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Anna de Simone; Pavel Pták

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GROUP-VALUED MEASURES ON COARSE-GRAINED QUANTUM LOGICS

Anna De Simone, Napoli, and Pavel Pták, Praha

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Abstract. In [3] it was shown that a (real) signed measure on a cyclic coarse-grained quantum logic can be extended, as a signed measure, over the entire power algebra. Later ([9]) this result was re-proved (and further improved on) and, moreover, the non-negative measures were shown to allow for extensions as non-negative measures. In both cases the proof technique used was the technique of linear algebra. In this paper we further generalize the results cited by extending group-valued measures on cyclic coarse-grained quantum logics (or non-negative group-valued measures for lattice-ordered groups). Obviously, the proof technique is entirely different from that of the preceding papers. In addition, we provide a new combinatorial argument for describing all atoms of cyclic coarse-grained quantum logics.

Keywords: coarse-grained quantum logic, group-valued measure, measure extension

MSC 2000: 06C15, 81P10, 28A99, 28A55

1. Introduction

In this paper we investigate the problem of extending a measure on a coarse-grained quantum logic over the entire power algebra. The problem is motivated by the quantum logic theory and the measurement theory (see [3], [4] and [12]). In our generalized setup we take up measures ranging in a group. The intention is to enlarge the area of potential applications (the values measured may not allow for scalar multiplication, a typical example being the values expressed by integers) as well as mathematical curiosity (can we prove the extension results without the use of real scalars?).

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Let $n \geqslant 2$ and $l \geqslant 2$ be natural numbers. Let $\Omega = \{0, 1, 2, \ldots, nl-1\}$ and let us denote by $\Delta_{n,l}$ the smallest system of subsets of Ω that contains all sets of the type $I_h = \{h, h+1, \ldots, h+l-1\}$ ($h \in \Omega$), where the sum is understood modulo nl, and which is closed under the formation of complements in Ω and under the formation of disjoint unions. Let us call the collection $\Delta_{n,l}$ the *coarse-grained quantum logic* and the sets I_h the *generating sets* of $\Delta_{n,l}$. (It should be noted that any $\Delta_{n,l}$ constitutes a so called quantum logic (abbr. a logic, alternatively also called an orthomodular poset). Moreover, $\Delta_{n,l}$ belongs to the important class within quantum theories of the so called "concrete quantum logics", see [5] and [12]. Various measure-theoretic results on concrete logics motivated by quantum logic considerations can be found in [2], [5], [6], [7], [8], [10], [11], etc.)

2. Results

Let $(G, +, 0)$ be an abelian group. Let $m\colon \Delta_{n,l} \to G$ be a mapping such that $m(0) = 0$, and $m(A \cup B) = m(A) + m(B)$ for any pair of disjoint sets $A, B \in \Delta_{n,l}$. Then m is said to be a coarse-grained-group-valued measure with values in G. If G is given, we shall simply call m a *G-measure* on $\Delta_{n,l}$.

Suppose that $m\colon \Delta_{n,l} \to G$ is a G-measure on $\Delta_{n,l}$. Let us again write $\Omega = \{0, 1, \ldots, nl-1\}$ and let us denote by $\exp\Omega$ the (Boolean) power algebra on Ω (i.e., let $\exp\Omega$ stand for the set of all subsets of Ω). A natural question arises whether m can be extended, as a G-measure p, over the entire $\exp\Omega$. Since all singletons belong to $\exp\Omega$, p is in fact given by a function $f_p\colon \Omega \to G$ in such a way that $p(A) = \sum\limits_{a \in A} f_p(a)$. In [3] and [9] the authors show by making use of certain linear algebra reasonings that m can indeed be extended if $G = \mathbb{R}$ (= the additive group of real numbers). Our first result says that a generalized extension theorem for group-valued measures can be obtained. Moreover, uniqueness can be guaranteed when initial conditions are preassigned. Prior to formulating the result, observe that if two G-valued measures on $\Delta_{n,l}$ agree on all generating sets, they have to agree on the entire $\Delta_{n,l}$ (indeed, if two G-measures agree on a collection $\mathscr{C}$ of sets then they have to agree on any set obtained either as a complement of a set of $\mathscr{C}$ or as a disjoint union of sets of $\mathscr{C}$).

Theorem 2.1. *Let G be an abelian group. Let $n, l \in \mathbb{N}$, $n \geqslant 2$, $l \geqslant 2$. Let $m\colon \Delta_{n,l} \to G$ be a G-measure, where the underlying set of $\Delta_{n,l}$ is the set $\Omega = \{0, 1, 2, \ldots, nl-1\}$. Let $g_0, g_1, \ldots, g_{l-2}$ be elements of G. Then there is exactly one G-measure $p\colon \exp\Omega \to G$ such that p extends m and $p(\{i\}) = g_i$ for any $i = 0, 1, \ldots, l-2$.*

P r o o f. Let $p(\{i\}) = g_i$ for $i = 0, 1, \ldots, l-2$. Further, let $p(\{h\}) = m(I_{h-l+1}) - \sum_{r=h-l+1}^{h-1} p(\{r\})$, where $h = l-1, l, l+1, \ldots, nl-1$. We claim that p, uniquely extended to a G-measure on $\exp\Omega$, $p(A) = \sum_{a\in A} p(\{a\})$, extends the measure m. Consider p restricted to $\Delta_{n,l}$. We have to verify that it agrees with m on the generating sets I_h ($h = 1, \ldots, nl-1$) of $\Delta_{n,l}$. This is obviously true for $h \leqslant (n-1)l$—we have constructed p to have this property. Suppose $h > (n-1)l$. Then I_h is a complement of a disjoint union of generating sets I_k with $k \leqslant (n-1)l$. Since both m and p are measures and since they both agree on the generating sets I_k with $k \leqslant (n-1)l$, they have to agree on I_h, too. The proof is complete. $\square$

A natural question arises whether Theorem 2.1 remains valid if G is a semigroup. The following example shows that it does not.

Example. Let $n = 2$ and $l = 3$ (thus, $\Omega = \{0, 1, 2, 3, 4, 5\}$). Let $\mathbb{R}^+$ be the additive semigroup of all non-negative numbers. Let $t\colon \Omega \to R$ be defined as follows: $t(0) = 1, t(1) = -1, t(2) = 1, t(3) = t(4) = t(5) = 0$. This t defines an R-measure, $\tilde{t}$, on $\exp\Omega$. When restricted to $\Delta_{n,l}$, the measure $\tilde{t}$ is a (two-valued) measure which ranges in R^+. By Theorem 2.1 every extension $\tilde{t}\colon \exp\Omega \to \mathbb{R}$ of t has the property $\tilde{t}(\{1\}) = -1$ and therefore there is no extension of t which ranges in $\mathbb{R}^+$.

It is worth noting that in the previous theorem we did *not* need to know which sets the logic $\Delta_{n,l}$ actually consists of. However, if we want to know which exact "initial conditions" we can preassign, the complete description of $\Delta_{n,l}$ is needed. Since other problems we want to address also require a complete description of $\Delta_{n,l}$, we will now present it. (It should be noted that the description was found in [9] where a fairly involved combinatorial argument was employed. We want to demonstrate a rather straightforward inductive reasoning that provides another proof of the description and that gives a good insight of how $\Delta_{n,l}$ can be obtained from the generating sets I_h ($h = 0, 1, \ldots, nl-1$). Moreover, some of this reasoning will be used in our main result of Theorem 2.6.

Let us first introduce some terminology. As before, let $\Omega = \{0, 1, \ldots, nl-1\}$ and let $\Delta_{n,l}$ denote the corresponding coarse-grained logic. By a *segment* in Ω we mean a set of "consecutive" points in Ω, $\{a, a+1, \ldots, a+h\}$, where addition is understood modulo nl. By an *equidistributed set* in Ω (abbr., an *ED* set) we mean a subset of Ω which consists of rl points ($r \leqslant n$, $r \in \mathbb{N}$) distributed in such a manner that exactly r points lie in one class of modulo l-equivalence. If r is specified, we call such a set an ED_r set. In particular, an ED_1 set consists of l points each point belonging to a different class modulo l. (Since the collection of all ED sets is closed under the formation of the complement in Ω and under the formation of disjoint unions, it is obvious that this collection constitutes a quantum logic).

We want to show that if n is at least 3, the family of ED_1 sets consists of all atoms of $\Delta_{n,l}$. It is sufficient to show that each ED_1 set belongs to $\Delta_{n,l}$ as the following proposition asserts.

Proposition 2.2. *Let $n \geqslant 3, l \geqslant 2$, and let $\Delta_{n,l}$ denote the coarse-grained logic on $\Omega = \{1, 2, \ldots, nl - 1\}$. Let each ED_1 set belong to $\Delta_{n,l}$. Then $\Delta_{n,l}$ consists of all ED sets.*

Proof. Let Δ be the logic consisting of all ED sets. Then $\Delta_{n,l} \subset \Delta$ since all generating sets $I_h (h = 0, 1, \ldots, nl - 1)$ are ED_1 sets and forming the complements of ED sets in Ω and the disjoint unions of ED sets in Ω always produces an ED set. But if any ED_1 set belongs to $\Delta_{n,l}$, we have $\Delta \subset \Delta_{n,l}$ and therefore $\Delta = \Delta_{n,l}$. □

Proposition 2.3. *If n is at least 3, then each ED_1 set belongs to $\Delta_{n,l}$.*

Proof. We will proceed by induction over the number of segments in a given ED_1 set. Let B be an ED_1 set. Let p be the number of segments the set B consists of. Thus, let $B = B_1 \cup B_2 \cup \ldots \cup B_p$, where B_i $(p \geqslant 2)$ are clockwise ordered segments in Ω such that B cannot be expressed as a union of strictly less than p segments. We will show that if all ED_1 sets consisting of strictly less than p segments belong to $\Delta_{n,l}$, then so does B. Since each ED_1 set consisting of exactly one segment is necessary a generating set of $\Delta_{n,l}$ and therefore it belongs to $\Delta_{n,l}$, this will constitute the proof of Proposition 2.3. Our task is therefore to verify the inductive step.

We first need an auxiliary result. □

Lemma 2.4. *Assume that $n \geqslant 3$. If any ED_1 set consisting of at most p segments belongs to $\Delta_{n,l}$, then any ED set consisting of at most p segments belongs to $\Delta_{n,l}$.*

Proof. Suppose that $p \geqslant 2$ (for $p = 1$ the lemma is trivial). Let B be an ED set in $\Delta_{n,l}$ and let us suppose that B consists of at most p segments, $B = B_1 \cup B_2 \cup \ldots \cup B_p$. We may (and will) suppose that each of the sets B_i $(i \leqslant p)$ is shorter than l (if some B_i is longer, we would subtract from B_i the appropriate disjoint union of generating sets). We claim that B can be obtained as a disjoint union of an ED_1 set and an ED set, each consisting of at most $p-1$ segments. This by induction proves our lemma.

Consider $B_1 = \{b, b+1, \ldots, b+k\}$. Then among the segments B_i $(2 \leqslant i \leqslant p)$ there must be one, say B_1^2, the left-most end point of which is an element equivalent to $b + k + 1 \pmod l$. Indeed, if all sets B_j which contain such an element had a predecessor, an element equivalent to $b + k \pmod l$, we would have more elements of the class $b + k \pmod l$ in B than the elements of the class $b + k + 1 \pmod l$. This is excluded. Let us consider the set $B_1 \cup B_1^2$. If it contains an ED_1 set of the type

$B_1 \cup C$, we are done. If not, we consider the set $B_1 \cup B_1^2$ and denote its right-most end point by s. Then there is a set, B_1^3, among the sets $B_1, B_2, \ldots, B_p$ such that $B_1^3 \neq B_1$, $B_1^3 \neq B_1^2$ and the left-most end point of B_1^3 is equivalent to $s+1 \pmod{l}$, etc. As soon as we construct in this way a set which contains an ED_1, we can easily arrange for writing the original ED set as a disjoint union of an ED_1 an ED set consisting of at most $p-1$ segments. This completes the proof of the lemma. $\square$

Let us now return to the proof of Proposition 2.3. Let $B = B_1 \cup B_2 \cup \ldots \cup B_p$. Let us first "translate" the segments $B_1, B_2, \ldots, B_p$ on the set $\{0, 1, \ldots, l-1\}$. In other words, let us write $\{0, 1, \ldots, l-1\} = A_1 \cup A_2 \cup \ldots \cup A_p$, where for any $i = 1, 2, \ldots, p$ there is exactly one j ($j \leqslant p$) and h ($h \leqslant n$) such that $A_i + hl = B_j$. We now "copy" these segments $A_1, \ldots, A_p$ on all the sets $I_{k,l} = \{kl, \ldots, l(k+1)-1\}$, $(1 \leqslant k \leqslant n-1)$. Expressed formally, we set $A_{i+kp} = A_i + kl$. In this way we obtain a partition of $\Omega = \{0, 1, \ldots, nl-1\}$ in the segments $A_1, A_2, \ldots, A_{np}$. Before going on with the proof, let us observe that if we organize these sets in a natural manner in an $n \times p$ matrix M, a choice of $A_{k_1}, A_{k_2}, \ldots, A_{k_p}$ would mean an ED_1 set, $A_{k_1} \cup A_{k_2} \cup \ldots \cup A_{k_p}$, just in case when we have chosen exactly one element from each column of M.

Let us denote by $i_1, i_2, \ldots, i_p$ the indices such that

$$B_1 = A_{i_1},\ B_2 = A_{i_2},\ \ldots,\ B_p = A_{i_p}.$$

For $j \in \{1, 2, \ldots, p\}$ let us denote by r_j the number of segments between B_j and B_{j+1}. Thus, let us define

$$r_j = \operatorname{card}\{A_{i_j+1}, A_{i_j+2}, \ldots, A_{i_{j+1}-1}\}.$$

Of course, $r_j = i_{j+1} - i_j - 1$ for $j \neq 1$ and $r_1 = i_1 - i_p - 1 + nl$. Denote now by c_j the index of the column of matrix M to which B_j belongs. Then $(c_1, c_2, \ldots, c_p)$ is a permutation of $(1, 2, \ldots, p)$. Observe also that $r_j > 1$ for at least one index j (otherwise B will be a complement of an ED_1 set, which is impossible in view of $n > 2$).

We will proceed by induction. Let us assume that B consists of at most p segments and assume that any ED_1 set consisting of strictly less than p segments belongs to $\Delta_{n,l}$. We will show that there is an ED set, C, such that $B \cap C = \emptyset$ and both the sets $B \cup C$ and C consist of strictly less than p segments. By Lemma 2.4 this will prove the theorem.

We shall consider the following three cases.

Case 1. There exists $h \in \{1, 2, \ldots, p\}$ such that $r_h = mp$ ($m \in \mathbb{N}$).

Case 2. There exists $h \in \{1, 2, \ldots, p\}$ such that $r_h = mp + r$ with $m \in \mathbb{N}_0$ and $1 < r < p$.

Case 3. $r_j \equiv 1 \bmod p$ for all $j \in \{1, 2, \ldots, p\}$.

It is obvious that one of the above cases must occur. We will discuss the cases in order. For the sake of simplicity, let us rename the "follower" of any B_j by writing $A_{i_j+1} = C_j$. Then $\{C_1, C_2, \ldots C_p\}$ is a family of p segments such that each of them lies in exactly one column of the matrix M and therefore $\bigcup_{i=1}^{p} C_i$ is an ED_1 set.

Case 1. We simply set

$$C = \bigcup_{s=i_h+1}^{i_{h+1}-1} A_s.$$

Thus, in this case C is the union of all segments between B_h and B_{h+1}. Of course, C is a generating set and $B \cup C$ is an ED set consisting of $p-1$ segments.

Case 2. Without any loss of generality we can assume that $m = 0$ (otherwise we pass to this case by subtracting from B a connected ED set). For $k = r_h$, let us denote by $\{D_1, D_2, \ldots, D_k\}$ the segments between B_h and B_{h+1}. These segments are in k different columns of the matrix M described above. Let us extend the system $\{D_1, D_2, \ldots, D_k\}$ by adding to it $p-k$ sets of the collection $\{C_1, C_2, \ldots C_p\}$ chosen from the remaining columns of the matrix. We will obtain a collection $\{D_1, D_2, \ldots, D_k, C_{i_{k+1}} \ldots, C_{i_p}\}$ such that its union C is an ED set and C consists of $p-k+1$ segments.

As a result, C consists of strictly less than p segments and so does $B \cup C$. We have verified Case 2.

Case 3. As already seen, it is impossible that $r_j = 1$ for all $j = 1, 2, \ldots, p$. Assume $r_j = 1$ for at least one index j. We then choose two segments B_h and B_k among the sets B_j's such that $r_h = 1$, $r_k > 1$ and $c_k = c_h + 1$ (B_h and B_k lie in two consecutive columns of M). Let us set $E = \bigcup_{s=r_k+1}^{r_k+p-1} A_s$ and write

$$C = E \cup C_h.$$

Since the set C is the union of $p-1$ consecutive segments with C_h, it consists of two segments. Moreover, the choice of C_h decreases the number of segments in $B \cup C$.

Finally, in case $r_j > 1$ for all $j = 1, 2, \ldots, p$, it is sufficient to take $C = D \cup E$, where

$$D = \bigcup_{s=i_1+1}^{i_2-1} A_s, \qquad E = \bigcup_{s=i_h+1}^{i_h+p-1} A_s$$

for $c_h = c_1 + 1$. The proof is complete.

The first consequence of the description of $\Delta_{n,l}$ is the following slight improvement of Theorem 2.1 (the case of $n = 2$, $l = 2$ which we did not consider above is trivial).

Theorem 2.5. *Let $n \geqslant 2$ and $l \geqslant 2$ and let G be an abelian group. Let $B = \{b_1, b_2, \ldots, b_l\}$ be an ED_1 set of $\Delta_{n,l}$. Let $m\colon \Delta_{n,l} \to G$ be a measure and let $g_1, g_2, \ldots, g_{l-1} \in G$. Then there is exactly one measure $p\colon \exp\Omega \to G$ such that p is an extension of m over $\exp\Omega$ and such that $p(\{b_i\}) = g_i$ for any $i = 1, 2, \ldots, l-1$.*

In the second consequence (the main result of the paper) we ask when a state (= a non-negative G-measure where G is a lattice-ordered group) on $\Delta_{n,l}$ extends as a state to $\exp\Omega$. This question has been considered in [3] and [9], first partially and then completely and affirmatively solved for real-valued states. We want to show that an affirmative answer is in force also in this generalized case (again, we cannot use the linear algebra technique utilized for real states, of course). Let G be a lattice-ordered group. That is, let G be an abelian group which is given a partial order making G a lattice-ordered group (see [1] for more detail; we shall only use simple properties of the lattice-ordered group calculus). Let $m\colon \Delta_{n,l} \to G$ be a non-negative measure (i.e., let $m(A) \geqslant 0$ for any $A \in \Delta_{n,l}$). Then m is said to be a *G-valued state*. Our result reads as follows.

Theorem 2.6. *Let $n \geqslant 3$ and $l \geqslant 2$. Let $\Omega = \{0, 1, \ldots, nl-1\}$ and let $\Delta_{n,l}$ be the coarse-grained quantum logic on Ω. Let G be a lattice-ordered group and let $m\colon \Delta_{n,l} \to G$ be a G-valued state. Then there is a G-valued state $t\colon \exp\Omega \to G$ such that t is an extension of m.*

P r o o f. We will first prove a lemma (we use the assumptions of Theorem 2.6 and the notation established above). □

Lemma 2.7. *Let $\mathscr{A} = \{a \in \Omega\colon a \equiv 0 \bmod l\}$ and let $d_a = m(\{0, 1, \ldots, l-1\}) - m(\{a, 1, 2, \ldots, l-1\})$. Let $\{0, x_1, x_2, \ldots, x_{l-1}\}$ be an ED_1 set. Then $d_a = m(\{0, x_1, x_2, \ldots, x_{l-1}\}) - m(\{a, x_1, \ldots, x_{l-1}\})$.*

P r o o f. This is obviously true if $a = 0$. Suppose $a \neq 0$. Write $X_0 = \{0, x_1, \ldots, x_{l-1}\}$ and $X_a = \{a, x_1, \ldots, x_{l-1}\}$. Take sets Y_0, Y_a such that $Y_0 = \{0, y_1, y_2, \ldots, y_{l-1}\}$, $Y_a = \{a, y_1.y_2, \ldots, y_{l-1}\}$ and $\{1, 2, \ldots, l-1\} \cap \{y_1, y_2, \ldots, y_{l-1}\} = \emptyset$, $\{x_1, x_2, \ldots, x_{l-1}\} \cap \{y_1, y_2, \ldots, y_{l-1}\} = \emptyset$. This can be done since $n \geqslant 3$. Then we have $X_0 \cap Y_a = X_a \cap Y_0 = \emptyset$ and $X_0 \cup Y_a = X_a \cup Y_0$. It follows that $m(X_0) + m(Y_a) = m(X_a) + m(Y_0)$ and therefore $m(X_0) - m(X_a) = m(Y_0) - m(Y_a) = d_a$. The proof of the lemma is complete. □

Let us return to the proof of Theorem 2.6. Let $\mathscr{A}$ and d_a be as in Lemma 2.7. Denote by s the supremum of d_a in G, $s = \bigvee\{d_a\colon a \in \mathscr{A}\}$. Consider the function

$f\colon \mathscr{A} \to G$ such that $f(a) = s - d_a$. Obviously, $f(a) \geqslant 0$ for each $a \in \mathscr{A}$. We claim that for any $X \in \Delta_{n,l}$ such that $a \in X$ we have $f(a) \leqslant m(X)$. Indeed, suppose that X_1 is an ED_1 set with $a \in X_1, X_1 \subset X$. Since m must be order-preserving, it is sufficient to show that $f(a) \leqslant m(X_1)$. Suppose first that $a = 0$. By Lemma 2.7, we have $m(X_1) \geqslant f(0) = s$. Assume therefore $a \neq 0$. Thus, $X_1 = \{a, x_1, x_2, \ldots, x_{l-1}\}$. Setting $Y = \{0, x_1, x_2, \ldots, x_{l-1}\}$, we have

$$f(a) = s - d_a = s - m(Y) + m(X) \leqslant m(X)$$

as asserted.

Let us pass from our original logic $\Delta_{n,l}$ to a new logic $\Delta_{n,l-1}$ by excluding all elements of $\mathscr{A}$ from Ω (i.e., let us delete all elements congruent to 0 modulo l). We have now n "segments" each of the length $l-1$ which generate the logic $\Delta_{n,l-1}$. Let us define a state m_1 on this new logic $\Delta_{n,l-1}$ in the following way: Consider an element X in $\Delta_{n,l-1}$. It is an ED_r set (i.e., it contains $r(l-1)$ points equally distributed). Let us take r elements $\{a_1, a_2, \ldots, a_r\}$ in the set $\mathscr{A}$ and put

$$m_1(X) = m(X \cup \{a_1, a_2, \ldots, a_r\}) - \sum_{i=1}^{r} f(a_i).$$

Let us first prove that the definition is correct. For arbitrary subsets $\{a_1, a_2, \ldots, a_r\}$ and $\{b_1, b_2, \ldots, b_r\}$ of $\mathscr{A}$, let us show that

$$m(X \cup \{a_1, a_2, \ldots, a_r\}) - \sum_{i=1}^{r} f(a_i) = m(X \cup \{b_1, b_2, \ldots, b_r\}) - \sum_{i=1}^{r} f(b_i).$$

Write X as a disjoint union of ED_1 sets $X_1, X_2, \ldots, X_r$. Then we obtain

$$m(X \cup \{a_1, a_2, \ldots, a_r\}) = \sum_{i=1}^{r} m(X_i \cup \{a_i\}).$$

Analogously,

$$m(X \cup \{b_1, b_2, \ldots, b_r\}) = \sum_{i=1}^{r} m(X_i \cup \{b_i\}).$$

The rest follows from the observation that

$$m(X_i \cup \{a_i\}) - f(a_i) = m(X_i \cup \{b_i\}) - f(b_i) \quad \text{for any} \quad i = 1, 2, \ldots, r.$$

(Note that, by the definition of d_a and f, $m(X \cup \{a\}) - f(a) = m(X \cup \{0\}) - f(0)$ for all $a \in \mathscr{A}$.) The function m_1 is obviously non-negative. Let us prove that it is additive. Let us take disjoint sets X and Y in $\Delta_{n,l-1}$. Then, for any $\{a_1, a_2\} \subseteq \mathscr{A}$, we have

$$m_1(X \cup Y) = m(X \cup Y \cup \{a_1, a_2\}) - f(a_1) - f(a_2).$$

Then it follows that

$$m_1(X \cup Y) = m(X \cup \{a_1\}) + m(Y \cup \{a_2\}) - f(a_1) - f(a_2) = m_1(X) + m_1(Y).$$

We have a group-valued non-negative measure m_1 on $\Delta_{n,l-1}$. If $l-1 > 1$, we repeat the procedure, obtaining a value $f(a)$ for n elements of Ω, and a measure m_2 on $\Delta_{n,l-2}$. After a finite number of steps, we reach atoms of length one (and then we put $f(a) = m_{l-1}(\{a\})$). As a result, we assign to each point $a \in \Omega$ a non-negative element of the group G.

It only remains to prove that the function $f\colon \exp\Omega \to G$ defined by $f(X) = \sum\limits_{x \in X} f(x)$ is the required extension of m. Let us take an ED_1 set $X = \{x_0, x_1, \ldots, x_{l-1}\}$ in $\Delta_{n,l}$ and compute the value $m(X)$. We obtain

$$\begin{aligned} m(X) &= m(\{x_0, x_1, \ldots, x_{l-1}\}) = f(x_0) + m_1(\{x_1, x_2, \ldots, x_{l-1}\}) \\ &= f(x_0) + f(x_1) + m_2(\{x_2, x_3, \ldots, x_{l-1}\}) \\ &= f(x_0) + f(x_1) + f(x_2) + m_3(\{x_3, x_4, \ldots, x_{l-1}\}) \\ &= \sum_{i=0}^{l-2} f(x_i) + m_{l-1}(\{x_{l-1}\}) = \sum_{i=0}^{l-1} f(x_i). \end{aligned}$$

We have shown that $m(X) = \sum\limits_{i=1}^{l-1} f(x_i)$ and this completes the proof.

Let us note in concluding that the only case not covered by Theorem 2.6, the extension question for $n = 2$ and $l = 3$, answers in the negative. Indeed, the state t constructed in Example before Prop. 2.2 cannot be extended as a state to $\exp\Omega$.

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Authors' addresses: Anna De Simone, Dipartimento di Matematica, e Statistica, Università degli Studi, "Federico II" di Napoli, 80126 Napoli, Italy, e-mail: `annades@unina.it`; Pavel Pták, Czech Technical University, Faculty of Electrical Eng., Department of Mathematics, 166 27 Prague 6, Czech Republic, e-mail: `ptak@math.feld.cvut.cz`.