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THE KATO-TYPE SPECTRUM AND LOCAL SPECTRAL THEORY

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Abstract. Let $T \in \mathscr{L}(X)$ be a bounded operator on a complex Banach space X. If V is an open subset of the complex plane such that $\lambda - T$ is of Kato-type for each $\lambda \in V$, then the induced mapping $f(z) \mapsto (z-T)f(z)$ has closed range in the Fréchet space of analytic X-valued functions on V. Since semi-Fredholm operators are of Kato-type, this generalizes a result of Eschmeier on Fredholm operators and leads to a sharper estimate of Nagy's spectral residuum of T. Our proof is elementary; in particular, we avoid the sheaf model of Eschmeier and Putinar and the theory of coherent analytic sheaves.

Keywords: decomposable operator, semi-Fredholm operator, semi-regular operator, Kato decomposition, Bishop's property (β), property (δ)

MSC 2000: 47A11, 47A53

1. Introduction and motivation

For a complex Banach space X, let $\mathscr{L}(X)$ denote the space of all bounded linear operators on X. For $T \in \mathscr{L}(X)$, let, as usual, $\sigma(T)$, $\sigma_e(T)$, and $\sigma_{ap}(T)$ denote, respectively, the spectrum, essential spectrum, and approximate point spectrum of T, and let $\sigma_{su}(T) := \{\lambda \in \mathbb{C}\colon\ \lambda - T \text{ is not surjective}\}$. The complements of these sets in $\mathbb{C}$ are denoted, respectively, by $\varrho(T)$, $\varrho_e(T)$, $\varrho_{ap}(T)$, and $\varrho_{su}(T)$.

The present article centers around certain localized versions of some basic concepts of local spectral theory, with emphasis on decomposability in the sense of Foiaş and on Bishop's property (β); see [3], [5], [9], [11], [12], and [14]. An operator $T \in \mathscr{L}(X)$ is said to be decomposable on an open subset U of $\mathbb{C}$ provided that, for every finite open cover $\{V_1, \ldots, V_n\}$ of $\mathbb{C}$ with $\mathbb{C} \setminus U \subseteq V_1$, there exist T-invariant closed linear subspaces $X_1, \ldots, X_n$ of T for which

$$X = X_1 + \ldots + X_n \quad \text{and} \quad \sigma(T|_{X_k}) \subseteq V_k \quad \text{for } k = 1, \ldots, n.$$

Classical decomposability occurs when $U = \mathbb{C}$. Moreover, T is said to possess Bishop's property (β) on the open set U if, for every open subset V of U and every sequence of analytic functions $f_n\colon V \to X$ for which $(\lambda - T)f_n(\lambda) \to 0$ as $n \to \infty$ locally uniformly on V, it follows that $f_n(\lambda) \to 0$ as $n \to \infty$, again locally uniformly on V.

Albrecht and Eschmeier proved the remarkable fact that an operator has property (β) on U precisely when it is the restriction to a closed invariant subspace of an operator that is decomposable on U, [3, Theorem 10]. Moreover, by Theorems 8 and 21 of [3], T is decomposable on U if and only if T and its adjoint T^* share property (β) on U. Evidently, there exists a largest open set on which T has property (β); its complement, denoted by $\mathscr{S}_\beta(T)$, is a closed, possibly empty, subset of $\sigma(T)$. It follows that $\mathscr{S}_r(T) := \mathscr{S}_\beta(T) \cup \mathscr{S}_\beta(T^*)$ is the complement of the largest open set on which T is decomposable. The existence of the spectral residuum $\mathscr{S}_r(T)$ was first discovered by Nagy, [12].

These results make it of interest to identify large open sets on which property (β) holds. For this it is convenient to reformulate this condition as follows. For an open subset V of $\mathbb{C}$, denote by $H(V,X)$ the space of all analytic X-valued functions on V. Then $H(V,X)$ is a Fréchet space with generating semi-norms given by $p_K(f) := \sup\{\|f(\lambda)\| : \lambda \in K\}$, where K runs through the compact subsets of V. Every operator $T \in \mathscr{L}(X)$ induces a continuous linear mapping T_V on $H(V,X)$, defined by $T_V f(\lambda) := (\lambda - T)f(\lambda)$ for all $f \in H(V,X)$ and $\lambda \in V$. It is not difficult to see that T has property (β) on U precisely when, for each open subset V of U, the operator T_V is injective and has closed range in $H(V,X)$; see [9, Prop. 1.2.6].

The injectivity issue is addressed by the classical single-valued extension property (SVEP), [1] and [11]. An operator $T \in \mathscr{L}(X)$ is said to have SVEP at a point $\lambda \in \mathbb{C}$ provided that, for every open disc V centered at λ, the mapping T_V is injective on $H(V,X)$. If $U \subseteq \mathbb{C}$ is open, then T is said to have SVEP on U if T has SVEP at every $\lambda \in U$, equivalently, if T_V is injective for each open set $V \subseteq U$. The set $\mathfrak{S}(T)$ of all $\lambda \in \mathbb{C}$ at which T fails to have SVEP is an open subset of the point spectrum $\sigma_p(T)$. Note that, if T_V has closed range for every open set $V \subseteq \mathbb{C}$, then, by [9, Prop. 3.3.5], T has SVEP and thus property (β) on $\mathbb{C}$.

Clearly, T has property (β) on $\varrho_{ap}(T)$, since it is well known and easily seen that, for each compact subset K of $\varrho_{ap}(T)$, there exists a constant $c > 0$ with the property that $\|(\lambda - T)x\| \geqslant c\,\|x\|$ for all $x \in X$ and $\lambda \in K$; see also Lemma 3.1.10 of [9] for a more general result. Moreover, if V is an open subset of $\varrho_{su}(T)$, then T_V is surjective as a consequence of a result due to Allan and Leiterer; see Theorem 3.2.1 of [9] for an elementary proof. On the other hand, using sheaf-theoretic tools, Eschmeier established in [5] that T_V has closed range for every open subset V of the Fredholm region $\varrho_e(T)$ and then derived interesting new proofs of results on Fredholm operators originally due to Herrero, [6], Putinar, [13], and the first two authors, [10].

In this article, we extend Eschmeier's result to a more general setting that includes, for instance, the case of open subsets of the semi-Fredholm region. Our approach avoids the explicit use of sheaf theory. In fact, our main strategy is to combine the above-mentioned results on $\varrho_{ap}(T)$ and $\varrho_{su}(T)$ with some basic facts on semi-regular operators and operators of Kato-type. In the main result of the next section, it is established that T_V has closed range for every open subset V of the Kato-type resolvent set $\varrho_{kt}(T)$, while Section 3 is devoted to a more sophisticated weak-$*$ version of this result for the adjoint. As a consequence, we obtain duality formulas for certain spectral subspaces of T, and we are able to identify the components of $\varrho_{kt}(T)$ on which T enjoys property (β) or is even decomposable.

2. Semi-regular and Kato-type operators

Given an operator $T \in \mathscr{L}(X)$, we denote by $\ker(T)$ and $\operatorname{ran}(T)$ the kernel and range of T, respectively, and define the hyper-kernel and hyper-range of T to be the sets $\mathscr{N}^\infty(T) := \bigcup_{n=1}^{\infty} \ker(T^n)$ and $T^\infty X := \bigcap_{n=1}^{\infty} \operatorname{ran}(T^n)$. An operator $T \in \mathscr{L}(X)$ is said to be semi-regular provided that $\operatorname{ran}(T)$ is closed and $\mathscr{N}^\infty(T) \subseteq T^\infty X$. This containment is equivalent to the condition that $\mathscr{N}^\infty(T) \subseteq \operatorname{ran}(T)$ or $\ker(T) \subseteq T^\infty X$, [1, Cor. 1.6], and the latter condition implies that $\operatorname{ran}(T^n)$ is closed for all n, [9, Prop. 3.1.5]. The equivalence of these conditions also implies that T is semi-regular if and only if the adjoint $T^* \in \mathscr{L}(X^*)$ is semi-regular, [9, Prop. 3.1.6]. Semi-regular operators were introduced by Kato, [7], and accordingly we define the Kato resolvent set of T to be the set of complex numbers

$$\varrho_K(T) := \{\lambda \in \mathbb{C} \colon \ \lambda - T \text{ is semi-regular}\}.$$

$\varrho_K(T)$ is an open subset of the complex plane and evidently contains both $\varrho_{ap}(T)$ and $\varrho_{su}(T)$. Moreover, if G is a component of $\varrho_K(T)$ and $\mu, \lambda \in G$, then $(\mu - T)^\infty X = (\lambda - T)^\infty X$, and $G \subseteq \varrho_{su}(T|_{(\lambda - T)^\infty X})$; see Propositions 3.1.5, 3.1.9, and 3.1.11 of [9].

A generalized Kato decomposition of an operator $T \in \mathscr{L}(X)$ is a pair of closed, T-invariant subspaces (M, N) such that $X = M \oplus N$, $T|_M$ is semi-regular and $T|_N$ is quasinilpotent. If $T|_N$ is nilpotent, then T is said to be of Kato-type. As pointed out by the referee, such operators were introduced and studied by Labrousse, [8], in the setting of Hilbert spaces under the name *quasi-Fredholm* operators. However, since the name quasi-Fredholm is also used for a different class of Banach space operators, we prefer to avoid this terminology here. A thorough discussion of operators of Kato-type may be found in the recent monograph by Aiena, [1]. Define the Kato-type resolvent set of T to be the set

$$\varrho_{kt}(T) := \{\lambda \in \mathbb{C} \colon \ \lambda - T \text{ is of Kato-type}\}.$$

Clearly $\varrho_K(T) \subseteq \varrho_{kt}(T)$. Moreover, by Theorems 1.43, 1.44 and Corollary 1.45 of [1], $\varrho_{kt}(T)$ is open, $\varrho_{kt}(T) \subseteq \varrho_{kt}(T^*)$, and $\varrho_{kt}(T) \setminus \varrho_K(T)$ is a discrete subset of $\varrho_{kt}(T)$, in the sense that $F \setminus \varrho_K(T)$ is finite whenever F is a compact subset of $\varrho_{kt}(T)$. We denote the complements of $\varrho_K(T)$ and $\varrho_{kt}(T)$ by $\sigma_K(T)$ and $\sigma_{kt}(T)$, respectively. By [2, Theorem 2.4], $\sigma_{kt}(T) = \emptyset$ if and only if T is algebraic.

We begin with a version of the three-space lemma for property (β), [9, Lemma 2.2.1].

Proposition 2.1. *Consider Banach space operators $R \in \mathscr{L}(X)$, $S \in \mathscr{L}(Y)$, and $T \in \mathscr{L}(Z)$ for which there exist $J \in \mathscr{L}(X,Y)$ and $Q \in \mathscr{L}(Y,Z)$ such that the diagram*

$$\begin{CD}
0 @>>> X @>J>> Y @>Q>> Z @>>> 0\\
@. @VRVV @VSVV @VTVV @.\\
0 @>>> X @>J>> Y @>Q>> Z @>>> 0
\end{CD}$$

is commutative with exact rows. If V is an open subset of $\mathbb{C}$ such that R_V has closed range in $H(V,X)$ and for which T_V is injective and with closed range in $H(V,Z)$, then S_V has closed range in $H(V,Y)$.

P r o o f. If $j\colon H(V,X) \to H(V,Y)$ and $q\colon H(V,Y) \to H(V,Z)$ are the composition mappings $jf := J \circ f$ and $qf := Q \circ f$, then $jR_V = S_V j$ and $qS_V = T_V q$, and Gleason's theorem, [9, Prop. 2.1.5], implies that the diagram

$$\begin{CD}
0 @>>> H(V,X) @>j>> H(V,Y) @>q>> H(V,Z) @>>> 0\\
@. @VR_VVV @VS_VVV @VT_VVV @.\\
0 @>>> H(V,X) @>j>> H(V,Y) @>q>> H(V,Z) @>>> 0
\end{CD}$$

is commutative with exact rows. Suppose now that $S_V f_n \to 0$ in $H(V,Y)$ as $n \to \infty$. Then $T_V q f_n = q S_V f_n \to 0$ in $H(V,Z)$, and, since T_V is injective with closed range, $qf_n \to 0$ in $H(V,Z)$. Thus, by exactness, there exists a sequence $(g_n)_n \subset H(V,X)$ so that $f_n - jg_n \to 0$ in $H(V,Y)$. Therefore $S_V j g_n = S_V(jg_n - f_n) + S_V f_n \to 0$ and so $jR_V g_n = S_V j g_n \to 0$ in $H(V,Y)$. The fact that j is injective and with closed range implies that $R_V g_n \to 0$, and, because R_V has closed range, it follows that there exists $(h_n)_n \subset \ker(R_V)$ so that $g_n - h_n \to 0$. Then $(jh_n)_n \subset \ker(S_V)$ and $f_n - jh_n = f_n - jg_n + j(g_n - h_n) \to 0$ in $H(V,Y)$. Thus $S_V f_n \to 0$ if and only if $[f_n] \to 0$ in $H(V,Y)/\ker(S_V)$; equivalently, $\operatorname{ran}(S_V)$ is closed. $\square$

As a corollary, we obtain the following.

Proposition 2.2. *Suppose that $T \in \mathscr{L}(X)$ and $V \subseteq \mathbb{C}$ is open. If there exists a closed, T-invariant subspace M of X for which $V \subseteq \varrho_{su}(T|_M)$ and a discrete subset E of V such that $V \setminus E \subseteq \varrho_{ap}([T]_{X/M})$, then* $\operatorname{ran}(T_V)$ *is closed in $H(V, X)$.*

Proof. Since $U := V \setminus E \subseteq \varrho_{ap}([T]_{X/M}) \subseteq \varrho_K([T]_{X/M})$, for each compact $K \subseteq U$, there exists a $c > 0$ so that $\|[x]\| \leqslant c \inf_{\lambda \in K} \|(\lambda - [T])[x]\|$ for every $[x] \in X/M$, by [9, Lemma 3.1.10]. It follows that $[T]_U$ is injective and with closed range in $H(U, X/M)$. Now suppose that $(f_n)_n \subset H(V, X/M)$ is such that $[T]_V f_n \to 0$, and let K be a compact subset of V. Since E is discrete, we may choose a contour γ in U surrounding K, and since $[T]_U$ is injective with closed range, it follows that $f_n \to 0$ uniformly on γ and therefore on K as well, by Cauchy's formula. The surjectivity of $(T|_M)_V$ is a consequence of the Allan-Leiterer theorem [9, Theorem 3.2.1], and since the canonical sequence

$$0 \longrightarrow M \longrightarrow X \longrightarrow X/M \longrightarrow 0$$

is exact, the statement now follows from Proposition 2.1. □

An "all or nothing" relation between SVEP and components of $\varrho_K(T)$ was observed in [11, Theorem 13]. The following proposition provides a simple proof of this and a slight extension of Theorem 19, [11], as well. The second statement is, in fact, a special case of Theorem 2.5, our main result of this section.

Proposition 2.3. *Let $T \in \mathscr{L}(X)$, and let V be an open subset of $\varrho_K(T)$. Then:*

(1) $\varrho_K(T) \setminus \big(\sigma_p(T) \cup \sigma_p(T^*)\big) = \varrho(T)$.

(2) *T_V has closed range in $H(V, X)$.*

(3) *If in addition V is connected, then* $\ker(\lambda - T) = \{f(\lambda)\colon\ f \in \ker(T_V)\}$ *for every $\lambda \in V$.*

(4) *If V is connected, then $V \subseteq \mathfrak{S}(T) \Leftrightarrow V \subseteq \sigma_p(T) \Leftrightarrow V \cap \sigma_p(T) \neq \emptyset \Leftrightarrow V \cap \mathfrak{S}(T) \neq \emptyset$.*

(5) *T has property (β) on $V \Leftrightarrow T$ has SVEP on $V \Leftrightarrow V \cap \sigma_p(T) = \emptyset \Leftrightarrow V \subseteq \varrho_{ap}(T)$.*

Proof. (1) If $\lambda \in \varrho_K(T) \setminus \big(\sigma_p(T) \cup \sigma_p(T^*)\big)$, then $\lambda - T$ has closed range, is injective and has dense range. Thus $\lambda \in \varrho(T)$, and the converse is clear.

For the proofs of (2)–(5), let V be an open subset of $\varrho_K(T)$ with components $\{V_n\}_n$. Then T_V is injective if and only if T_{V_n} is injective for every n, and T_V has closed range if and only if each T_{V_n} has closed range in $H(V_n, X)$. Thus, without loss of generality, we may assume that V is connected.

(2) Let $\lambda \in V$ and set $M = (\lambda - T)^\infty X$. Then M is independent of λ, and $V \subseteq \varrho_{su}(T|_M)$, by Propositions 3.1.5 and 3.1.11 of [9]. Moreover, $V \subseteq \varrho_{ap}([T]_{X/M})$.

Indeed, let $\lambda \in V$, and suppose that $(x_n)_n \subset X$ is such that $(\lambda - [T])[x_n] \to 0$. Then there exists $(y_n)_n \subset M$ such that $(\lambda - T)x_n - y_n \to 0$ in X. Since $(\lambda - T)M = M$, we may write $y_n = (\lambda - T)w_n$ for some $w_n \in M$, and thus $(\lambda - T)(x_n - w_n) \to 0$. But $\operatorname{ran}(\lambda - T)$ is closed in X and therefore $x_n - w_n \to 0$; i.e., $[x_n] \to 0$ in X/M. It follows from Proposition 2.2 that $\operatorname{ran}(T_V)$ is closed.

(3) Clearly, $\{f(\lambda)\colon\ f \in \ker(T_V)\} \subseteq \ker(\lambda - T)$ for every $\lambda \in V$. On the other hand, for fixed $\lambda \in V$, $\ker(\lambda - T) \subseteq M$, and so $x \in \ker(\lambda - T)$ implies, by the Allan-Leiterer theorem, that $x = T_V g$ for some $g \in H(V, M)$. If $f \in H(V, M)$ is defined by $f(\mu) = (\lambda - T)g(\mu)$, then $f \in \ker(T_V)$ and $f(\lambda) = x$. Thus (3) holds.

(4) It is also clear that $V \subseteq \mathfrak{S}(T)$ implies that $V \subseteq \sigma_p(T)$, which in turn implies that $V \cap \sigma_p(T) \neq \emptyset$. If $V \cap \sigma_p(T) \neq \emptyset$, then it follows from (3) that $\ker(T_V) \neq \{0\}$. If $\lambda \in V \setminus \sigma_p(T)$, then there is a neighborhood U of λ contained in $\varrho_{ap}(T) \cap V$. But in this case, every $f \in \ker(T_V)$ must vanish identically on U and therefore on V as well. Thus $V \cap \sigma_p(T) \neq \emptyset$ implies that $V \subseteq \sigma_p(T)$ and, by (3) again, that $V \subseteq \mathfrak{S}(T)$. Thus (4) is established.

(5) is an immediate consequence of (2) and (4). □

Now, suppose that V is an open, connected subset of $\varrho_{kt}(T)$. For each $\lambda \in V$, let $(M_\lambda(T), N_\lambda(T))$ be a generalized Kato decomposition for $\lambda - T$ such that $\lambda - T|_{N_\lambda(T)}$ is nilpotent. If $\lambda \in V \cap \varrho_K(T)$, then the only possible decomposition is $M_\lambda(T) = X$ and $N_\lambda(T) = \{0\}$. Set $M(T, V) := \bigcap_{\lambda \in V} M_\lambda(T)$, $M_\lambda^\infty(T) := (\lambda - T)^\infty M_\lambda(T)$, and $M^\infty(T, V) := \bigcap_{\lambda \in V} M_\lambda^\infty(T)$. When the operator T and domain V are understood and there is no possible ambiguity, we write $M_\lambda = M_\lambda(T)$, $M = M(T, V)$, etc.

Of central importance for us will be the space M^∞. Clearly, this space is closed and invariant under T. Moreover, the following argument will show that $\ker(T_V) \subseteq H(V, M^\infty)$. If $\mu \in V$, then, by Proposition 2.1.6 of [9], the space $H(V, X)$ decomposes naturally as

$$H(V, X) = H(V, M_\mu) \oplus H(V, N_\mu),$$

so that $T_V = (T|_{M_\mu})_V \oplus (T|_{N_\mu})_V$ and $\ker(T_V) = \ker\big((T|_{M_\mu})_V\big) \oplus \ker\big((T|_{N_\mu})_V\big)$. Since $\mu - T|_{N_\mu}$ is nilpotent, $T|_{N_\mu}$ has SVEP, and therefore $\ker(T_V) = \ker\big((T|_{M_\mu})_V\big)$. Also, since the Kato resolvent set is open, there exists an open disc U for which $\mu \in U \subseteq V$ and $\lambda - T|_{M_\mu}$ is semi-regular for all $\lambda \in U$. By [9, Prop. 3.1.11], for every $\lambda \in U$, we obtain that $\ker(\lambda - T|_{M_\mu}) \subseteq (\lambda - T)^\infty M_\mu = M_\mu^\infty$. Thus, given an arbitrary $f \in \ker(T_V)$, we conclude that $f(\lambda) \in M_\mu^\infty$ for all $\lambda \in U$. Since V is connected, it then follows from Theorem A.3.2 of [9] that $f(\lambda) \in M_\mu^\infty$ actually for all $\lambda \in V$. This shows that $f(\lambda) \in M^\infty$ and therefore that $\ker(T_V) \subseteq H(V, M^\infty)$.

Lemma 2.4. *Suppose that $\mu, \lambda \in V$, an open, connected subset of $\varrho_{kt}(T)$.*

(1) *If $\mu \neq \lambda$, then $\mathscr{N}^\infty(\mu - T) \subseteq M_\lambda$.*
(2) *If $\mu \neq \lambda$, then $M_\lambda = (M_\lambda \cap M_\mu) \oplus N_\mu$.*
(3) *$(\mu - T)(M_\lambda \cap M_\mu)$ is closed.*
(4) *$\mathscr{N}^\infty(\mu - T) \cap M_\mu \subseteq (\mu - T)(M_\lambda \cap M_\mu)$.*
(5) *$\{\lambda, \mu\} \cup \big(V \cap \varrho_K(T)\big) \subseteq \varrho_K(T|_{M_\lambda \cap M_\mu})$.*
(6) *$M_\lambda^\infty \cap M_\mu = (\lambda - T)^\infty(M_\lambda \cap M_\mu) = (\mu - T)^\infty(M_\lambda \cap M_\mu) = M_\mu^\infty \cap M_\lambda$.*
(7) *$M_\lambda^\infty \cap M = M^\infty$.*
(8) *$V \subseteq \varrho_{su}(T|_{M^\infty})$.*
(9) *$V \cap \varrho_K(T) \subseteq \varrho_{ap}([T]_{X/M^\infty})$.*

P r o o f. If $\lambda, \mu \in V$, then, for every n, $(\mu - T)^n = (\mu - T|_{M_\lambda})^n \oplus (\mu - T|_{N_\lambda})^n$ on $M_\lambda \oplus N_\lambda$. If $\mu \neq \lambda$, then $\mu - T|_{N_\lambda}$ is invertible since $\lambda - T|_{N_\lambda}$ is nilpotent. Thus, if $x_1 \in M_\lambda$ and $x_2 \in N_\lambda$ are such that $x_1 + x_2 \in \ker(\mu - T)^n$, then $x_2 = 0$, and so (1) holds. Moreover, since $N_\mu \subseteq \ker(\mu - T)^n$ for all n sufficiently large, $N_\mu \subseteq M_\lambda$ whenever $\lambda \neq \mu$. If $x \in M_\lambda$, write $x = x_1 + x_2$ where $x_1 \in M_\mu$ and $x_2 \in N_\mu$. Then $x_1 = x - x_2 \in M_\lambda \cap M_\mu$, and (2) is established.

To prove (3) and (4), we may assume, without loss of generality, that $\lambda \neq \mu$. Then $\operatorname{ran}(\mu - T|_{M_\mu})$ is closed, since $\mu - T|_{M_\mu}$ is semi-regular, and $\ker(\mu - T|_{M_\mu}) = \ker(\mu - T) \cap M_\mu \subseteq M_\mu \cap M_\lambda$ by (1). Thus (3), $(\mu - T)(M_\mu \cap M_\lambda) = (\mu - T|_{M_\mu})(M_\mu \cap M_\lambda)$ is closed, by [9, Lemma 3.1.3]. It also follows from (1) that $\ker(\mu - T)^n \cap M_\mu \subseteq M_\mu \cap M_\lambda$ for every n, and, by (2),

$$\begin{aligned}\ker(\mu - T)^n \cap M_\mu \subseteq (\mu - T)M_\mu &= (\mu - T)\big((M_\mu \cap M_\lambda) \oplus N_\lambda\big)\\ &= (\mu - T)(M_\mu \cap M_\lambda) \oplus N_\lambda.\end{aligned}$$

Therefore, $\ker(\mu - T)^n \cap M_\mu \subseteq (\mu - T)(M_\mu \cap M_\lambda)$ for every n. This establishes (4), and the fact that $\lambda, \mu \in \varrho_K(T|_{M_\lambda \cap M_\mu})$ follows immediately from (3) and (4). Suppose now that $\omega \in V \cap \varrho_K(T) \setminus \{\lambda, \mu\}$. Then $\operatorname{ran}(\omega - T)$ closed, and $\ker(\omega - T) \subseteq M_\lambda \cap M_\mu$ implies that $(\omega - T)(M_\lambda \cap M_\mu)$ is closed, again by [9, Lemma 3.1.3]. It follows from (4) and (2) that

$$\ker(\omega - T)^n \subseteq (\omega - T)M_\mu = (\omega - T)\big((M_\lambda \cap M_\mu) \oplus N_\lambda\big) = \big((\omega - T)(M_\lambda \cap M_\mu)\big) \oplus N_\lambda,$$

since $\omega - T|_{N_\lambda}$ is invertible. If $x \in \ker(\omega - T)^n$ write $x = x_1 + x_2$ with $x_1 \in (\omega - T)(M_\lambda \cap M_\mu)$ and $x_2 \in N_\lambda$. Then $0 = (\omega - T)^n x = (\omega - T)^n x_1 + (\omega - T)^n x_2$ implies that $(\omega - T)^n x_2 = 0$ and thus $x_2 = 0$. Consequently, $\ker(\omega - T)^n \subseteq (\omega - T)(M_\lambda \cap M_\mu)$ for every n; i.e., $\omega - T|_{M_\lambda \cap M_\mu}$ is semi-regular. This establishes (5).

Since (6) is vacuous otherwise, suppose that $\lambda \neq \mu$. Then

$$M_\lambda^\infty = (\lambda - T)^\infty\big((M_\mu \cap M_\lambda) \oplus N_\mu\big) = (\lambda - T)^\infty(M_\lambda \cap M_\mu) \oplus N_\mu.$$

Thus $M_\lambda^\infty \cap M_\mu = (\lambda - T)^\infty(M_\lambda \cap M_\mu)$. Since $\big(V \cap \varrho_K(T)\big) \cup \{\lambda, \mu\}$ is open and connected, (5) and [9, Prop. 3.1.11] imply that, for all $\mu, \lambda \in V$, $(\lambda - T)^\infty(M_\mu \cap M_\lambda) = (\mu - T)^\infty(M_\mu \cap M_\lambda)$; thus (6), and so

$$\begin{aligned} M_\lambda^\infty \cap M &= \bigcap_{\mu\in V} (M_\lambda^\infty \cap M_\mu) = \bigcap_{\mu\in V} \big((\lambda - T)^\infty(M_\lambda \cap M_\mu)\big) \\ &= \bigcap_{\mu\in V} \big((\mu - T)^\infty(M_\lambda \cap M_\mu)\big) \subseteq \bigcap_{\mu\in V} (\mu - T)^\infty M_\mu = M^\infty. \end{aligned}$$

Since the other containment is obvious, (7) is obtained.

To prove (8), fix $\lambda \in V$ and suppose that $x \in M^\infty$. Then there exists $y \in M_\lambda^\infty$ so that $(\lambda - T)y = x$. Let $\mu \in V \setminus \{\lambda\}$, and write $y = y_1 + y_2$ where $y_1 \in M_\mu$ and $y_2 \in N_\mu$. Then $(\lambda - T)y_2 = x - (\lambda - T)y_1 \in M_\mu \cap N_\mu = \{0\}$ and, since $\lambda - T|_{N_\mu}$ is invertible, $y_2 = 0$. Thus $y \in M_\lambda^\infty \cap M_\mu$ for all $\mu \in V$; i.e., $y \in M^\infty$ by (7).

Finally, suppose that $\mu \in V \cap \varrho_K(T)$. Then a sequence $([x_n])_n \subset X/M^\infty$ satisfies $(\mu - [T])[x_n] \to 0$ as $n \to \infty$ if and only if there exists $(y_n)_n \subset M^\infty$ such that $(\mu - T)x_n - y_n \to 0$ in X. Since, by (8), $\mu - T|_{M^\infty}$ is surjective, we may write $y_n = (\mu - T)z_n$ for some $(z_n)_n \subset M^\infty$. Thus $(\mu - T)(x_n - z_n) \to 0$, and, because $\operatorname{ran}(\mu - T)$ is closed, there exists $(w_n)_n \subset \ker(\mu - T)$ such that $x_n - z_n - w_n \to 0$ in X. But, by (1), $\ker(\mu - T) \subseteq M_\lambda$ for every $\lambda \in V \setminus \{\mu\}$, while $M_\mu = X$ since $\mu \in \varrho_K(T)$, and therefore $\ker(\mu - T) \subseteq M^\infty$, by (7) and the definition of $\varrho_K(T)$. Thus $[x_n] \to 0$ in X/M^∞; $\mu - [T]_{X/M^\infty}$ is bounded below. □

Our first main result is an immediate consequence of the preceding lemma.

Theorem 2.5. *If V is an open subset of $\varrho_{kt}(T)$, then $\operatorname{ran}(T_V)$ is closed in $H(V, X)$.*

Proof. Again, we may assume without loss of generality that V is connected. Then, by Lemma 2.4 (8) and (9), $V \subseteq \varrho_{su}(T|_{M^\infty})$ and $E := V \setminus \varrho_K(T) \supseteq V \setminus \varrho_{ap}([T]_{X/M^\infty})$. Since E is discrete, the theorem now follows from Proposition 2.2. □

3. Duality and weak-∗ closed ranges

To establish the weak-∗ counterpart of Theorem 2.5, we need the duality theory for property (β). An operator $T \in \mathscr{L}(X)$ is said to have property (δ) on an open subset U of $\mathbb{C}$ if, for all open sets $V, W \subseteq \mathbb{C}$ for which $\mathbb{C} \setminus U \subseteq V \subseteq \overline{V} \subseteq W$, it follows that

$$X = \mathscr{X}_T(\mathbb{C} \setminus V) + \mathscr{X}_T(\overline{W}),$$

where $\mathscr{X}_T(F) := X \cap \operatorname{ran}(T_{\mathbb{C}\setminus F})$ denotes the glocal spectral subspace of T for a closed set $F \subseteq \mathbb{C}$.

Albrecht and Eschmeier, [3, Theorem 15], established that property (δ) on U characterizes the quotients by closed invariant subspaces of operators that are decomposable on U. By Theorem 8 of [3], T is decomposable on U precisely when T has both of the properties (β) and (δ) on U. Also, by Theorems 19 and 21 of [3], these two properties are completely dual to each other, in the sense that T has one of the properties (β) or (δ) exactly when T^* has the other one.

Moreover, property (δ) admits a characterization that is dual to the definition of property (β). To provide the details, let $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere, and, for a closed subset F of $\mathbb{C}_\infty$ with $\infty \in F$, let $P(F, X)$ denote the (LF)-space consisting of the germs of X-valued functions analytic in some open neighborhood of F with $f(\infty) = 0$. Any operator $T \in \mathscr{L}(X)$ induces a linear mapping on $P(F, X)$ through $(T^F f)(\lambda) := (\lambda - T)f(\lambda) - \lim_{\mu\to\infty} \mu f(\mu)$. As noted in the proof of Theorem 5 of [3], T has property (δ) on the open set U precisely when T^F is surjective for each closed set $F \subseteq \mathbb{C}_\infty$ with $\mathbb{C}_\infty \setminus U \subseteq F$; see also [9, Theorem 2.2.2]. Moreover, if $F = \mathbb{C}_\infty \setminus U$, then, by the Grothendieck-Köthe duality principle, $H(U, X^*)$ is canonically isomorphic to the strong dual of $P(F, X)$, and, in the sense of this identification, $T^*_U = (T^F)^*$; see Theorem 2.5.12 and Lemma 2.5.13 of [9]. Consequently, if T^* has property (β) on U, then T^F is surjective on $P(F, X)$ and hence, by a theorem of Köthe, [9, Theorem 2.5.9], T^*_U has weak-$*$ closed range in $H(U, X^*)$.

Proposition 3.1. *Let $T \in \mathscr{L}(X)$, and let M be a weak-$*$ closed, T^*-invariant subspace of X^*. Suppose that V is an open subset of $\varrho_{su}(T^*|_M)$ for which there exists a discrete set $E \subseteq V$ such that $V \setminus E \subseteq \varrho_{ap}([T^*]_{X^*/M})$. Then $\operatorname{ran}(T^*_V)$ is weak-$*$ closed in $H(V, X^*)$.*

Proof. For M, V, and E as in the hypotheses, set $S := T|_{^\perp M}$, so that $S^* = [T^*]_{X^*/M}$. Then, arguing as in Proposition 2.2, we obtain that, for every open subset W of V, S^*_W is injective with range closed in the Fréchet topology of $H(W, X^*/M)$. Indeed, suppose that $W \subseteq V$ is open, and let $U := W \setminus E$. Then $U \subseteq \varrho_{ap}(S^*)$, and thus S^*_U is injective and has closed range in $H(U, X^*/M)$. Since E is discrete, any compact subset of W may be surrounded by a contour in U, and so Cauchy's formula implies that S^*_W is injective and has closed range in $H(W, X^*/M)$. Thus S^* has property (β) on V, and therefore $\operatorname{ran}(S^*_V)$ is, in fact, weak-$*$ closed in $H(V, X^*/M)$, by the remarks preceding this proposition. Again, the assumption that $V \subseteq \varrho_{su}(T^*|_M)$ together with the Allan-Leiterer theorem implies that $(T^*|_M)_V$ is surjective on $H(V, M)$. Moreover, in the exact commutative diagram of Frèchet spaces

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H(V,M) & \xrightarrow{\ j\ } & H(V,X^*) & \xrightarrow{\ q\ } & H(V,X^*/M) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle (T^*|_M)_V} & & \Big\downarrow {\scriptstyle T_V^*} & & \Big\downarrow {\scriptstyle S_V^*} & & \\
0 & \longrightarrow & H(V,M) & \xrightarrow{\ j\ } & H(V,X^*) & \xrightarrow{\ q\ } & H(V,X^*/M) & \longrightarrow & 0
\end{array}
$$

the inclusion j and quotient q are adjoints of the quotient $P(F,X) \to P(F,X/{}^\perp M)$, $f \mapsto [f]_{X/{}^\perp M}$ and inclusion $P(F,{}^\perp M) \to P(F,X)$, $f \mapsto f$, respectively, and are therefore each weak-$*$ continuous.

Now, if $T_V^* f_\alpha \to f$ weak-$*$ in $H(V,X^*)$, then $S_V^* q f_\alpha \to qf$ in $H(V,X^*/M)$, and it follows that $qf = S_V^* qg$ for some $g \in H(V,X^*)$. Thus $f - T_V^* g \in \ker q = H(V,M) = \operatorname{ran}(T^*|_M)_V$, as noted above. Therefore $f \in \operatorname{ran}(T_V^*)$, and so $\operatorname{ran}(T_V^*)$ is weak-$*$ closed, as required. □

Theorem 3.2. *If V is an open subset of $\varrho_{kt}(T)$, then $\operatorname{ran}(T_V^*)$ is weak-$*$ closed in $H(V,X^*)$.*

P r o o f. Again, we may assume that V is connected. If $\big(M_\lambda(T), N_\lambda(T)\big)$ is the generalized Kato decomposition for $\lambda - T$ as in Lemma 2.4, then $\lambda - T^*$ has corresponding decomposition $\big(M_\lambda(T^*), N_\lambda(T^*)\big)$, where $M_\lambda(T^*) := M_\lambda(T)^* = N_\lambda(T)^\perp$ and $N_\lambda(T^*) := N_\lambda(T)^* = M_\lambda(T)^\perp$, [1, Theorem 1.43]. In particular, $M_\lambda(T^*)$ is weak-$*$ closed in X^*, and the closed range theorem implies that $(\lambda - T^*)^n M_\lambda(T^*)$ is weak-$*$ closed in $M_\lambda(T^*)$ and therefore in X^* for all n. Thus $M^\infty(T^*,V)$ is weak-$*$ closed in X^* as well. By Lemma 2.4 (8) and (9), $M^\infty(T^*,V)$ satisfies the hypotheses of Proposition 3.1, from which the desired conclusion now follows. □

Since every semi-Fredholm operator is of Kato-type, [1, Theorem 1.62 and page 24], Theorems 2.5 and 3.2 generalize the aforementioned theorem of Eschmeier, [5]. Moreover, Theorems 2.5 and 3.2 may also be used to characterize the annihilators and pre-annihilators of certain glocal spectral subspaces. If $U \subseteq \mathbb{C}$ is open, define $\mathscr{X}_T(U) := \bigcup\{\mathscr{X}_T(F)\colon\ F \subseteq U \text{ compact}\}$.

Corollary 3.3. *Let F be a closed subset of $\mathbb{C}$ for which $\sigma_{kt}(T) \subseteq F$. Then $\mathscr{X}_T(F) = {}^\perp\mathscr{X}^*_{T^*}(\mathbb{C}\setminus F)$ and $\mathscr{X}^*_{T^*}(F) = \mathscr{X}_T(\mathbb{C}\setminus F)^\perp$.*

P r o o f. If $V := \mathbb{C}\setminus F$, then $V \subseteq \varrho_{kt}(T)$, and so T_V and T_V^* have, respectively, closed and weak-$*$ closed ranges. The result now follows from [4, Lemma 2.5 (c), (d)]; alternatively, one can argue as in the proof [9, Prop. 2.5.14]. □

Finally, the notion of operators of Kato-type provides a unification of Corollaries 20 and 21 of [11]. The "all or nothing" relation between SVEP and components of $\varrho_K(T)$ in Proposition 2.3 extends trivially to components of $\varrho_{kt}(T)$, a fact first established by different methods in [2, Theorems 2.2 and 2.3].

Proposition 3.4. *Let V be a component of $\varrho_{kt}(T)$, and let $U := V \cap \varrho_K(T)$. Then*

$$V \subseteq \mathfrak{S}(T) \Leftrightarrow V \cap \mathfrak{S}(T) \neq \emptyset \Leftrightarrow U \cap \mathfrak{S}(T) \neq \emptyset \Leftrightarrow U \cap \sigma_p(T) \neq \emptyset \Leftrightarrow U \subseteq \mathfrak{S}(T).$$

Thus, if T has SVEP on U, then T has property (β) on V, while T has property (δ) on V provided T^* has SVEP on U. In particular, if both T and T^* have SVEP on $\varrho_{kt}(T)$, then $\mathscr{S}_r(T) \subseteq \sigma_{kt}(T)$.

Proof. Since U inherits connectedness from V, the list of equivalences follows from that in Proposition 2.3 (3). If T has SVEP on U, then Theorem 2.5 implies that T has property (β) on V. The last statement follows from the facts that $\varrho_{kt}(T) \subseteq \varrho_{kt}(T^*)$ and that $T \in \mathscr{L}(X)$ is decomposable on an open set G if and only if both T and T^* have Bishop's property (β) on G, equivalently, that T has both property (β) and (δ) on G, [3]. $\square$

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