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# ON THE EXISTENCE AND THE STABILITY OF SOLUTIONS FOR HIGHER-ORDER SEMILINEAR DIRICHLET PROBLEMS

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*Abstract.* We investigate the existence and stability of solutions for higher-order two-point boundary value problems in case the differential operator is not necessarily positive definite, i.e. with superlinear nonlinearities. We write an abstract realization of the Dirichlet problem and provide abstract existence and stability results which are further applied to concrete problems.

*Keywords*: Dirichlet problem

*MSC 2000*: 35J20

## 1. Introduction

We are interested in the existence results as well as in the continuous dependance on the functional parameter for a two-point boundary value problem

$$\begin{cases} Lx(t) = F_x(t, x(t)), \\ x(0) = x(1) = \dot{x}(0) = \dot{x}(1) = \ldots = x^{(n-1)}(0) = x^{(n-1)}(1) = 0, \end{cases}$$

where $L = \sum\limits_{i=0}^{n} a_i x^{(2i)}$ not necessarily satisfies inequality

$$\int_0^T \Big\langle \sum_{i=0}^{n} a_i x^{(2i)}, x \Big\rangle \,\mathrm{d}t \geqslant \alpha \int_0^T x^2(t)\,\mathrm{d}t \tag{1.1}$$

for some $\alpha > 0$. Here $F_x$ denotes the derivative of $F$ with respect to the second variable. We assume that $F$ satisfies certain general growth-type conditions, e.g.

(1) There exist $d > 0$ such that $F_x(t,-d), F_x(t,d) \in Y$ and $|F_x(t,-d)| \leqslant |F_x(t,d)|$ for a.e. $t \in [0,1]$. Moreover

$$\int_0^T F_x^2(t,d)\,\mathrm{d}t \leqslant \beta d^2,$$

where $\beta$ is a certain constant which depends on a type of a differential operator.

(2) $F, F_x\colon [0,1] \times [-d,d] \to \mathbb{R}$ are Carathéodory functions, $F$ is convex in second variable and $F(t,x) = +\infty$ for $(t,x) \in [0,1] \times (\mathbb{R} \setminus [-d,d])$.

For instance, when $L = -x^{(6)} + x^{(4)} + 4\ddot{x}$ we put $\beta = [4\sqrt{3}(\pi^2(4\pi^2-1)-1)]^2$.

As stated, these growth conditions are rather general and are not restricted to at most quadratic growth type usually assumed on $F$, see [15] and references therein. Compare the examples in Section 6 where we show that our growth assumptions in concrete applications concern only behaviour of $F$ in the neighbourhood of 0. Therefore we may consider both sub- and super-linear cases. Two-point boundary value problems have attracted attention in the last few years, see [1], [2], [11], [12], [13], [14], [15]. The approaches of the mentioned cites differ from ours. Since it is either applied a method of upper and lower solutions (in the case of second order equations) [14] or a Mawhin's degree theory [12], [13] or some other topological arguments [1], [2]. For variational method in sublinear case see [15]. It is also assumed in case of higher order equation, see [11], that a kind of inequality (1.1) is satisfied.

Our approach consists in writing the abstract version of the semilinear Dirichlet problem and then in proving, by constructing a suitable variational method, the existence of its solutions and their stability.

Therefore we will take up the problem of the existence and the stability of solution to the following family of problems

$$Lx = \nabla G_k(x), \tag{1.2}$$

where $L$ is a self-adjoint operator defined on a real Hilbert space $D(L)$ which is dense in a real Hilbert space $Y$. $G_k\colon Y \to \mathbb{R}$ is a function satisfying suitable growth conditions. We are interested in the existence results in case $L$ is not necessarily positive definite. Although the case of $L$ positive definite is also covered by our method. Similar has been considered in [7] but now we take up a different approach since we look for minimizers of a dual action functional while in [7] the primal one is minimized. The existence results are based on application of the so-called dual method which was first introduced in [17] for O.D.E. and later developed for abstract problems in case the differential operator was positive definite in [5]. However, the method from [5] applies only in case $L$ is positive definite. The dual least action [18] principle will not work due to the fact that the dual action functional in the sense of Clarke is not bounded in a supercritical case.

We say that a family of equations (1.2) is stable provided that from a sequence $\{x_k\}_{k=1}^{\infty}$ being a solution to (1.2), we may choose a subsequence weakly convergent to a certain $\overline{x}$ which is a solution to the problem

$$L\overline{x} = \nabla G_0(\overline{x}),$$

under the assumption that $\nabla G_{k_i}(x) \rightharpoonup \nabla G_0(x)$ weakly in $Y$ for all $x$ from a certain subset of $D(L)$. Stability for abstract problems satisfying quadratic growth conditions is considered in [8]. This approach is based on pioneering works [19], [20], where the question of stability in case of non unique solution is properly stated. The dual method was first applied to proving the continuous dependance on parameters in [16], where a certain type of differential equation with a nonlinearity being separated in the state variable and a parameter. Later, using the ideas from [8] and [16] for the stability of solution, the problem similar to ours and with the additional assumption that $L$ is positive definite, has been considered in [6]. Our approach being somehow different allows us to prove the stability of the solutions and contrary to [6] we do need to use the spectral theory. This is possible just because of the duality which we develop here. Thus an approach towards stability is somehow different. The method from [6] does not apply and since we also do not require the sublinear growth on the nonlinearity, we may not use the approach of [8]. We also prove, under a mild additional assumption, the statement of Remark 1, [6] is valid which is not demonstrated in [6], see Theorem 4.1.

As in [7] we consider an equivalent problem

$$(L + A)x - Ax = \nabla G_k(x),$$

where $A\colon D(A) \to Y$ is such an linear, densely defined, self-adjoint operator that $L + A$ is positive definite and $D(L + A) = D(L)$. The action functional $J_k\colon D((L + A)^{1/2}) \to \mathbb{R}$ and the dual action functional $J_{D_k}\colon D((L + A)^{1/2}) \to D(A^{1/2}) \to \mathbb{R}$ read

$$J_k(x) = \frac{1}{2}\langle (L + A)^{1/2}x, (L + A)^{1/2}x\rangle - \frac{1}{2}\langle A^{1/2}x, A^{1/2}x\rangle - G_k(x),$$
$$J_{D_k}(p, q) = G_k^*((L + A)^{1/2}p - A^{1/2}q) + \frac{1}{2}\langle q, q\rangle - \frac{1}{2}\langle p, p\rangle,$$

where $G_k^*\colon Y \to \mathbb{R}$ denotes the Fenchel-Young conjugate of $G_k$, [4] and $(L + A)^{1/2}$ denotes the (unique) square root operator, [10].

We impose the following assumptions, where $\|\cdot\|$ is the norm induced by the scalar product $\langle\cdot,\cdot\rangle$ in the space $Y$:

(A1) $D((L+A)^{1/2})$ is compactly imbedded and dense in $D(A^{1/2})$; $D(A^{1/2})$ is compactly imbedded and dense in $Y$.

(A2) If there exists a constant $C_1$ such that $\|(L+A)^{1/2}x\| < C_1$, then there exists a constant $C_2$ such that $\|Ax\| < C_2$.

(A3) $G_k\colon Y \to \mathbb{R}$ is convex, lower semicontinuous, $G_k(0) < \infty$. $\nabla G_k(0) \neq 0$.

The main idea of our variational method relies on the fact that we seek critical points and critical values of $J_k$ and $J_{D_k}$ on a suitably constructed sets $X_k$ and $X_k^d$. The definition of $X_k$ uses nothing else but a kind of a linearisation trick. It is also apparent that in the first stage we must show that a certain set is nonempty and invariant. These ideas usually come from topological methods, compare [3]. That is why this approach unites in a certain sense topological and variational methods.

We are interested in finding solutions in a form of a triple $(x_k, p_k, q_k) \in D(L) \times D((L+A)^{1/2}) \times D(A^{1/2})$ satisfying the following relations

$$
\begin{aligned}
(L+A)^{1/2}x_k &= p_k,\\
A^{1/2}x_k &= q_k,\\
(L+A)^{1/2}p_k - A^{1/2}q_k &= \nabla G_k(x_k).
\end{aligned}
\tag{1.3}
$$

Both duality and variational principle will provide the above relations describing connections between critical points and critical values of $J_k$, $J_{D_k}$ considered on suitably constructed subsets of $D(L)$, $D((L+A)^{1/2}) \times D(A^{1/2})$, respectively.

## 2. Duality results

In this section, as in the Sections 3 and 4, we fix the subscript $k$ for simplicity.

Now we define sets on which the critical points and critical values for the action and the dual action functional will be investigated.

(A4) There exists a set $X \subset D(L)$ such that for all $x_2 \in X$ the relation

$$
(L+A)x_1 - Ax_2 = \nabla G(x_2) \tag{2.1}
$$

implies that $x_1 \in X$. Moreover, $\nabla G(X)$ is relatively weakly compact in $Y$, $X$ is weakly compact in $D((L+A)^{1/2})$ and $\nabla G(x_n) \rightharpoonup \nabla G(\overline{x})$ for all sequences $\{x_n\} \subset X$, $x_n \to \overline{x}$ in $Y$.

The dual action functional is now considered on the following set: We say that $(p,q) \in X^d \subset D((L+A)^{1/2}) \times D(A^{1/2})$ if and only if there exist $x_1, x_2 \in X$ satisfying relation (2.1) and such that

$$p = (L + A)^{1/2} x_1 \quad \text{and} \quad q = A^{1/2} x_2.$$

By definition it follows that

$$(L + A)^{1/2} p - A^{1/2} q = \nabla G(x_2).$$

From now on $J$ and $J_D$ are considered on $X$ and $X^d$, respectively.

We have the following duality principle

**Theorem 2.1.**

$$\inf_{x \in X} J(x) = \inf_{(p,q) \in X^d} J_D(p, q).$$

Proof. We shall first consider a perturbation functional $J_{D,\text{pert}}\colon X^d \times Y \to \mathbb{R}$ given by the formula

$$J_{D,\text{pert}}(p, q, v) = -G^*((L + A)^{1/2} p - A^{1/2} q) - \frac{1}{2}\langle q, q\rangle + \frac{1}{2}\langle p + v, p + v\rangle.$$

Since $J_{D,\text{pert}}$ is convex with respect to $v$ for any fixed $(p, q)$ we may define its Fenchel-Young transform $J_D^{\#}\colon X^d \times X \to \mathbb{R}$, [4] (but with domain restricted to $X$ instead of $Y$)

$$\begin{aligned} J^{\#}_{D,\text{pert}}(p, q, x) = {} & \sup_{v \in Y} \Big\{ \langle (L + A)^{1/2} x, v\rangle - \frac{1}{2}\langle p + v, p + v\rangle \Big\} + \frac{1}{2}\langle q, q\rangle \\ & + G^*((L + A)^{1/2} p - A^{1/2} q). \end{aligned}$$

It reads

$$\begin{aligned} J^{\#}_{D,\text{pert}}(p, q, x) = {} & \frac{1}{2}\langle (L + A)^{1/2} x, (L + A)^{1/2} x\rangle - \langle (L + A)^{1/2} x, p\rangle + \frac{1}{2}\langle q, q\rangle \\ & + G^*((L + A)^{1/2} p - A^{1/2} q). \end{aligned}$$

Now we prove the following two relations.

(1) For any $(p, q) \in X^d$

$$\inf_{x \in X} J^{\#}_{D,\text{pert}}(p, q, x) = J_D(p, q),$$

(2) for any $x \in X$

$$\inf_{(p,q) \in X^d} J^{\#}_{D,\text{pert}}(p, q, x) = J(x).$$

For a given $(p,q)\in X^d$ there exists $x_p\in X$ satisfying $(L+A)^{1/2}x_p=p$. We have then the following equality

$$\frac{1}{2}\langle p,p\rangle=\langle x_p,(L+A)^{1/2}p\rangle-\frac{1}{2}\langle (L+A)^{1/2}x_p,(L+A)^{1/2}x_p\rangle.$$

Thus

$$\begin{aligned}\frac{1}{2}\langle p,p\rangle&\leqslant \sup_{x\in X}\Big\{\langle (L+A)^{1/2}x,p\rangle-\frac{1}{2}\langle (L+A)^{1/2}x,(L+A)^{1/2}x\rangle\Big\}\\&\leqslant \sup_{v\in Y}\Big\{\langle v,p\rangle-\frac{1}{2}\langle v,v\rangle\Big\}=\frac{1}{2}\langle p,p\rangle.\end{aligned}$$

This implies relation (1). To prove relation (2) let us fix $x\in X$. By the definition of $X^d$ for a given $x$ there exists $(p_x,q_x)\in X^d$ such that $(L+A)^{1/2}\tilde{x}=p_x$ and $A^{1/2}x=q_x$ where $\tilde{x}\in X$ is such that $(L+A)\tilde{x}-Ax=\nabla G(x)$. It follows that

$$(L+A)^{1/2}p_x-A^{1/2}q_x=\nabla G(x).$$

By the properties of Fenchel-Young transformation we have

$$G(x)+G^*((L+A)^{1/2}p_x-A^{1/2}q_x)=\langle x,(L+A)^{1/2}p_x-A^{1/2}q_x\rangle$$

and

$$\frac{1}{2}\langle A^{1/2}x,A^{1/2}x\rangle+\frac{1}{2}\langle q_x,q_x\rangle=\langle A^{1/2}x,q_x\rangle.$$

Hence

$$\begin{aligned}&G(x)+\frac{1}{2}\langle A^{1/2}x,A^{1/2}x\rangle\\&\quad=\langle A^{1/2}x,q_x\rangle-\frac{1}{2}\langle q_x,q_x\rangle+G(x)\\&\quad=\langle A^{1/2}x,q_x\rangle-\frac{1}{2}\langle q_x,q_x\rangle+\langle x,(L+A)^{1/2}p_x-A^{1/2}q_x\rangle\\&\qquad-G^*((L+A)^{1/2}p_x-A^{1/2}q_x)\\&\quad\leqslant \sup_{(p,q)\in X^d}\Big\{\langle x,(L+A)^{1/2}p-A^{1/2}q\rangle-G^*((L+A)^{1/2}p-A^{1/2}q)\\&\qquad\qquad+\langle A^{1/2}x,q\rangle-\frac{1}{2}\langle q,q\rangle\Big\}\\&\quad\leqslant \sup_{(p,q)\in X^d}\{\langle x,(L+A)^{1/2}p-A^{1/2}q\rangle-G^*((L+A)^{1/2}p-A^{1/2}q)\}\\&\qquad+\sup_{(p,q)\in X^d}\Big\{\langle A^{1/2}x,q\rangle-\frac{1}{2}\langle q,q\rangle\Big\}\\&\quad\leqslant \sup_{v\in Y}\{\langle x,v\rangle-G^*(v)\}+\frac{1}{2}\langle A^{1/2}x,A^{1/2}x\rangle=G(x)+\frac{1}{2}\langle A^{1/2}x,A^{1/2}x\rangle.\end{aligned}$$

This asserts that relation (2) holds. Both relations imply the following

$$\inf_{x\in X} J(x) = \inf_{x\in X} \inf_{(p,q)\in X^d} J^{\#}_{D,\mathrm{pert}}(p,q,x)$$
$$= \inf_{(p,q)\in X^d} \inf_{x\in X} J^{\#}_{D,\mathrm{pert}}(p,q,x) = \inf_{(p,q)\in X^d} J_D(p,q).$$

□

**Remark 2.2.** Due to the duality relations which we have introduced it was possible in above calculations to apply Fenchel-Young transform to functionals defined only on subsets.

## 3. Variational principles

Now we provide the necessary existence conditions.

**Theorem 3.1.** *Let* $(\overline{p},\overline{q}) \in X^d$ *be such that* $J_D(\overline{p},\overline{q}) = \inf_{(p,q)\in X^d} J_D(p,q)$. *There exist* $\overline{x} \in X$ *such that*

$$\inf_{x\in X} J(x) = J(\overline{x}) = J_D(\overline{p},\overline{q}) = \inf_{(p,q)\in X^d} J_D(p,q), \tag{3.1}$$

$$(L+A)^{1/2}\overline{x} = \overline{p}, \tag{3.2}$$

$$A^{1/2}\overline{x} = \overline{q}, \tag{3.3}$$

$$(L+A)^{1/2}\overline{p} - A^{1/2}\overline{q} = \nabla G(\overline{x}). \tag{3.4}$$

P r o o f. By the definitions of $X$ and $X^d$ there exist $\overline{x}, \tilde{x} \in X$ satisfying the following relations

$$(L+A)^{1/2}\overline{x} = \overline{p}, \quad A^{1/2}\tilde{x} = \overline{q},$$
$$(L+A)\overline{x} - A\tilde{x} = \nabla G(\tilde{x}).$$

Thus assertion (3.2) follows.

Now by Theorem 2.1 it follows that

$$J_D(\overline{p},\overline{q}) = \inf_{(p,q)\in X^d} J_D(p,q) = \inf_{x\in X} J(x) \leqslant J(\overline{x}),$$

so $J_D(\overline{p},\overline{q}) \leqslant J(\overline{x})$. By (3.2) and Fenchel-Young inequality we have

$$\begin{aligned}
-J(\overline{x}) + J_D(\overline{p},\overline{q}) = & -\frac{1}{2}\langle (L+A)^{1/2}\overline{x}, (L+A)^{1/2}\overline{x}\rangle + \frac{1}{2}\langle A^{1/2}\overline{x}, A^{1/2}\overline{x}\rangle + G(\overline{x}) \\
& -\frac{1}{2}\langle \overline{p},\overline{p}\rangle + \frac{1}{2}\langle \overline{q},\overline{q}\rangle + G^*((L+A)^{1/2}\overline{p} - A^{1/2}\overline{q}) \\
= & -\langle (L+A)^{1/2}\overline{x}, \overline{p}\rangle + G(\overline{x}) + G^*((L+A)^{1/2}\overline{p} - A^{1/2}\overline{q}) \\
& + \frac{1}{2}\|\overline{q}\|^2 + \frac{1}{2}\|A^{1/2}\overline{x}\|^2 \\
\geqslant & -\langle (L+A)^{1/2}\overline{x}, \overline{p}\rangle + \langle \overline{x}, (L+A)^{1/2}\overline{p} - A^{1/2}\overline{q}\rangle \\
& + \frac{1}{2}\|\overline{q}\|^2 + \frac{1}{2}\|A^{1/2}\overline{x}\|^2 \\
= & -\langle \overline{x}, A^{1/2}\overline{q}\rangle + \frac{1}{2}\|\overline{q}\|^2 + \frac{1}{2}\|A^{1/2}\overline{x}\|^2 \geqslant 0.
\end{aligned}$$

As a consequence $J(\overline{x}) = J_D(\overline{p},\overline{q})$ and (3.1) is obtained. The same argument as applied in demonstration the $J(\overline{x}) \geqslant J_D(\overline{p},\overline{q})$ shows that

$$\begin{aligned}
\frac{1}{2}\|\overline{q}\|^2 + \frac{1}{2}\|A^{1/2}\overline{x}\|^2 - \langle A^{1/2}\overline{x}, \overline{q}\rangle + G(\overline{x}) + G^*((L+A)^{1/2}\overline{p} - A^{1/2}\overline{q}) \\
-\langle \overline{x}, (L+A)^{1/2}\overline{p} - A^{1/2}\overline{q}\rangle = 0.
\end{aligned}$$

This implies, by the Fenchel-Young inequalities, that

$$\frac{1}{2}\|\overline{q}\|^2 + \frac{1}{2}\|A^{1/2}\overline{x}\|^2 = \langle A^{1/2}\overline{x}, \overline{q}\rangle,$$

$$G(\overline{x}) + G^*((L+A)^{1/2}\overline{p} - A^{1/2}\overline{q}) = \langle \overline{x}, (L+A)^{1/2}\overline{p} - A^{1/2}\overline{q}\rangle. \tag{3.5}$$

By the properties of the norm $\overline{q} = A^{1/2}\overline{x}$ and $\tilde{x} = \overline{x}$, thus relation (3.3) holds. By convexity, from relation (3.5) we obtain (3.4). □

Now we give the version of the above results but for minimizing sequences.

**Theorem 3.2.** *Let $\{(p_j, q_j)\} \subset X^d$ be a minimizing sequence for $J_D$ and let*

$$\inf_{j\in\mathbb{N}} J_D(p_j, q_j) = \inf_{(p,q)\in X^d} J_D(p,q).$$

*Then there exists a sequence $\{x_j\} \subset X$ minimizing for $J$ and such that*

$$x_j = (L+A)^{-1/2}p_j, \tag{3.6}$$

$$\inf_{x\in X} J(x) = \inf_{j\in\mathbb{N}} J(x_j) = \inf_{j\in\mathbb{N}} J_D(p_j, q_j) = \inf_{(p,q)\in X^d} J_D(p,q). \tag{3.7}$$

*For any $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$, such that for all $j \geqslant j_0$ we have*

$$0 \leqslant J_D(p_j, q_j) - J(x_j) \leqslant \varepsilon. \tag{3.8}$$

*Moreover*

$$\liminf_{j\to\infty} \Big(\frac{1}{2}\|A^{1/2}x_j\|^2 + \frac{1}{2}\|q_j\|^2 - \langle A^{1/2}x_j, q_j\rangle\Big) = 0, \tag{3.9}$$

$$\liminf_{j\to\infty} \big(G(x_j) + G^*((L+A)^{1/2}p_j - A^{1/2}q_j) - \langle x_j, (L+A)^{1/2}p_j - A^{1/2}q_j\rangle\big) = 0. \tag{3.10}$$

P r o o f. The existence of the sequence $\{x_j\}$ satisfying (3.6) we obtain as in the proof of Theorem 3.1. Now, by the properties of $\{(p_j, q_j)\}$ and by Theorem 2.1 we have

$$\inf_{j\in\mathbb{N}} J_D(p_j, q_j) = \inf_{(p,q)\in X^d} J_D(p, q) = \inf_{x\in X} J(x) \leqslant \inf_{j\in\mathbb{N}} J(x_j).$$

As in the proof of Theorem 3.1 we obtain $J_D(p_j, q_j) \geqslant J(x_j)$, so

$$\inf_{j\in\mathbb{N}} J(x_j) \leqslant \inf_{j\in\mathbb{N}} J_D(p_j, q_j).$$

Thus $\inf_{j\in\mathbb{N}} J(x_j) = \inf_{j\in\mathbb{N}} J_D(p_j, q_j)$ and as a result $\{x_j\}$ is a minimizing sequence of $J$ and relation (3.7) is satisfied.

Since $\{(p_j, q_j)\}$ minimizes $J_D$ and (3.7) holds, for a given $\varepsilon > 0$ we may choose $j_0 \in \mathbb{N}$ such that for all $j \geqslant j_0$

$$J(x_j) \leqslant J_D(p_j, q_j) \leqslant \inf_{j\in\mathbb{N}} J_D(p_j, q_j) + \varepsilon \leqslant J(x_j) + \varepsilon$$

which implies (3.8). This and (3.6) imply that

$$\begin{aligned} 0 &\leqslant G^*((L+A)^{1/2}p_j - A^{1/2}q_j) + G(x_j) + \frac{1}{2}\|q_j\|^2 + \frac{1}{2}\|A^{1/2}x_j\|^2 \\ &\quad - \frac{1}{2}\langle (L+A)^{1/2}x_j, p_j\rangle - \frac{1}{2}\langle (L+A)^{1/2}x_j, (L+A)^{1/2}(L+A)^{-1/2}p_j\rangle \leqslant \varepsilon, \\ 0 &\leqslant G^*((L+A)^{1/2}p_j - A^{1/2}q_j) + G(x_j) + \frac{1}{2}\|q_j\|^2 + \frac{1}{2}\|A^{1/2}x_j\|^2 \\ &\quad - \langle (L+A)^{1/2}x_j, p_j\rangle \leqslant \varepsilon. \end{aligned}$$

From the above, by a Fenchel-Young inequality, we have

$$\begin{aligned} &0 \leqslant \frac{1}{2}\|A^{1/2}x_j\|^2 + \frac{1}{2}\|q_j\|^2 - \langle A^{1/2}x_j, q_j\rangle \leqslant \varepsilon, \\ &0 \leqslant G(x_j) + G^*((L+A)^{1/2}p_j - A^{1/2}q_j) - \langle x_j, (L+A)^{1/2}p_j - A^{1/2}q_j\rangle \leqslant \varepsilon. \end{aligned}$$

Passing to lower limits we obtain (3.9) and (3.10). □

## 4. Existence of solutions

We shall prove now the existence of a triple $(\overline{x}, \overline{p}, \overline{q}) \in D(L) \times D((L+A)^{1/2}) \times D(A^{1/2})$ satisfying relations (1.3).

**Theorem 4.1.** *There exists a triple* $(\overline{x}, \overline{p}, \overline{q}) \in D(L) \times D((L+A)^{1/2}) \times D(A^{1/2})$ *such that*

$$(L+A)^{1/2}\overline{x} = \overline{p}, \tag{4.1}$$

$$A^{1/2}\overline{x} = \overline{q}, \tag{4.2}$$

$$(L+A)^{1/2}\overline{p} - A^{1/2}\overline{q} = \nabla G(\overline{x}), \tag{4.3}$$

$$J(\overline{x}) = \inf_{x\in X} J(x) = \inf_{(p,q)\in X^d} J_D(p,q) = J_D(\overline{p}, \overline{q}). \tag{4.4}$$

Proof. By (A4) there exists a constant $M > 0$ such that

$$\|(L+A)^{1/2}u\| \leqslant M \quad \text{and} \quad \|A^{1/2}u\| \leqslant M, \tag{4.5}$$

for every $u \in X$. This and the definition of $X^d$ imply that $\|p\|, \|q\| \leqslant M$ for every $p, q \in Y$. Hence we obtain that $X^d$ is relatively weakly compact in $Y \times Y$. By (A3) and Fenchel-Young inequality

$$G^*((L+A)^{1/2}p - A^{1/2}q) + G(0) \geqslant 0$$

we conclude that $J_D$ is bounded from below on $X^d$ and thus we may choose its minimizing sequence $\{(p_j, q_j)\}$ which may be assumed to be weakly convergent in $Y \times Y$. We denote its limit by $(\overline{p}, \overline{q})$. By Theorem 3.2 it follows that there exists a sequence $\{x_j\} \subset X$ satisfying

$$(L+A)^{1/2}x_j = p_j. \tag{4.6}$$

From the above and (4.5) we have that $\{x_j\}$ is weakly convergent in $D((L+A)^{1/2})$, thus by (A1) strongly in $Y$. We denote its limit by $\overline{x}$. Therefore (4.1) holds. By (A1) and the weak convergence of $\{x_j\}$ in $D((L+A)^{1/2})$ we have that it converges strongly in $D(A^{1/2})$.

By the definition of $X^d$ there exist a sequence $\{\tilde{x}_j\} \subset X$ such that

$$q_j = A^{1/2}\tilde{x}_j. \tag{4.7}$$

The first inequality in (4.5) and (A2) imply that $\{A\tilde{x}_j\}$ is bounded, so $\{q_j\}$ is weakly convergent in $D(A^{1/2})$, thus strongly in $Y$. By Theorem 3.2 and a Fenchel-Young inequality we get

$$\begin{aligned}
0 &= \liminf_{j\to\infty}\Big(\frac{1}{2}\|A^{1/2}x_j\|^2+\frac{1}{2}\|q_j\|^2-\langle A^{1/2}x_j,q_j\rangle\Big)\\
&\geqslant \lim_{j\to\infty}\Big(\frac{1}{2}\|A^{1/2}x_j\|^2\Big)+\lim_{j\to\infty}\Big(\frac{1}{2}\|q_j\|^2\Big)-\lim_{j\to\infty}\langle A^{1/2}x_j,q_j\rangle\\
&= \frac{1}{2}\|A^{1/2}\overline{x}\|^2+\frac{1}{2}\|\overline{q}\|^2-\langle A^{1/2}\overline{x},\overline{q}\rangle \geqslant 0.
\end{aligned}$$

This implies that (4.2) follows.

Now we shall show that (4.3) holds. By (4.7), (4.2), convergence $q_j \rightharpoonup \overline{q}$ and continuity of $A^{-1/2}$ we have

$$\tilde{x}_j = A^{-1/2}q_j \rightharpoonup A^{-1/2}\overline{q} = \overline{x},$$

thus $\nabla G(\tilde{x}_j) \rightharpoonup \nabla G(\overline{x})$. Moreover, by (4.6), (4.7) and the definition of $X^d$ we have

$$(L+A)^{1/2}p_j - A^{1/2}q_j = \nabla G(\tilde{x}_j). \tag{4.8}$$

For $f \in D((L+A)^{1/2})$, from (4.1) and (4.2) we obtain

$$\begin{aligned}
\lim_{j\to\infty}\langle (L+A)^{1/2}p_j - A^{1/2}q_j, f\rangle &= \langle \overline{p},(L+A)^{1/2}f\rangle - \langle \overline{q}, A^{1/2}f\rangle\\
&= \langle (L+A)^{1/2}\overline{x},(L+A)^{1/2}f\rangle - \langle A^{1/2}\overline{x}, A^{1/2}f\rangle\\
&= \langle (L+A)\overline{x}, f\rangle - \langle A\overline{x}, f\rangle\\
&= \langle (L+A)^{1/2}\overline{p} - A^{1/2}\overline{q}, f\rangle,
\end{aligned}$$

so $(L+A)^{1/2}p_j - A^{1/2}q_j \rightharpoonup (L+A)^{1/2}\overline{p} - A^{1/2}\overline{q}$. The uniqueness of the weak limit asserts (4.3). By relation (4.8) it now follows that $\{(L+A)^{1/2}p_j\}$ is bounded in $Y$. Thus $\{p_j\}$ is, up to a subsequence, strongly convergent in $Y$.

To conclude, we shall show that $J_D(\overline{p},\overline{q}) = \inf_{(p,q)\in X^d} J_D(p,q)$. Observe that $J_D$ is weakly lower semicontinuous on $\{(p_j,q_j)\}$. Indeed, the functional

$$D((L+A)^{1/2})\times D(A^{1/2}) \ni (p,q) \mapsto G^*((L+A)^{1/2}p - A^{1/2}q) + \frac{1}{2}\langle q,q\rangle \in \mathbb{R}$$

is, by (A3), weakly lower semicontinuous on $D((L+A)^{1/2})\times D(A^{1/2})$. Moreover, by the above argument $\{p_j\}$ is strongly convergent to $\overline{p}$ in $Y$. This implies the weak lower semicontinuity of $J_D$

$$\liminf_{j\to\infty} J_D(p_j,q_j) \geqslant J_D(\overline{p},\overline{q}).$$

Thus

$$J_D(\overline{p},\overline{q}) = \inf_{(p,q)\in X^d} J_D(p,q).$$

By equality $J(\overline{x}) = J_D(\overline{p},\overline{q})$, the above relation and Duality Principle we have (4.4). □

The following corollary is a consequence of Theorem 4.1 and the definition of $X$.

**Corollary 4.2.** *There exists* $\overline{x} \in X$ *such that*

$$L\overline{x} = \nabla G(\overline{x}). \tag{4.9}$$

Proof. From (4.1), (4.2) and (4.3) we easily obtain that

$$(L + A)\overline{x} = A\overline{x} + \nabla G(\overline{x}),$$

thus (4.9) holds. Moreover, since $(L + A)$ is invertible, we have from the above equality that

$$\overline{x} = (L + A)^{-1}(A\overline{x} + \nabla G(\overline{x}),$$

so by definition of $X$ we conclude that $\overline{x} \in X$. □

## 5. Stability results

We shall prove the stability of solutions to the problems

$$Lx = \nabla G_k(x), \tag{5.1}$$

where for each $k = 0, 1, 2, \dots$ $G_k \colon Y \to Y$ satisfies (A3).

Let us recall that the family of equations (5.1) is said to be stable provided that from a sequence $\{x_k\}_{k=1}^{\infty}$, $x_k \in X_k$ of solutions to (5.1), one can choose a subsequence strongly convergent in $Y$ to a certain $\overline{x}$ being a solution to the problem

$$L\overline{x} = \nabla G_0(\overline{x}).$$

We assume (A1)–(A4) and that for each $k = 0, 1, 2, \dots$

(A5) There exists a weakly compact convex set $B \subset D((L + A)^{1/2})$ such that $X_k \subset B$ for $k = 0, 1, 2, \dots$ and $\{\nabla G_k\}$ is uniformly bounded on $B$.

**Theorem 5.1.** *Assume* (A1)–(A5) *and that for any $x \in B$ there exists a subsequence $\{k_i\}$ such that*

$$\nabla G_{k_i}(x) \rightharpoonup \nabla G_0(x)$$

*weakly in $Y$. For each $k = 1, 2, \dots$ there exists a solution $x_k$ to* (5.1)*. Moreover, there exists $\overline{x} \in D(L)$ being a solution to*

$$L\overline{x} = \nabla G_0(\overline{x})$$

*and such that $\lim\limits_{n\to\infty} x_{k_n} = \overline{x}$ strongly in $Y$, where $\{x_{k_n}\}$ is a certain subsequence of $\{x_k\}$.*

P r o o f. By the Theorem 4.1, for each $k = 1, 2, \dots$ there exists a triple $(x_k, p_k, q_k) \in D(L) \times D((L+A)^{1/2}) \times D(A^{1/2})$ such that

$$\begin{aligned}(L+A)^{1/2}x_k &= p_k,\\ A^{1/2}x_k &= q_k,\\ (L+A)^{1/2}p_k - A^{1/2}q_k &= \nabla G_k(x_k).\end{aligned}$$

By (A5) it follows that from a sequence $\{x_k\}$ we may choose a subsequence weakly converging in $D((L+A)^{1/2})$ to a certain $\overline{x} \in D((L+A)^{1/2})$. This sequence has a subsequence, still denoted by $\{x_k\}$, which by (A1) converges strongly in $Y$. Using the argument applied in the proof of Theorem 4.1 we obtain that the sequences $\{p_k\}$, $\{q_k\}$ are weakly convergent in $Y$. We denote their limits by $\overline{p}$, $\overline{q}$, respectively. Let us take a subsequence $\{k_i\}$ such that $\lim\limits_{i\to\infty} \nabla G_{k_i}(\overline{x}) = \nabla G_0(\overline{x})$ weakly. We denote all the subsequences by the subscript $k$ for simplicity.

We will begin with proving that

$$(L+A)^{1/2}\overline{x} = \overline{p}.$$

By (A2) and (A5) the sequence $A^{1/2}q_k = Ax_k$ is bounded. Thus, since $\{x_k\}$ is bounded, the sequence

$$(L+A)^{1/2}p_k = A^{1/2}q_k + \nabla G_k(x_k)$$

is also bounded. We can infer then the weak convergence of $\{p_k\}$ in $D((L+A)^{1/2})$ and thus strong in $Y$. By the uniqueness of the weak limit

$$(L+A)^{1/2}\overline{x} = \overline{p}.$$

The proof that $A^{1/2}q_k \rightharpoonup A^{1/2}\overline{q}$ weakly and $A^{1/2}\overline{x} = \overline{q}$ follow by the same argument.

We only need to prove that

$$(L+A)^{1/2}\overline{p} - A^{1/2}\overline{q} = \nabla G_0(\overline{x}).$$

By convexity of $G_k$ we get for any $x \in Y$

$$\begin{aligned} 0 &\leqslant \langle \nabla G_k(x_k) - \nabla G_k(x), x_k - x\rangle \\ &= \langle Lx_k - \nabla G_k(x), x_k - x\rangle \\ &= \langle Lx_k + (\nabla G_0(x) - \nabla G_k(x)) - \nabla G_0(x), x_k - x\rangle. \end{aligned}$$

Hence, by the strong convergence $x_k \to \overline{x}$, weak $\nabla G_k(x) \rightharpoonup \nabla G_0(x)$ we have

$$\langle (\nabla G_0(x) - \nabla G_k(x)) - \nabla G_0(x), x_k - x\rangle \to \langle -\nabla G_0(x), \overline{x} - x\rangle.$$

In addition, by the weak convergence $(L+A)^{1/2}p_k \rightharpoonup (L+A)^{1/2}\overline{p}$ and $A^{1/2}q_k \rightharpoonup A^{1/2}\overline{q}$ we obtain

$$\begin{aligned} \langle Lx_k, x_k - x\rangle &= \langle (L+A)x_k - Ax_k, x_k - x\rangle \\ &= \langle (L+A)^{1/2}p_k, x_k - x\rangle - \langle A^{1/2}q_k, x_k - x\rangle \\ &\quad \to \langle (L+A)^{1/2}\overline{p}, \overline{x} - x\rangle - \langle A^{1/2}\overline{q}, \overline{x} - x\rangle. \end{aligned}$$

We may conclude that for any $x \in D(L)$

$$\langle (L+A)^{1/2}\overline{p} - A^{1/2}\overline{q} - \nabla G_0(x), \overline{x} - x\rangle \geqslant 0.$$

Now we apply the Minty "trick" i.e. we consider the points $\overline{x}+tx$, where $x \in D(L)$ and $t > 0$. The last inequality provides that

$$\langle (L+A)^{1/2}\overline{p} - A^{1/2}\overline{q} - \nabla G_0(\overline{x}+tx), x\rangle \leqslant 0.$$

By convexity of the function $[-1,1] \ni t \mapsto G_0(\overline{x}+tx) \in \mathbb{R}$ it follows that

$$\begin{aligned} 0 &\geqslant \lim_{t\to 0}\langle (L+A)^{1/2}\overline{p} - A^{1/2}\overline{q} - \nabla G_0(\overline{x}+tx), x\rangle \\ &= \langle (L+A)^{1/2}\overline{p} - A^{1/2}\overline{q} - \nabla G_0(\overline{x}), x\rangle \end{aligned}$$

for any $x \in D(L)$. Finally, by the fact that $D(L)$ is dense in $Y$ we conclude that

$$(L+A)^{1/2}\overline{p} - A^{1/2}\overline{q} = \nabla G_0(\overline{x})$$

and therefore

$$L\overline{x} = \nabla G_0(\overline{x}).$$

□

We observe that, by Theorem 4.1 and Corollary 4.2, there exists $x_0 \in X_0$ such that

$$Lx_0 = \nabla G_0(x_0) \quad \text{and} \quad \min_{x\in X_0} J(x) = J(x_0).$$

The following corollary shows that under some additional assumptions $\overline{x}$ minimizes $J_0$ on $X_0$.

**Corollary 5.2.** *Under the assumptions of Theorem* 5.1, *if $X_k \subset X_0$ holds only for almost all $k$, $X_0$ is convex and $\limsup_{k\to\infty}(G_k(x_0) - G_0(x_0)) \leqslant 0$, then $\overline{x} \in X_0$ and it minimizes $J_0$ on $X_0$.*

P r o o f. Due to the assumptions of the corollary we may put $X_0 = B$ in Theorem 5.1. By the weak compactness of $X_0$ it follows that from the sequence $\{x_k\}$ one can choose a subsequence, still denoted by $\{x_k\}$, weakly converging in $X_0$ to $\overline{x}$.

Let us suppose that $\overline{x}$ does not minimize $J_0$ on $X_0$ i.e.

$$J_0(\overline{x}) - J_0(x_0) > 0, \tag{5.2}$$

where $x_0$ is a point minimizing $J_0$ on $X_0$, provided by Theorem 3.1. Due to weak lower semicontinuity of $J_0$ we have

$$\liminf_{k\to\infty}(J_0(x_k) - J_0(\overline{x})) \geqslant 0. \tag{5.3}$$

Hence, by the following equality

$$J_0(\overline{x}) - J_0(x_0) = (J_k(x_k) - J_0(x_0)) - (J_k(x_k) - J_0(x_k)) - (J_0(x_k) - J_0(\overline{x}))$$

the proof will be finished by showing that

$$\lim_{k\to\infty}(J_k(x_k) - J_0(x_k)) = 0 \tag{5.4}$$

and

$$\liminf_{k\to\infty}(J_k(x_k) - J_0(x_0)) \leqslant 0. \tag{5.5}$$

We have

$$|G_0(x_k) - G_k(x_k)| \leqslant |G_0(x_k) - G_0(\overline{x})| + |G_k(\overline{x}) - G_0(\overline{x})| + |G_k(x_k) - G_k(\overline{x})|.$$

By (A5) we obtain the following inequalities

$$G_0(x_k) - G_0(\overline{x}) \geqslant \langle \nabla G_0(\overline{x}), x_k - \overline{x}\rangle,$$
$$G_0(\overline{x}) - G_0(x_k) \geqslant \langle \nabla G_0(x_k), \overline{x} - x_k\rangle$$

and

$$G_k(x_k) - G_k(\overline{x}) \geqslant \langle \nabla G_k(\overline{x}), x_k - \overline{x} \rangle,$$
$$G_k(\overline{x}) - G_k(x_k) \geqslant \langle \nabla G_k(x_k), \overline{x} - x_k \rangle.$$

Thus

$$|G_0(x_k) - G_0(\overline{x})| \leqslant \max\{|\langle \nabla G_0(\overline{x}), x_k - \overline{x} \rangle|, |\langle \nabla G_0(x_k), \overline{x} - x_k \rangle|\},$$
$$|G_k(x_k) - G_k(\overline{x})| \leqslant \max\{|\langle \nabla G_k(\overline{x}), x_k - \overline{x} \rangle|, |\langle \nabla G_k(x_k), \overline{x} - x_k \rangle|\}.$$

Consequently, by the strong convergence $x_k \to \overline{x}$, weak convergence $G_k(\overline{x}) \rightharpoonup G_0(\overline{x})$ and by the boundedness of $\nabla G_k(X_k)$

$$\lim_{k\to\infty} (J_k(x_k) - J_0(x_k)) = \lim_{k\to\infty} (G_0(x_k) - G_k(x_k)) = 0,$$

so (5.4) is shown.

Now since $x_k$ minimizes $J_k$ and $\limsup_{k\to\infty}(G_k(x_0) - G_0(x_0)) \leqslant 0$ we have

$$\begin{aligned}\liminf_{k\to\infty}(J_k(x_k) - J_0(x_0)) &\leqslant \liminf_{k\to\infty}(J_k(x_0) - J_0(x_0)) \\ &= \liminf_{k\to\infty}(G_0(x_0) - G_k(x_0)) \leqslant 0,\end{aligned}$$

so (5.5) is proved. □

## 6. Applications

We are now on the point of giving an example of the equation with superlinear growth conditions imposed on its right-hand side and such that our method can be applied.

### 6.1. Existence of solutions

We begin with general idea of construction of set $X$ in concrete application. Here $Y = L^2(0,1;\mathbb{R})$. Consider the following Dirichlet problem

$$\text{(6.1)} \qquad \begin{cases} -x^{(6)} + x^{(4)} + 4\ddot{x} = G_x(t, x(t)), \\ x(0) = x(1) = \dot{x}(0) = \dot{x}(1) = \ddot{x}(0) = \ddot{x}(1) = 0, \end{cases}$$

where $L = -x^{(6)} + x^{(4)} + 4\ddot{x}$, $D(L) = H^6(0,1) \cap H_0^3(0,1)$. The operator $L$ is self-adjoint but not positive definite. We may put $Ax = -4\ddot{x}$ which is clearly positive definite, $D(A) = H^2(0,1) \cap H_0^1(0,1)$. Now $L + A = -x^{(6)} + x^{(4)}$ is positive definite as well.

Assumptions (A1), (A2) are clearly satisfied due to Poincaré inequalities and properties of the spaces. Let us assume as follows:

(Ap1) There exist $d > 0$ such that

$$G_x(t,-d), G_x(t,d) \in L^2(0,1;\mathbb{R}) \quad \text{and} \quad |G_x(t,-d)| \leqslant |G_x(t,d)|$$

for a.e. $t \in [0,1]$. Moreover

$$\|G_x(\cdot,d)\| \leqslant 4\sqrt{3}\pi d(\pi^2(4\pi^2-1)-1). \tag{6.2}$$

(Ap2) $G, G_x\colon [0,1]\times[-d,d] \to \mathbb{R}$ are Carathéodory functions, $G$ is convex in second variable and $G(t,x) = +\infty$ for $(t,x) \in [0,1]\times(\mathbb{R}\setminus[-d,d])$.

Therefore (A3) also holds. We need to construct a certain set $X$ and assert that the relation (2.1) is satisfied. We put

$$\tilde{X} = \{x \in H^6(0,1)\cap H_0^3(0,1)\colon\ \|x^{(3)}\| \leqslant 2\sqrt{3}\pi^2 d,\ |x(t)| \leqslant d \text{ for } t \in (0,1)\}$$

and now prove

**Proposition 6.1.** $X = \tilde{X}$.

Consider the equation

$$-x^{(6)} + x^{(4)} = -4\ddot{u} + G_x(t,u)$$

for an arbitrary $u \in \tilde{X}$. We will show that $x \in \tilde{X}$. From the above equation we obtain

$$\|-x^{(6)} + x^{(4)}\| \leqslant 4\|u^{(2)}\| + \|G_x(t,u)\|. \tag{6.3}$$

By the Poincaré inequality and convexity of $G$ together with $|G_x(t,-d)| \leqslant |G_x(t,d)|$ we get

$$4\|u^{(2)}\| + \|G_x(t,u)\| \leqslant \frac{2}{\pi}\|u^{(3)}\| + \|G_x(t,d)\| \tag{6.4}$$

and

$$\|-x^{(6)} + x^{(4)}\| \geqslant \|x^{(6)}\| - \|x^{(4)}\| \geqslant (4\pi^2-1)\|x^{(4)}\| \geqslant 2\pi(4\pi^2-1)\|x^{(3)}\|. \tag{6.5}$$

Combining (6.4) and (6.5) with (6.3) we have

$$2\pi(4\pi^2-1)\|x^{(3)}\| \leqslant \frac{2}{\pi}\|u^{(3)}\| + \|G_x(t,d)\|$$

which leads to

$$\|x^{(3)}\| \leqslant \frac{1}{\pi^2(4\pi^2-1)}\|u^{(3)}\| + \frac{1}{2\pi(4\pi^2-1)}\|G_x(t,d)\|.$$

Since $u \in \tilde{X}$ it follows by (6.2) that

$$\|x^{(3)}\| \leqslant \frac{2\sqrt{3}d}{4\pi^2-1} + \frac{2\sqrt{3}d(\pi^2(4\pi^2-1)-1)}{4\pi^2-1} = 2\sqrt{3}\pi^2 d. \tag{6.6}$$

To conclude that $x \in \tilde{X}$ we need to show that $|x(t)| \leqslant d$ for $t \in (0,1)$. By the Wirtinger inequality and the Poincaré inequality we obtain

$$\|x\|_\infty^2 \leqslant \frac{1}{12}\|\dot{x}\|^2 \leqslant \frac{1}{12}\cdot\frac{1}{16\pi^4}\|x^{(3)}\|^2.$$

Thus

$$2\sqrt{3}\cdot 4\pi^2\|x\|_\infty \leqslant \|x^{(3)}\|,$$
$$2\sqrt{3}\cdot 4\pi^2 \max_{t\in[0,1]} |x(t)| \leqslant \|x^{(3)}\|.$$

In consequence, by (6.6) we have for each $t \in (0,1)$ that $|x(t)| \leqslant d$. So the relation $x \in \tilde{X}$ is shown and we may put $X = \tilde{X}$.

To demonstrate (A4) we need to show that $\nabla G(X)$ is weakly compact (which holds by (6.2) and the construction of $X$) and that for all sequences $\{x_n(\cdot)\}$ strongly convergent in $L^2(0,1;\mathbb{R})$ the sequence $G_x(\cdot,x_n(\cdot))$ converges weakly in $L^2(0,1;\mathbb{R})$. The last assertion also follows by (6.2) and by the application of the generalized Krasnoselskii Theorem and the continuity of the Nemytskii operator, see [9].

Therefore we have the main result of this section

**Theorem 6.2.** *Assume* (Ap1) *and* (Ap2). *Then there exists a solution to the Dirichlet problem* (6.1).

We present now some examples of functions $G$ for which the constant $d$ in (Ap1) will be given. The acceptable range of $d$ depends strictly on $G$ and must be estimated or calculated separately in each case.

**Example 6.3.** Let $G(t,x) = \frac{1}{4}x^4 + g(t)x$, where $g \in L^2(0,1;\mathbb{R})$ is such that $0 \leqslant g(t) \leqslant d$. The constant $d > 0$ may be chosen arbitrary from these satisfying the inequality

$$d^2 \leqslant 4\sqrt{3}\pi\left(\pi^2(4\pi^2-1)-1\right)-1. \tag{6.7}$$

Clearly, $G$ satisfies (Ap2). We will show that (Ap1) also holds.

Since $g$ is nonnegative we have

$$|G_x(t,-d)| = |-d^3 + g(t)| \leqslant |d^3 + g(t)| = |G_x(t,d)|$$

for all $t \in [0,1]$. The relation $g \in L^2(0,1;\mathbb{R})$ implies that both $G_x(t,-d)$ and $G_x(t,d)$ are in $L^2(0,1;\mathbb{R})$, so only (6.2) remains to be shown. Since $0 \leqslant g(t) \leqslant d$, it follows by (6.7) that

$$\begin{aligned}\|G_x(\cdot,d)\|^2 &= \|d^3 + g(\cdot)\|^2 \leqslant |d^3+d|^2 \\ &= d^2(d^2+1)^2 \leqslant d^2 \cdot 48\pi^2(\pi^2(4\pi^2-1)-1)^2.\end{aligned}$$

This implies (6.2).

Consider the following subcritical case

**Example 6.4.** Let $G(t,x) = \frac{2}{3}|x|^{3/2} \cdot g(t) + h(t)x$. Suppose $g, h \in L^2(0,1;\mathbb{R})$, $g(t)\cdot h(t) \geqslant 0$ and $|g(t)|, |h(t)| \leqslant 4\sqrt{3}\pi d$ for a.e. $t \in [0,1]$, where $d > 0$ satisfies

$$\sqrt{d} \leqslant \pi^2(4\pi^2-1)-2. \tag{6.8}$$

$G_x$ now reads

$$G_x(t,x) = \operatorname{sgn}(x)\sqrt{|x|}\cdot g(t) + h(t).$$

Of course (Ap2) holds and $G_x(t,-d), G_x(t,d) \in L^2(0,1;\mathbb{R})$. Since $g$ and $h$ are either both nonnegative or nonpositive it follows that $|G_x(t,-d)| \leqslant |G_x(t,d)|$, so only (6.2) needs to be proved. By (6.8) and estimation of $|g(\cdot)|$, $|h(\cdot)|$ we have

$$\|G_x(\cdot,d)\| = \|\sqrt{d}\cdot g(\cdot) + h(\cdot)\| \leqslant 4\sqrt{3}\pi d\left(\sqrt{d}+1\right) \leqslant \sqrt{3}\pi d(\pi^2(4\pi^2-1)-1),$$

so (6.2) holds.

Our method can be applied in case the exponential growth is imposed on the potential.

**Example 6.5.** Let $G(t,x)=\frac{1}{2}x^2g(t)+\mathrm{e}^x$. Assume $d>0$ satisfies

$$d^2[48\pi^2(\pi^2(4\pi^2-1)-1)^2\mathrm{e}^{-d}-1]\geqslant 1 \tag{6.9}$$

and $g\in L^2(0,1;\mathbb{R})$ is such that $0\leqslant g(t)\leqslant d\mathrm{e}^d$.

First of all, let us present two properties of the family of functions $f_M(d)=d^2(M\mathrm{e}^{-d}-1)-1$ for $M>0$, i.e.

(1) $\lim\limits_{d\to\infty} f_M(d)=-\infty$ for each $M>0$.

(2) There exists a constant $M_0>0$ such that for each $M>M_0$ the function $f_M$ is positive only on some bounded interval $(d_0,d_1)$.

Numerical experiments provide that $M_0=5.5$ approximately and the diameter of the aforementioned interval increases as $M$ increases. The same experiments show that for $M=48\pi^2(\pi^2(4\pi^2-1)-1)^2$ the interval on which $f_M$ is positive includes $[10^{-3},18]$.

As previously we have that $G_x(t,-d),G_x(t,d)\in L^2(0,1;\mathbb{R})$ and by the positivity of $g$ we conclude $|G_x(t,-d)|\leqslant|G_x(t,d)|$. As for (6.2) we have by (6.9) and the estimation of $g$ that

$$\|G_x(\cdot,d)\|^2=\|dg(\cdot)+\mathrm{e}^d\|^2\leqslant \mathrm{e}^d(d^2+1)\leqslant \mathrm{e}^d\cdot 48\pi^2d^2(\pi^2(4\pi^2-1)-1)^2\cdot \mathrm{e}^{-d}$$

so (6.2) is now shown.

### 6.2. Stability of solutions

We consider the problem

$$\begin{cases} -x^{(6)}+x^{(4)}+4\ddot{x}=\nabla G_k(t,x(t)),\\ x(0)=x(1)=\dot{x}(0)=\dot{x}(1)=\ddot{x}(0)=\ddot{x}(1)=0. \end{cases} \tag{6.10}$$

We assume

(Sk1) There exists a sequence $\{d_k\}$ such that $0<d_k\leqslant d_0$, $\nabla G_k(\cdot,-d_k)$, $\nabla G_k(\cdot,d_k)\in L^2(0,1;\mathbb{R})$ and $|\nabla G_k(t,-d_k)|\leqslant|\nabla G_k(t,d_k)|$ for a.e. $t\in[0,1]$. Moreover

$$\|\nabla G_k(\cdot,d_k)\|\leqslant 4\sqrt{3}\pi d_k(\pi^2(4\pi^2-1)-1).$$

(Sk2) $G_k$, $\nabla G_k$ are Carathéodory functions, $G_k$ is convex in second variable on the interval

$$I=[-d_0,d_0]$$

and equals $+\infty$ outside $I$. $G_k(t,0)<+\infty$, $\nabla G_k(t,0)\neq 0$ for a.e. $t\in[0,1]$.

We need to construct sets $X_k$, $B$ in order to show that (A4) and (A5) are satisfied. We put

$$\tilde{X}_k = \{x \in H^6(0,1) \cap H_0^3(0,1)\colon \ \|x^{(3)}\| \leqslant 2\sqrt{3}\pi^2 d_k, \ |x(t)| \leqslant d_k \text{ for } t \in (0,1)\}.$$

For a fixed $k$ it follows by Proposition 6.1 that we may take $X_k = \tilde{X}_k$. Again, $X_k$ and $\nabla G_k(X_k)$ are weakly compact in $L^2(0,1;\mathbb{R})$. To conclude our reasoning notice that due to (Sk1) the set

$$B = \{x \in H^6(0,1) \cap H_0^3(0,1)\colon \ \|x^{(3)}\| \leqslant 2\sqrt{3}\pi^2 d_0, \ x(t) \in I \text{ for } t \in (0,1)\}$$

satisfies the conditions in (A5). Thus we have the following

**Theorem 6.6.** *Assume* (Sk1) *and* (Sk2). *For each* $k = 1, 2, \ldots$ *there exists a solution* $x_k$ *to the problem* (6.10). *Moreover, there exist a subsequence* $\{x_{k_i}\}$ *of the sequence* $\{x_k\}$ *and* $\overline{x} \in H^6(0,1) \cap H_0^3(0,1)$ *such that* $\lim\limits_{i\to\infty} x_{k_i} = \overline{x}$ *strongly in* $L^2(0,1;\mathbb{R})$ *and*

$$-\overline{x}^{(6)} + \overline{x}^{(4)} + 4\ddot{\overline{x}} = \nabla G_0(\cdot, \overline{x}(\cdot)).$$

P r o o f. In order to apply Theorem 5.1 we need to show that for any $x \in B$ there exists a subsequence $\{k_i\}$ such that

$$\nabla G_{k_i}(\cdot, x(\cdot)) \rightharpoonup \nabla G_0(\cdot, x(\cdot))$$

weakly in $Y$. Indeed, it follows by the generalization of Krasnoselskii Theorem [9] and the fact that convexity of $G_k$ and (Sk1) imply that the sequence $\{\nabla G_k(\cdot, x(\cdot))\}$ is bounded in $L^2(0,1;\mathbb{R})$. $\square$

### 6.3. Dependence on parameters

We will now concentrate on similar problem as above but with some additional parameter taken from a certain set. Let us consider the problem

$$(6.11) \qquad \begin{cases} -x^{(6)} + x^{(4)} + 4\ddot{x} = \nabla G(t, x(t), u(t)), \\ x(0) = x(1) = \dot{x}(0) = \dot{x}(1) = \ddot{x}(0) = \ddot{x}(1) = 0, \end{cases}$$

where $\nabla G(t, x(t), u(t))$ denotes the derivative with respect to the second variable. Assume $M \subset \mathbb{R}^m$ is a given bounded set and the parameter $u(\cdot)$ is a element of the set

$$\mathscr{U}_M = \{u\colon \ [0,1] \to \mathbb{R}^m : \ u \text{ is measurable}, u(t) \in M \text{ a.e.}\}.$$

We assume as follows

(Dk1) There exists a constant $d > 0$ such that $\nabla G(\cdot, -d, u(\cdot)), \nabla G(\cdot, d, u(\cdot)) \in L^2(0,1;\mathbb{R})$, $|\nabla G(t, -d, u(t))| \leqslant |\nabla G(t, d, u(t))|$ and

$$\|\nabla G(\cdot, d, u(\cdot))\| \leqslant 4\sqrt{3}\pi d(\pi^2(4\pi^2 - 1) - 1)$$

for all $u \in \mathscr{U}_M$ and a.e. $t \in [0,1]$.

(Dk2) $G, \nabla G\colon [0,1] \times \mathbb{R} \times M \to \mathbb{R}$ are Carathéodory functions, i.e. they are measurable in the first variable and continuous with respect to the last two variables. $G$ is convex in second variable on $I = [-d, d]$ and equals $+\infty$ outside $I$ for all $u \in \mathscr{U}_M$.

As above we put

$$X = \{x \in H^6(0,1) \cap H_0^3(0,1)\colon \ \|x^{(3)}\| \leqslant 2\sqrt{3}\pi^2 d, \ x(t) \in I \text{ for } t \in (0,1)\}$$

and take $B = X$. We have the following

**Theorem 6.7.** *Assume* (Dk1), (Dk2) *and* $\mathscr{U}_M \ni u_k \to \overline{u}$ *in* $L^2(0,1;\mathbb{R})$. *Then for each* $k \in \mathbb{N}$ *there exists a solution to* (6.11). *Moreover, there exists a subsequence* $\{x_{k_i}\}$ *of the sequence* $\{x_k\}$ *and* $\overline{x} \in H^6(0,1) \cap H_0^3(0,1)$ *such that* $\lim_{i\to\infty} x_{k_i} = \overline{x}$ *strongly in* $L^2(0,1;\mathbb{R})$ *and*

$$-\overline{x}^{(6)} + \overline{x}^{(4)} + 4\ddot{\overline{x}} = \nabla G(t, \overline{x}(t), \overline{u}(t)).$$

Proof. By convexity of $G$, (Dk1) and by the generalization of the Krasnoselskii Theorem [9] we have for all $x \in H^6(0,1) \cap H_0^3(0,1)$

$$\lim_{i\to\infty} \nabla G(\cdot, x(\cdot), u_{k_i}(\cdot)) = \nabla G(\cdot, x(\cdot), \overline{u}(\cdot)).$$

Now it suffices to put $G_{k_i}(\cdot, x(\cdot)) = G(\cdot, x(\cdot), u_{k_i}(\cdot))$ in Theorem 5.1. □

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