_{n-1}$ for some $x_1, x_2 \in G$. Further, let $\overline{K}_n$ be the set of all $b \in G$ which can be expressed in the form $b = a_1 + \ldots + a_m$ for some $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in K_n$. We denote

$$\bigcup_{n=1}^{\infty} \overline{K}_n = \overline{K}, \quad \overline{K}_0 = A \cap \overline{K}.$$

Further, we denote by $\bar{\bar{\varrho}}$ the least element of S. In view of the results of Section 3 of [8] we easily obtain the relation

$$0(\bar{\bar{\varrho}}) = \overline{K}_0;$$

hence for each $z \in A$ we have

$$z(\bar{\bar{\varrho}}) = z \oplus \overline{K}_0.$$

The question of characterizing $\bar{\bar{\varrho}}$ internally (in terms of elements of A and operations in $\mathscr{A}$) remains open.

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