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LEXICOGRAPHIC PRODUCT DECOMPOSITIONS OF HALF LINEARLY ORDERED LOOPS

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Abstract. In this paper we prove for an hl-loop Q an assertion analogous to the result of Jakubík concerning lexicographic products of half linearly ordered groups. We found conditions under which any two lexicographic product decompositions of an hl-loop Q with a finite number of lexicographic factors have isomorphic refinements.

Keywords: half linearly ordered quasigroup, half linearly ordered loop, lexicographic product, isomorphic refinements

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1. Introduction

The notion of a half linearly ordered group has been introduced by Giraudet and Lucas [6]. Lexicographic products of half linearly ordered groups were discussed by Jakubík in [9].

In the present paper we define the Φ-lexicographic product of half linearly ordered loops. This definition includes as a particular case the lexicographic product of half linearly ordered groups and also the lexicographic product of linearly ordered loops. Here we will prove the following assertion analogous to [9; Theorem 4.5]: Let Q be a half linearly ordered loop and let there exist a set of representatives R of Q such that R is a subgroupoid of Q. Then any two lexicographic product decompositions of Q with a finite number of lexicographic factors have isomorphic refinements.

The analogous theorem for lexicographic product decompositions of linearly ordered groups was proved by Maltsev [10]; this result was generalized by Fuchs [5]. Further, lexicographic product decompositions of some types of ordered algebraic structures were dealt with in the papers [2], [3], [7], [8].

2. Preliminaries

General information concerning quasigroups can be found in [1]. Recall that a quasigroup Q is defined as an algebra having a binary operation "$\cdot$" which satisfies the condition that for any $a, b \in Q$ the equations $ax = b$ and $ya = b$ have unique solutions x and y. A quasigroup Q having an identity element 1 (i.e., such that $1.x = x.1 = x$ for each $x \in Q$) is called a loop. If Q is a quasigroup, then we define $a/b = c$ if and only if $a = cb$; in this case we also put $c \setminus a = b$.

Let Q be a quasigroup. An equivalence relation θ on Q is called a normal congruence relation on Q, if it satisfies the following conditions

$$a\theta b \Leftrightarrow ac\theta bc \Leftrightarrow ca\theta cb.$$

A subquasigroup (subloop) H of a quasigroup (loop) Q is called a normal subquasigroup (subloop) of Q if H is a class with respect to some normal congruence relation on Q. If Q is a loop, then a subloop H is normal in Q (see [1]) if and only if $xH = Hx$, $xy \cdot H = x \cdot yH$, $H \cdot xy = Hx \cdot y$ for all $x, y \in Q$. It is routine to verify that for loops the following assertion (analogous to that for groups) is valid.

2.1. Lemma. *Let H be a normal subloop of a loop Q. Then a relation θ on Q defined by the rule*

$$x\theta y \Leftrightarrow x/y \in H$$

is a normal congruence relation on Q.

Now, let Q be a quasigroup and at the same time let $\leqslant$ be a partial order on Q. We denote by $Q\uparrow$ (or $Q\downarrow$) the set of all $x \in Q$ such that whenever $y, z \in Q$, then $y \leqslant z$ if and only if $xy \leqslant xz$ (or $y \leqslant z$ if and only if $xy \geqslant xz$, respectively).

2.2. Definition. Q is said to be a half linearly ordered quasigroup (hl-quasigroup) if the following conditions are satisfied:
(i) the partial order $\leqslant$ on Q is nontrivial;
(ii) if $x, y, z \in Q$, then $y \leqslant z$ if and only if $yx \leqslant zx$;
(iii) $Q = Q\uparrow \cup\ Q\downarrow$;
(iv) $Q\uparrow$ is a linearly ordered set.

In particular, if Q is a loop, then Q is called a half linearly ordered loop (hl-loop).

Let Q be an hl-quasigroup. If $Q\downarrow = \emptyset$, then Q is a linearly ordered quasigroup. If Q is a group under the binary operation, then, by the definition in [6], Q is a half linearly ordered group (hl-group). In this case, $Q\downarrow \neq \emptyset$ yields that $Q\uparrow$ is a normal subgroup of Q with index 2 (see [6]). The situation is different if we consider quasigroups instead of groups. There exists an hl-quasigroup Q such that $Q\uparrow$ is not a subquasigroup of Q (for an example see [4]). On the other hand there exist hl-quasigroups Q such that $Q\!\uparrow$ is normal in Q and the number of classes modulo $Q\!\uparrow$ is greater than 2; moreover, their number can be infinite (see [4; Theorem 2]). We will apply the following results which were proved in [4].

2.3. Proposition. *Let Q be an hl-loop with the identity 1, $Q\!\downarrow \neq \emptyset$. Then*

(i) *$p \in Q\!\uparrow$ if and only if p and 1 are comparable;*

(ii) *if $p \in Q\!\uparrow$, $q \in Q\!\downarrow$, then p and q are incomparable.*

2.4. Proposition. *Let Q be an hl-loop. Then $Q\!\uparrow$ is a normal subloop of Q.*

Let Q be an hl-quasigroup. For each $a, b \in Q$ we put

$$a \varrho b \Leftrightarrow a, b \text{ are comparable.} \tag{1}$$

2.5. Proposition (Cf. [4]). *Let Q be an hl-quasigroup such that $Q\!\uparrow$ is a subquasigroup of Q. Then ϱ is a normal congruence relation on Q and $Q\!\uparrow$ is normal in Q.*

2.6. Notation. *Let Q be an hl-quasigroup. Let ϱ be a congruence relation on a quasigroup Q defined by (1). For each $a \in Q$ we denote $T_a = \{x \in Q\colon x\varrho a\}$. Since ϱ is a normal congruence relation on Q, the sets T_a are elements of the quotient-quasigroup Q/ϱ with an operation defined by $T_a \cdot T_b = T_{ab}$ (cf. [1]). The cardinal* $\operatorname{card} Q/\varrho$ *will be called the index of an hl-quasigroup Q.*

If Q is an hl-loop, then, by 2.3, $T_1 = Q\!\uparrow$. For the quotient-loop Q/ϱ we will use the notation $Q/Q\!\uparrow$.

2.7. Definition. Let Q_1 and Q_2 be hl-quasigroups and ϱ_i be a normal congruence relation on Q_i, $i = 1, 2$, defined by (1). We say that hl-quasigroups Q_1 and Q_2 are h-equivalent, written $Q_1 \sim_h Q_2$, if Q_1/ϱ_1 and Q_2/ϱ_2 are isomorphic quasigroups.

2.8. Remark. The relation $\sim_h$ is obviously reflexive, symmetric and transitive.

2.9. Remark. All hl-quasigroups Q with $Q\!\downarrow = \emptyset$ are h-equivalent and their index is 1. All hl-groups G with $G\!\downarrow \neq \emptyset$ are h-equivalent and they have index 2.

3. The lexicographic product of hl-loops

Let $I=\{1,2,\dots,n\}$. Let Q_i be an hl-loop for each $i\in I$. We denote by $Q^{(1)}$ the direct product of the loops Q_i. The elements of $Q^{(1)}$ will be expressed as $\overline{g}=(g_1,g_2,\dots,g_n)$; g_i is the component of $\overline{g}$ in Q_i. For the components of an identity $\overline{1}\in Q^{(1)}$ we will use the unit notation 1. By $A^{(1)}$ (or $B^{(1)}$) we denote the set of all elements $\overline{g}\in Q^{(1)}$ such that for each $i\in I$ $g_i\in Q_i\!\uparrow$ (or $g_i\in Q_i\!\downarrow$, respectively).

Let H be a subset of $Q^{(1)}$. We say that a relation $\leqslant$ on H is a lexicographic order on H if for arbitrary elements $\overline{g},\bar{r}\in H$ we have $\overline{g}\leqslant\bar{r}$ if and only if $\overline{g}=\bar{r}$ or $g_i<r_i$ for the least $i\in I$ with $g_i\neq r_i$. It is easy to verify that $\leqslant$ is a partial order on H.

Finally, let us denote by $\mathcal{L}_{\,Q^{(1)}}$ the set of all H such that

(i$_0$) H is a subloop of $Q^{(1)}$;

(ii$_0$) $A^{(1)}\subseteq H$;

(iii$_0$) under the lexicographic order $\leqslant$, H is an hl-loop.

3.1. Lemma. *Let $H\in\mathcal{L}_{\,Q^{(1)}}$. Then $H\!\uparrow=A^{(1)}$ and $H\!\downarrow\subseteq B^{(1)}$.*

Proof. Assume that $\overline{g}\in H$ has components $g_j\in Q_j\!\uparrow$ and $g_k\in Q_k\!\downarrow$ for some $j,k\in I$. There exist elements $\bar{r},\bar{s}\in A^{(1)}$ such that $r_j<s_j$ and $r_i=s_i=1$ for each $i\in I$, $i\neq j$. By (ii$_0$) $\bar{r},\bar{s}\in H$. Clearly $\bar{r}<\bar{s}$ and $\overline{g}\cdot\bar{r}<\overline{g}\cdot\bar{s}$. Thus $\overline{g}\in H\!\uparrow$. Now, let $\bar{r'}$, $\bar{s'}$ be the elements of $A^{(1)}$ such that $r'_k<s'_k$ and $r'_i=s'_i=1$ for each $i\in I$, $i\neq k$ (such elements exist and belong to H). Since $\bar{r'}<\bar{s'}$ and $g_k\in Q_k\!\downarrow$, we have $\overline{g}\cdot\bar{r'}>\overline{g}\cdot\bar{s'}$. Hence $\overline{g}\in H\!\downarrow$, which contradicts the fact that $\overline{g}\in H\!\uparrow$. So, either $\overline{g}\in A^{(1)}$ or $\overline{g}\in B^{(1)}$ and this yields that $H\!\uparrow=A^{(1)}$ and $H\!\downarrow\subseteq B^{(1)}$. $\square$

In view of 2.6 we will use the notations $T_{\overline{g}}=\{\bar{x}\in H\colon\ \bar{x},\overline{g}\text{ are comparable}\}$ (or $T_{g_i}=\{x\in Q_i\colon\ x,g_i\text{ are comparable}\}$) for elements of $H/H\!\uparrow$ (or $Q_i/Q_i\!\uparrow$, respectively).

3.2. Lemma. *Let $H\in\mathcal{L}_{\,Q^{(1)}}$, $\overline{g},\bar{r}\in H$. Let there exist an index $j\in I$ such that $T_{g_j}=T_{r_j}$. Then $T_{g_i}=T_{r_i}$ for each $i\in I$.*

Proof. Assume that $\overline{g},\bar{r}\in H$, $T_{g_j}=T_{r_j}$, $T_{g_k}\neq T_{r_k}$. There exists $\bar{y}\in H$ such that $\bar{r}\cdot\bar{y}=\overline{1}$. Denote $\bar{s}=\overline{g}\cdot\bar{y}$. Obviously $\bar{s}\in H$ and

$$T_{s_j}=T_{g_jy_j}=T_{g_j}T_{y_j}=T_{r_j}T_{y_j}=T_{r_jy_j}=T_1=Q_j\!\uparrow.$$

Thus $s_j\in Q_j\!\uparrow$. At the same time

$$T_{s_k}=T_{g_k}T_{y_k}\neq T_{r_k}T_{y_k}=Q_k\!\uparrow,$$

therefore $s_k\in Q_k\!\downarrow$, which contradicts 3.1. $\square$

3.3. Lemma. *Let $H \in \mathcal{L}_{Q^{(1)}}$, $\overline{g}, \bar{r} \in H$. Then $T_{\overline{g}} = T_{\bar{r}}$ if and only if $T_{g_i} = T_{r_i}$ for each $i \in I$.*

P r o o f. From $T_{\overline{g}} = T_{\bar{r}}$ it follows that $\overline{g}, \bar{r}$ are comparable. Therefore there exists $k \in I$ such that g_k and r_k are comparable, i.e., $T_{g_k} = T_{r_k}$. Then, by 3.2, $T_{g_i} = T_{r_i}$ for each $i \in I$. Conversely, $T_{g_i} = T_{r_i}$ yields that g_i, r_i are comparable. Thus $\overline{g}$ and $\bar{r}$ are comparable, i.e., $T_{\overline{g}} = T_{\bar{r}}$. $\square$

In the remaining part of the present section we assume that for each $i, j \in I$, Q_i and Q_j are h-equivalent hl-loops. This means that for each $i \in I$ there exists an isomorphism (with respect to the loop operation)

$$\varphi_i\colon\ Q_1/Q_1\!\uparrow\ \to Q_i/Q_i\!\uparrow. \tag{1}$$

Let $\Phi = (\varphi_i, i \in I)$ be a system of isomorphisms (1) such that $\varphi_1 = id$, where id is the identity transformation of $Q_1/Q_1\!\uparrow$. We denote by $Q^{(0)}$ the subset of $Q^{(1)}$ such that

$$\overline{g} \in Q^{(0)} \text{ if and only if } \quad T_{g_i} = \varphi_i(T_{g_1}) \text{ for each } i \in I.$$

3.4. Lemma. *$Q^{(0)}$ is a subloop of $Q^{(1)}$.*

P r o o f. Obviously $\bar{1} \in Q^{(0)}$. Let $\overline{g}, \bar{r} \in Q^{(0)}$ and $\bar{s} = \overline{g} \cdot \bar{r}$. Since $\varphi_i \in \Phi$ is an isomorphism with respect to the loop operation, we have (for each $i \in I$)

$$\varphi_i(T_{s_1}) = \varphi_i(T_{g_1 r_1}) = \varphi_i(T_{g_1} T_{r_1}) = \varphi_i(T_{g_1})\varphi_i(T_{r_1}) = T_{g_i} T_{r_i} = T_{s_i}.$$

Thus $\bar{s} \in Q^{(0)}$. Analogously $\overline{g}/\bar{r} \in Q^{(0)}$ and $\bar{r} \setminus \overline{g} \in Q^{(0)}$. $\square$

3.5. Lemma. *Under the lexicographic order $\leqslant$, $Q^{(0)}$ is an hl-loop.*

P r o o f. By 3.4, $Q^{(0)}$ is a loop. Clearly, under $\leqslant$, $Q^{(0)}$ is a partially ordered set. Since Q_1 is an hl-loop, there exists $p \in Q_1\!\uparrow$, $p > 1$. Let $\bar{r}$ be an element of $Q^{(1)}$ such that $r_1 = p$ and $r_i = 1$ for each $i \in I$, $i \neq 1$. It is obvious that $\bar{r} \in Q^{(0)}$ and $\bar{1} < \bar{r}$. Thus $\leqslant$ is a nontrivial partial order on $Q^{(0)}$. Likewise, it is trivial to see that if $\overline{g}, \bar{r}, \bar{s} \in Q^{(0)}$, then $\overline{g} \leqslant \bar{r}$ if and only if $\overline{g} \cdot \bar{s} \leqslant \bar{r} \cdot \bar{s}$. We are going to show that $Q^{(0)} = Q^{(0)}\!\uparrow \cup\ Q^{(0)}\!\downarrow$. Evidently $Q^{(0)}\!\uparrow \cup\ Q^{(0)}\!\downarrow \subseteq Q^{(0)}$. Assume that $\overline{g} \in Q^{(0)}$. If $g_1 \in Q_1\!\uparrow$, then for each $i \in I$ we have $T_{g_i} = \varphi_i(T_{g_1}) = \varphi_i(Q_1\!\uparrow) = Q_i\!\uparrow$. This yields that $g_i \in Q_i\!\uparrow$ for each $i \in I$ and therefore $\overline{g} \in Q^{(0)}\!\uparrow$. Similarly, if $g_1 \in Q_1\!\downarrow$, then $\overline{g} \in Q^{(0)}\!\downarrow$. Therefore $Q^{(0)} = Q^{(0)}\!\uparrow \cup\ Q^{(0)}\!\downarrow$. Further, it is easy to see that $Q^{(0)}\!\uparrow$ is a linearly ordered set, thus we can conclude that $Q^{(0)}$ is an hl-loop. $\square$

3.6. Theorem. *Let $Q^{(0)}$ be as above. Then $Q^{(0)} \in \mathcal{L}_{Q^{(1)}}$ and $Q^{(0)}$, Q_i are h-equivalent hl-loops for each $i \in I$.*

P r o o f. By 3.4 and 3.5 (i_0) and (iii_0) hold. Also, it is easy to verify that $\bar{p} \in A^{(1)}$ implies $\bar{p} \in Q^{(0)}$, thus (ii_0) is valid. We have that $Q^{(0)} \in \mathcal{L}_{Q^{(1)}}$ and we are going to show that $Q^{(0)} \sim_h Q_i$. Define

$$\psi\colon\ Q^{(0)}/Q^{(0)}\!\uparrow\ \to Q_1/Q_1\!\uparrow;\quad \psi(T_{\overline{g}}) = T_{g_1}.$$

In view of 3.2 and 3.3 we have

$$T_{\overline{g}} = T_{\bar{r}} \ \text{ if and only if } \ T_{g_1} = T_{r_1}.$$

Hence ψ is an injective map. To prove that ψ is a surjection take $T_r \in Q_1/Q_1\!\uparrow$. For each $i \in I$, $i \neq 1$ there exists $r_i \in Q_i$ such that $T_{r_i} = \varphi_i(T_r)$, where $\varphi_i \in \Phi$. Let $\overline{g}$ be an element of $Q^{(1)}$ such that $g_1 = r$ and $g_i = r_i$ for each $i \in I$, $i \neq 1$. Clearly $\overline{g} \in Q^{(0)}$ and $\psi(T_{\overline{g}}) = T_r$. Thus ψ is a surjection. Evidently ψ preserves the loop operation, therefore ψ is an isomorphism of $Q^{(0)}/Q^{(0)}\!\uparrow$ onto $Q_1/Q_1\!\uparrow$. We have shown that $Q^{(0)} \sim_h Q_1$. Now, since $Q_1 \sim_h Q_i$ for all $i \in I$, we can conclude, by 2.8, that $Q^{(0)} \sim_h Q_i$ for each $i \in I$. $\square$

3.7. Definition. Let Q_i ($i \in I$) and $Q^{(0)}$ be as above. Then $Q^{(0)}$ is said to be the Φ-lexicographic product of hl-loops Q_i and we express this fact by writing

$$Q^{(0)} = (\Phi) \mathop{\Gamma}_{i=1}^{n} Q_i$$

or

$$Q^{(0)} = (\Phi)(Q_1 \circ Q_2 \circ \ldots \circ Q_n).$$

The hl-loops Q_i are called lexicographic factors of $Q^{(0)}$.

3.8. Remark. The Φ-lexicographic product of hl-loops Q_i depends on the system Φ. There exist hl-loops Q_i and systems of isomorphisms Φ and Ψ such that $\Phi \neq \Psi$ and hl-loops $(\Phi) \mathop{\Gamma}\limits_{i=1}^{n} Q_i$ and $(\Psi) \mathop{\Gamma}\limits_{i=1}^{n} Q_i$ are not isomorphic (see Example 3.9). If Q_i are hl-groups, then there exists exactly one system of isomorphisms (1) and $Q^{(0)}$ is the lexicographic product of hl-groups Q_i (cf. [9]). If Q_i are linearly ordered loops (or groups), then $Q^{(0)} = Q^{(1)}$ and $Q^{(0)}$ is the lexicographic product of linearly ordered loops (or groups, respectively) Q_i.

3.9. Example. Let $(\mathbb{Z}_4, \oplus)$ be the additive group of residues modulo 4. Let $Q = \mathbb{Z}_4 \times \mathbb{R}$ ($\mathbb{R}$ is the set of all real numbers) and let $\leqslant$ be the relation on Q defined by

$$(i, x) \leqslant (j, y) \Leftrightarrow i = j \text{ and } x \leqslant y.$$

Put

$$(i,x)\cdot(j,y)=\begin{cases}(i\oplus j, x+y) & \text{if } i=0,\\ \\ (i\oplus j, x-iy) & \text{if } i\neq 0.\end{cases}$$

It is routine to verify that $(Q,\cdot,\leqslant)$ is an hl-loop and

1. $Q\!\uparrow=\{(0,x);\ x\in\mathbb{R}\}$ and $Q\!\downarrow=\{(i,x);\ i\in\mathbb{Z}_4, i\neq 0, x\in\mathbb{R}\}$;
2. $Q\!\uparrow$ is normal in Q and $Q/Q\!\uparrow=\{T_{(0,0)},T_{(1,0)},T_{(2,0)},T_{(3,0)}\}$.

We take a map $\psi\colon Q/Q\!\uparrow\to Q/Q\!\uparrow$ such that

$$\psi\colon\ T_{(0,0)}\mapsto T_{(0,0)};\ T_{(1,0)}\mapsto T_{(3,0)};\ T_{(2,0)}\mapsto T_{(2,0)};\ T_{(3,0)}\mapsto T_{(1,0)}.$$

It is trivial to see that ψ is an isomorphism of $Q/Q\!\uparrow$ onto $Q/Q\!\uparrow$ with respect to the loop operation. Let us put

$$Q^{(0)}=(\Phi)(Q\circ Q),\ \text{where } \Phi=\{\mathrm{id},\mathrm{id}\},$$
$$G^{(0)}=(\Psi)(Q\circ Q),\ \text{where } \Psi=\{\mathrm{id},\psi\}.$$

Clearly $Q^{(0)}=\{((0,x),(0,y)),((1,x),(1,y)),((2,x),(2,y)),(3,x),(3,y))\colon\ x,y\in\mathbb{R}\}$ and $G^{(0)}=\{((0,x),(0,y)),((1,x),(3,y)),((2,x),(2,y)),((3,x),(1,y))\colon\ x,y\in\mathbb{R}\}$.

Now we consider the following condition for hl-loops.

(C) There exists T_a such that for each $b\in T_a$ the assertion $a\cdot a=b\cdot b$ holds.

Since $Q^{(0)}$ satisfies (C) (taking $a=((1,x),(1,y))$ for any $x,y\in\mathbb{R}$) and for $G^{(0)}$ (C) fails to hold, the hl-loops $Q^{(0)}$ and $G^{(0)}$ are not isomorphic.

Let Q be an hl-loop. The isomorphism

$$\alpha\colon\ Q\to(\Phi)\mathop{\Gamma}_{i=1}^{n}Q_i$$

with respect to the loop operation and the partial order is said to be a Φ-lexicographic product decomposition of Q.

3.10. Remark. Let $\alpha_0\colon\ Q\to Q_1$ be an isomorphism of the hl-loop Q onto the hl-loop Q_1. We regard α_0 as a lexicographic product decomposition of Q and Q_1 as a Φ-lexicographic product with one factor Q_1, where Φ contains only the identity transformation of $Q_1/Q_1\!\uparrow$.

Let

$$\beta\colon\ Q\to(\Psi)\mathop{\Gamma}_{i=1}^{m}G_i$$

be a Ψ-lexicographic product decomposition of an hl-loop Q. We say that α,β are isomorphic decompositions if $m=n$ and Q_i, G_i are isomorphic hl-loops for each $i=1,2,\dots,n$.

4. Two-factor Φ-lexicographic product decompositions

The lexicographic product decompositions of a partially ordered quasigroup with an idempotent element h were discussed in [3]. Putting $h = 1$ we can apply these results to the linearly ordered loops, especially for $Q\!\uparrow$, where Q is an hl-loop. We start this section by recalling some notions from [3], formulated for the case of Q a linearly ordered loop.

Let Q be a linearly ordered loop and let A be a subloop of Q. A linear order on Q induces a linear order on A under which A is again a linearly ordered loop; A will be called a linearly ordered subloop of Q.

Let A, B be the linearly ordered subloops of Q such that (cf. [3, Section 4], where we put $h = 1$):

(C1) for each $p \in Q$ there exists exactly one pair (a, b) such that $a \in A, b \in B$ and $p = ab$;

(C2) if $p_1, p_2 \in Q$, $p_1 = a_1b_1$, $p_2 = a_2b_2$, $a_1, a_2 \in A$, $b_1, b_2 \in B$, then

$$p_1p_2 = (a_1a_2) \cdot (b_1b_2);$$

(C3) under the notation as in (C2), the relation $p_1 \leqslant p_2$ is valid if and only if either $a_1 < a_2$ or $a_1 = a_2$ and $b_1 \leqslant b_2$.

Then we write

$$Q = A \circ B.$$

From [3, Section 4] we have that if $Q = A \circ B$, then Q is isomorphic to the lexicographic product of A and B (with respect to the loop operation and the linear order). Conversely, if Q is a lexicographic product of linearly ordered loops Q_1, Q_2, then there exist linearly ordered subloops A, B of Q such that $Q = A \circ B$. We say that $Q = A \circ B$ defines the lexicographic product decomposition of Q.

Now, let Q be an hl-loop. We take one element from every class $T_r \in Q/\,Q\!\uparrow$; from $T_1 = Q\!\uparrow$ we choose an identity element 1. We denote by R the set of all elements chosen from the respective T_r; R will be called the set of representatives of an hl-loop Q. In what follows we assume that R is any fixed set of representatives of Q.

4.1. Lemma. *If $Q\!\uparrow = A \circ B$, then each element $x \in Q$ can be uniquely written in the form $x = ab \cdot r$, where $a \in A, b \in B$ and $r \in R$.*

Proof. For each element $x \in Q$ there exists exactly one element $r \in R$ such that $r \in T_x$. Since $x/r \in Q\!\uparrow$, by (C1) there exists exactly one pair of elements $a \in A, b \in B$ such that $x = ab \cdot r$. □

In view of 4.1 we employ the following notation.

4.2. Notation. *Let $Q\uparrow = A \circ B$ and let R be a set of representatives of Q. For each $x \in Q$ we denote $a_x \in A$, $b_x \in B$ and $r_x \in R$ the elements which fulfil $x = a_x b_x \cdot r_x$. By* 4.1 *these elements are uniquely determined* (*for a fixed set R*).

Obviously $r_x = r_y$ if and only if $T_x = T_y$ (i.e., x and y are comparable).

4.3. Lemma. *Let $Q\uparrow = A \circ B$, where A, B are normal subloops of Q. Then for each $x, y, z \in Q$ the following conditions are satisfied:*

(i) $r_{xz} = r_{yz} \Leftrightarrow r_x = r_y$;
(ii) *if $r_x = r_y$, then $b_x \leqslant b_y \Leftrightarrow b_{xz} \leqslant b_{yz}$.*

Proof. (i) This is obvious. (ii) Put $r = r_x = r_y$. Since A, B are normal subloops of Q, there exist $a_x^{(1)}, a_x^{(2)}, a_x^{(3)} \in A$ and $b_x^{(1)}, b_x^{(2)} \in B$ such that:

$$\begin{aligned} xz &= (a_x b_x \cdot r)z = (a_x^{(1)} \cdot b_x r)z = a_x^{(2)} \cdot (b_x r \cdot z) = a_x^{(2)}(b_x^{(1)} \cdot rz) \\ &= a_x^{(2)}[b_x^{(1)} \cdot (a_{rz} b_{rz} \cdot r_{rz})] = a_x^{(2)}[(b_x^{(2)} \cdot a_{rz} b_{rz}) \cdot r_{rz}]. \end{aligned}$$

Hence, applying (C2), we obtain

$$xz = a_x^{(2)}[(a_{rz} \cdot b_x^{(2)} b_{rz}) \cdot r_{rz}] = [a_x^{(3)}(a_{rz} \cdot b_x^{(2)} b_{rz})]r_{rz} = (a_x^{(3)} a_{rz} \cdot b_x^{(2)} b_{rz})r_{rz}.$$

Analogously

$$\begin{aligned} yz &= (a_y b_y \cdot r)z = (a_y^{(1)} \cdot b_y r)z = a_y^{(2)} \cdot (b_y r \cdot z) = a_y^{(2)}(b_y^{(1)} \cdot rz) \\ &= a_y^{(2)}[b_y^{(1)} \cdot (a_{rz} b_{rz} \cdot r_{rz})] = a_y^{(2)}[(b_y^{(2)} \cdot a_{rz} b_{rz}) \cdot r_{rz}] \\ &= a_y^{(2)}[(a_{rz} \cdot b_y^{(2)} b_{rz}) \cdot r_{rz}] = [a_y^{(3)}(a_{rz} \cdot b_y^{(2)} b_{rz})]r_{rz} = (a_y^{(3)} a_{rz} \cdot b_y^{(2)} b_{rz})r_{rz}. \end{aligned}$$

By 4.1, we have

$$b_{xz} = b_x^{(2)} b_{rz}, \quad b_{yz} = b_y^{(2)} b_{rz}.$$

Using 2.2 (ii) and the above equations we can conclude:

$$\begin{aligned} b_x \leqslant b_y &\Leftrightarrow b_x r \cdot z \leqslant b_y r \cdot z \\ &\Leftrightarrow b_x^{(1)} \cdot rz \leqslant b_y^{(1)} \cdot rz \Leftrightarrow (b_x^{(2)} \cdot a_{rz} b_{rz})r_{rz} \leqslant (b_y^{(2)} \cdot a_{rz} b_{rz})r_{rz} \\ &\Leftrightarrow b_x^{(2)} \leqslant b_y^{(2)} \Leftrightarrow b_x^{(2)} b_{rz} \leqslant b_y^{(2)} b_{rz} \Leftrightarrow b_{xz} \leqslant b_{yz}. \end{aligned}$$

□

Using similar methods as in the proof of 4.3 we obtain

4.4. Lemma. *Let $Q\!\uparrow\, = A \circ B$, where A, B are normal subloops of Q. Then for each $x, y, z \in Q$ the following conditions are satisfied:*

(i) $r_{zx} = r_{zy} \Leftrightarrow r_x = r_y$;

(ii) *if* $z \in Q\!\uparrow$ *and* $r_x = r_y$, *then* $b_x \leqslant b_y \Leftrightarrow b_{zx} \leqslant b_{zy}$;

(iii) *if* $z \in Q\!\downarrow$ *and* $r_x = r_y$, *then* $b_x \leqslant b_y \Leftrightarrow b_{zx} \geqslant b_{zy}$.

Let $Q\!\uparrow\, = A \circ B$. For $x = a_x b_x$ and $y = a_y b_y$ from $Q\!\uparrow$ we put

$$x \,\tau_1\, y \Leftrightarrow a_x = a_y,\ x \,\tau_2\, y \Leftrightarrow b_x = b_y. \tag{2}$$

It is routine to verify that τ_1, τ_2 are normal congruence relations on $Q\!\uparrow$. For $i = 1, 2$ and $x \in Q\!\uparrow$ we set $\tau_i[x] = \{y \in Q\!\uparrow\colon\ y \,\tau_i\, x\}$. Clearly $\tau_1[1] = B$ and $\tau_2[1] = A$.

Now, for $i = 1, 2$ we define a relation Θ_i on Q:

$$x \,\Theta_i\, y \Leftrightarrow x/y \in Q\!\uparrow \text{ and } x/y \,\tau_i\, 1. \tag{3}$$

If A and B (i.e., $\tau_2[1]$ and $\tau_1[1]$) are normal subloops of Q, then, by 2.1, Θ_1, Θ_2 are normal congruence relations on Q. Analogously as above, for each $i = 1, 2$ and $x \in Q$ we set $\Theta_i[x] = \{y \in Q\colon\ y \,\Theta_i\, x\}$.

4.5. Lemma. *Let $Q\!\uparrow\, = A \circ B$, where A, B are normal subloops of Q. Let $x, y \in Q$. Then*

$$x \,\Theta_1\, y \Leftrightarrow r_x = r_y \text{ and } a_x = a_y,$$
$$x \,\Theta_2\, y \Leftrightarrow r_x = r_y \text{ and } b_x = b_y.$$

P r o o f. Assume that $x \,\Theta_1\, y$. Then $x/y \,\tau_1\, 1$, i.e., $x = by, b \in B$. Using the notations from 4.2 and the assumption that B is a normal subloop of Q we can write:

$$a_x b_x \cdot r_x = b(a_y b_y \cdot r_y) = (b' \cdot a_y b_y) r_y,$$

where $b' \in B$. By (C2) we obtain $a_x b_x \cdot r_x = (a_y \cdot b' b_y) r_y$ and hence, in view of 4.1, we get $r_x = r_y$ and $a_x = a_y$. Conversely, let x, y be elements of Q such that $a_x = a_y, r_x = r_y$. From $r_x = r_y$ we have $x/y \in Q\!\uparrow$. Therefore $x = py$, where $p \in Q\!\uparrow$. Thus $a_x b_x \cdot r_x = p(a_x b_y \cdot r_x)$. Since $Q\!\uparrow$ is a normal subloop of Q, there exists $z \in Q\!\uparrow$ such that

$$a_x b_x \cdot r_x = p(a_x b_y \cdot r_x) = (z \cdot a_x b_y) r_x. \tag{4}$$

Hence $a_x b_x = z \cdot a_x b_y$. From $z \in Q\!\uparrow$ we have $z = ab$, where $a \in A, b \in B$. Then $a_x b_x = ab \cdot a_x b_y$ and hence, in view of (C2) and (C1), we get $a_x = a a_x$. Thus $a = 1$, and therefore $z \in B$. Since B is a normal subloop of Q and $z \in B$, we have, by (4), $p \in B$ $(= \tau_1[1])$. Hence $x/y \,\tau_1\, 1$, i.e., $x \,\Theta_1\, y$. The proof for Θ_2 is analogous. □

4.6. Lemma.

(i) *If* $A = \{1\}$, *then* $(x\ \Theta_1\ y \Leftrightarrow T_x = T_y)$ *and* $(x\ \Theta_2\ y \Leftrightarrow x = y)$.
(ii) *If* $B = \{1\}$, *then* $(x\ \Theta_2\ y \Leftrightarrow T_x = T_y)$ *and* $(x\ \Theta_1\ y \Leftrightarrow x = y)$.

P r o o f. This is a consequence of 4.5. □

In what follows we assume that $Q\!\uparrow = A \circ B$, A, B are normal subloops of Q and $A, B \neq \{1\}$. For each $i = 1, 2$ we denote

$$\overline{Q}_i = \{\Theta_i[x] \colon\ x \in Q\}.$$

Since Θ_i is a normal congruence relation on Q, $\overline{Q}_i$ with the operation $\Theta_i[x] \cdot \Theta_i[y] = \Theta_i[xy]$ is a loop. Put

$$\Theta_1[x] \leqslant \Theta_1[y] \Leftrightarrow r_x = r_y \text{ and } a_x \leqslant a_y, \tag{5}$$

and

$$\Theta_2[x] \leqslant \Theta_2[y] \Leftrightarrow r_x = r_y \text{ and } b_x \leqslant b_y. \tag{6}$$

It is easy to verify that the relation $\leqslant$ is correctly defined on $\overline{Q}_i$ $(i = 1, 2)$, i.e., it does not depend on the choice of the elements from $\Theta_i[x]$. Further, we immediately obtain

4.7. Lemma. *The relation* $\leqslant$ *is a partial order on* $\overline{Q}_i$, $i = 1, 2$.

4.8. Lemma.

(i) $\Theta_1[x]$ *and* $\Theta_1[y]$ *are comparable* (*by the relation* $\leqslant$) *if and only if* $\Theta_2[x]$ *and* $\Theta_2[y]$ *are comparable;*
(ii) $x = y$ *if and only if* $\Theta_i[x] = \Theta_i[y]$ *for* $i = 1, 2$.

P r o o f. Since arbitrary two elements of $Q\!\uparrow$ are comparable, (i) follows from (5) and (6). The assertion (ii) is an immediate consequence of 4.5. □

4.9. Lemma.

(i) *If* $\Theta_1[x] < \Theta_1[y]$, *then* $x < y$.
(ii) *If* $x \leqslant y$, *then* $\Theta_1[x] \leqslant \Theta_1[y]$.

P r o o f. (i) From 4.5 and (5) it follows that $\Theta_1[x] < \Theta_1[y]$ implies $r_x = r_y$, $a_x < a_y$. Thus, by (C3), $x < y$. (ii) $x \leqslant y \Rightarrow r_x = r_y$, $a_x b_x \leqslant a_y b_y \Rightarrow r_x = r_y$, $a_x \leqslant a_y \Rightarrow \Theta_1[x] \leqslant \Theta_1[y]$. □

Now, for each $i = 1, 2$ we denote

$$H_i^{\uparrow} = \{\Theta_i[x];\ x \in Q\!\uparrow\}, \quad H_i^{\downarrow} = \{\Theta_i[x];\ x \in Q\!\downarrow\}.$$

Clearly $\overline{Q}_i = H_i^{\uparrow} \cup H_i^{\downarrow}$.

4.10. Lemma. *For each $i = 1, 2$, the loop $\overline{Q}_i$ under the relation* (5) (*or* (6), *respectively*) *is an hl-loop with $\overline{Q}_i\!\uparrow = H_i^{\uparrow}$ and $\overline{Q}_i\!\downarrow = H_i^{\downarrow}$.*

P r o o f. We are going to prove that $\overline{Q}_1$ fulfills the conditions (i)–(iv) from 2.2. By 4.7, under the relation $\leqslant$, $\overline{Q}_1$ is a partially ordered set. Since $A \neq \{1\}$, there exists $x \in A$ such that $x < 1$. Then $\Theta_1[x] < \Theta_1[1]$, thus $\leqslant$ is a nontrivial partial order on $\overline{Q}_1$; hence (i) is valid. Let $x, y, z \in Q$. Clearly

$$\Theta_1[x] = \Theta_1[y] \Leftrightarrow \Theta_1[xz] = \Theta_1[yz]$$

and, in view of 4.9,

$$\begin{aligned}\Theta_1[x] < \Theta_1[y] &\Leftrightarrow x < y,\ \Theta_1[x] \neq \Theta_1[y] \Leftrightarrow \\ &\Leftrightarrow xz < yz,\ \Theta_1[xz] \neq \Theta_1[yz] \Leftrightarrow \Theta_1[xz] < \Theta_1[yz] \\ &\Leftrightarrow \Theta_1[x] \cdot \Theta_1[z] < \Theta_1[y] \cdot \Theta_1[z],\end{aligned}$$

thus (ii) is valid. Using a similar method as above we can prove that $\overline{Q}_1\!\uparrow = H_1^{\uparrow}$ and $\overline{Q}_1\!\downarrow = H_1^{\downarrow}$. Hence $\overline{Q}_1 = \overline{Q}_1\!\uparrow \cup \overline{Q}_1\!\downarrow$; thus (iii) holds. Finally, since $\overline{Q}_1\!\uparrow$ is obviously a linearly ordered set, we have that $\overline{Q}_1$ is an hl-loop.

The proof that (6) is a nontrivial partial order on $\overline{Q}_2$ is analogous to that for $\overline{Q}_1$. Let $x, y, z \in Q$. From (6) and 4.3 we obtain

$$\begin{aligned}\Theta_2[x] \leqslant \Theta_2[y] &\Leftrightarrow r_x = r_y,\ b_x \leqslant b_y \Leftrightarrow r_{xz} = r_{yz}, b_{xz} \leqslant b_{yz} \\ &\Leftrightarrow \Theta_2[x]\Theta_2[z] \leqslant \Theta_2[y]\Theta_2[z],\end{aligned}$$

thus 2.2(ii) is valid. We are going to show that $\overline{Q}_2\!\downarrow = H_2^{\downarrow}$. Let $\Theta_2[z] \in \overline{Q}_2\!\downarrow$. By way of contradiction, suppose that $z \in Q\!\uparrow$. Since $\leqslant$ is a nontrivial partial order on $\overline{Q}_2$, there exist $x, y \in Q$ such that $\Theta_2[x] < \Theta_2[y]$. Then $\Theta_2[zx] > \Theta_2[zy]$, and thus $b_{zx} > b_{zy}$, $r_{zx} = r_{zy}$ Hence, by 4.4, $b_x > b_y$, $r_x = r_y$, which contradicts the fact that $\Theta_2[x] < \Theta_2[y]$. Therefore $z \in Q\!\downarrow$, i.e., $\Theta_2[z] \in H_2^{\downarrow}$. To prove the converse inclusion take $\Theta_2[z] \in H_2^{\downarrow}$ (this means that $z \in Q\!\downarrow$). Then

$$\begin{aligned}\Theta_2[x] \leqslant \Theta_2[y] &\Leftrightarrow r_x = r_y,\ b_x \leqslant b_y \\ &\Leftrightarrow r_{zx} = r_{zy},\ b_{zx} \geqslant b_{zy} \Leftrightarrow \Theta_2[zx] \geqslant \Theta_2[zy].\end{aligned}$$

Thus $\Theta_2[z] \in \overline{Q}_2\!\downarrow$. We have $\overline{Q}_2\!\downarrow = H_2^{\downarrow}$. To prove that $\overline{Q}_2\!\uparrow = H_2^{\uparrow}$ we proceed similarly. Now it is easy to see that $\overline{Q}_2 = \overline{Q}_2\!\uparrow \cup \overline{Q}_2\!\downarrow$, and since $\overline{Q}_2\!\uparrow$ is a linearly ordered set, we can conclude that $\overline{Q}_2$ is an hl-loop. $\square$

The hl-loops $\overline{Q}_1$, $\overline{Q}_2$ are h-equivalent. Indeed, let

$$\varphi\colon\ \overline{Q}_1/\overline{Q}_1{\uparrow} \to \overline{Q}_2/\overline{Q}_2{\uparrow};\ T_{\Theta_1[x]} \mapsto T_{\Theta_2[x]}.$$

By 4.8 (i), $T_{\Theta_2[x]} = T_{\Theta_2[y]}$ if and only if $T_{\Theta_1[x]} = T_{\Theta_1[y]}$, thus φ is an injective mapping. Moreover, it is easy to see that φ is a surjection and φ preserves the loop operation. Thus $\overline{Q}_1 \sim_h \overline{Q}_2$.

Since φ is an isomorphism (with respect to the loop operation), we can construct Φ-lexicographic product

$$\overline{G} = (\Phi)(\overline{Q}_1 \circ \overline{Q}_2),\ \text{where } \Phi = \{\mathrm{id}, \varphi\}.$$

4.11. Lemma. $(\Theta_1[x], \Theta_2[y]) \in \overline{G}$ *if and only if* $T_x = T_y$.

P r o o f. $(\Theta_1[x], \Theta_2[y]) \in \overline{G} \Leftrightarrow \varphi(T_{\Theta_1[x]}) = T_{\Theta_2[y]} \Leftrightarrow T_{\Theta_2[x]} = T_{\Theta_2[y]} \Leftrightarrow \Theta_2[x], \Theta_2[y]$ are comparable $\Leftrightarrow T_x = T_y$. □

Let us put

$$\psi\colon\ Q \to \overline{G};\ \psi(x) = (\Theta_1[x], \Theta_2[x]).$$

4.12. Lemma. *ψ is an isomorphism of the hl-loop Q onto the hl-loop $\overline{G}$.*

P r o o f. By 4.11, $(\Theta_1[x], \Theta_2[x]) \in \overline{G}$ for each $x \in Q$. Using 4.8 (ii) it is easy to see that ψ is an injective mapping. We are going to show that ψ is a surjection. Let $(\Theta_1[x], \Theta_2[y]) \in \overline{G}$. By 4.11, $T_x = T_y$, and thus there exists $r \in R$ (R is the set of representatives of Q) such that $x = a_x b_x \cdot r$ and $y = a_y b_y \cdot r$. Put $z = a_x b_y \cdot r$. Since $\Theta_1[z] = \Theta_1[x]$ and $\Theta_2[z] = \Theta_2[y]$, we have $\psi(z) = (\Theta_1[x], \Theta_2[y])$. Thus ψ is a surjection. It is routine to verify that ψ preserves the loop operation. Finally,

$$\begin{aligned}\psi(x) \leqslant \psi(y) &\Leftrightarrow \Theta_1[x] < \Theta_1[y] \text{ or } (\Theta_1[x] = \Theta_1[y],\ \Theta_2[x] \leqslant \Theta_2[y])\\ &\Leftrightarrow (r_x = r_y,\ a_x < a_y) \text{ or } (r_x = r_y,\ a_x = a_y,\ b_x \leqslant b_y)\\ &\Leftrightarrow a_x b_x \cdot r_x \leqslant a_y b_y \cdot r_y \Leftrightarrow x \leqslant y.\end{aligned}$$

Thus ψ is an isomorphism with respect to the loop operation and the partial order. □

Summarizing, we have

4.13. Theorem. *Let Q be an hl-loop and let A, B be nontrivial normal subloops of Q such that $Q\upharpoonleft = A \circ B$. Then ψ is a Φ-lexicographic product decomposition of Q.*

5. Finite-factor Φ-lexicographic product decompositions

The finite-factor lexicographic product decomposition of a partially ordered quasigroup Q with an idempotent element h has been studied by author in [3]. Analogously as in Section 4, putting $h = 1$, we can apply these results to a linearly ordered loop $Q\upharpoonleft$ in case Q is an hl-loop.

Firstly, assume that Q is a linearly ordered loop. Let A_1, A_2, A_3 be linearly ordered subloops of Q. Then (cf. [3, Lemma 4.5]) $Q = (A_1 \circ A_2) \circ A_3$ if and only if $Q = A_1 \circ (A_2 \circ A_3)$. Hence, by induction, we can conclude that the finite-factor lexicographic product decomposition of Q does not depend on the setting of parentheses. Moreover, putting $h = 1$ in [3; (4.4)] we immediately obtain

5.1. Lemma. *Let $Q = A_1 \circ A_2 \circ A_3$. Then $a^{(1)} \cdot (a^{(2)} \cdot a^{(3)}) = (a^{(1)} \cdot a^{(2)}) \cdot a^{(3)}$ for arbitrary elements $a^{(i)} \in A_i, i = 1, 2, 3$.*

For the lexicographic product decomposition of the linearly ordered loop Q with lexicographic factors $A_1, A_2, \dots, A_n$ we use the notation

$$Q = A_1 \circ A_2 \circ \dots \circ A_n.$$

By 5.1, provided $Q = A_1 \circ A_2 \circ \dots \circ A_n$ the parentheses in the product $a^{(1)} a^{(2)} \cdot \ldots \cdot a^{(n)}$ of elements $a^{(i)} \in A_i$ can be omitted. Moreover, by (C1), arbitrary elements $x, y \in Q$ can be uniquely written in the form $x = a^{(1)} a^{(2)} \dots a^{(n)}$, $y = b^{(1)} b^{(2)} \dots b^{(n)}$, where $a^{(i)}, b^{(i)} \in A_i$ and, by (C2), $xy = (a^{(1)} b^{(1)}) \cdot (a^{(2)} b^{(2)}) \cdot \ldots \cdot (a^{(n)} b^{(n)})$.

Now, let Q be an hl-loop, R be a set of representatives of Q. Suppose that

$$Q\upharpoonleft = A_1 \circ A_2 \circ \dots \circ A_n \tag{1}$$

is a lexicographic product decomposition of the linearly ordered loop $Q\upharpoonleft$. It is easy to verify that the generalization of 4.1 is valid, i.e., each element $x \in Q$ can be uniquely written in the form $(a^{(1)} a^{(2)} \dots a^{(n)}) \cdot r$, where $a^{(i)} \in A_i$ and $r \in R$. In view of this fact we will employ the notations $x = a_x^{(1)} a_x^{(2)} \dots a_x^{(n)} \cdot r_x$, $a_x^{(i)} \in A_i$, $r_x \in R$, $y = a_y^{(1)} a_y^{(2)} \dots a_y^{(n)} \cdot r_y$, $a_y^{(i)} \in A_i$, $r_y \in R$, $xy = a_{xy}^{(1)} a_{xy}^{(2)} \dots a_{xy}^{(n)} \cdot r_{xy}$, $a_{xy}^{(i)} \in A_i$, $r_{xy} \in R$, etc. (we recall that the relations $a_{xy}^{(i)} = a_x^{(i)} a_y^{(i)}$, $r_{xy} = r_x r_y$ don't hold in general).

5.2. Lemma. *Let Q be an hl-loop. Let A_i, $i = 1, 2, \ldots, n$, be normal subloops of Q such that $Q\uparrow = A_1 \circ A_2 \circ \ldots \circ A_n$. Then $B = A_{i_1} \circ A_{i_2} \circ \ldots \circ A_{i_k}$ is a normal subloop of Q for arbitrary $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$, $i_1 < i_2 < \ldots < i_k$.*

Proof. The assertion of the lemma is trivial for $k = 1$. Let $k \in N$, $1 < k \leqslant n$. We are going to show that if $B^* = A_{i_1} \circ A_{i_2} \circ \ldots \circ A_{i_{k-1}}$ is normal in Q, then B is a normal subloop of Q. It is routine to verify that B is a subloop of Q. Let $x \in Q$, $b \in B$. Clearly $b = ca$, where $c \in B^*$ and $a \in A_{i_k}$. Since A_{i_k}, B^* are normal subloops of Q, there exist $a' \in A_{i_k}$, $c' \in B^*$ such that $xb = x \cdot ca = c'a' \cdot x$. Therefore $xb \in Bx$. Analogously we obtain $bx \in xB$, thus we have $xB = Bx$ for each $x \in Q$. Similarly, if $x, y \in Q$, then $xy \cdot B = x \cdot yB$ and $B \cdot xy = Bx \cdot y$. Therefore B is a normal subloop of Q. □

Suppose that (1) is valid and A_i is normal in Q for each $i = 1, 2, \ldots, n$. We define a relation τ_i $(i = 1, 2, \ldots, n)$ on $Q\uparrow$:

$$x \; \tau_i \; y \Leftrightarrow a_x^{(i)} = a_y^{(i)}. \tag{2}$$

It is easy to see that τ_i is a normal congruence relation on $Q\uparrow$. By 5.2, $\tau_i[1] = \{x \in Q\uparrow \colon \; x \; \tau_i \; 1\;\}$ is a normal subloop of Q. Now, for $x, y \in Q$ and for each $i = 1, 2, \ldots, n$ we put

$$x\Theta_i y \Leftrightarrow x/y \in Q\uparrow \text{ and } x/y \; \tau_i \; 1. \tag{3}$$

In view of 2.1 Θ_i is a normal congruence relation on Q. Analogously as in Section 4 we denote $\Theta_i[x] = \{z \in Q \colon \; z\Theta_i x\}$ and $\overline{Q}_i = \{\Theta_i[x] \colon \; x \in Q\}$. Recall that under the operation $\Theta_i[x] \cdot \Theta_i[y] = \Theta_i[xy]$, $\overline{Q}_i$ is a loop.

5.3. Lemma. *For each $i = 1, 2, \ldots, n$ the following holds*

$$x\Theta_i y \Leftrightarrow a_x^{(i)} = a_y^{(i)} \text{ and } r_x = r_y.$$

Proof. In view of 5.2 it suffices to apply the same method as in the proof of 4.5. □

Let us denote (for each $i = 1, 2, \ldots, n$)

$$H_i^{\uparrow} = \{\Theta_i[x]; \; x \in Q\uparrow\}, \quad H_i^{\downarrow} = \{\Theta_i[x]; \; x \in Q\downarrow\}.$$

Clearly $\overline{Q}_i = H_i^{\uparrow} \cup H_i^{\downarrow}$. For $\Theta_i[x]$, $\Theta_i[y] \in \overline{Q}_i$ we set

$$\Theta_i[x] \leqslant \Theta_i[y] \Leftrightarrow r_x = r_y \text{ and } a_x^{(i)} \leqslant a_y^{(i)}. \tag{4}$$

It is easy to see that the relation $\leqslant$ is a partial order on $\overline{Q}_i$.

5.4. Lemma. *Let Q be an hl-loop. Let $Q\!\uparrow = A_1 \circ A_2 \circ A_3$, where A_1, A_2, A_3 are nontrivial normal subloops of Q. Then for each $i = 1, 2, 3$ $\overline{Q}_i$ is an hl-loop and $\overline{Q}_i\!\uparrow = H_i^{\uparrow}$, $\overline{Q}_i\!\downarrow = H_i^{\downarrow}$.*

Proof. Denote $B = A_1 \circ A_2$. By 5.2, B is normal in Q. Clearly $Q\!\uparrow = B \circ A_3$, thus each element $x \in Q$ can be uniquely written in the form $x = ba \cdot r$, where $b \in B$, $a \in A_3$ and $r \in R$. Let η be a relation defined on Q by the rule

$$(ba \cdot r)\ \eta\ (b'a' \cdot r') \Leftrightarrow a = a',\ r = r'$$

and let

$$\eta[ba \cdot r] \leqslant' \eta[b'a' \cdot r'] \Leftrightarrow a \leqslant a',\ r = r'.$$

Denote $\overline{G} = \{\eta[x]\colon\ x \in Q\}$. By 4.10, under the relation $\leqslant'$, $\overline{G}$ is an hl-loop. But it is easy to see that $\overline{G} = \overline{Q}_3$ and

$$\eta[x] \leqslant' \eta[y] \Leftrightarrow \Theta_3[x] \leqslant \Theta_3[y],$$

where $\leqslant$ is the relation defined by (4). Therefore we can conclude that $\overline{Q}_3$ is an hl-loop and $\overline{Q}_3\!\uparrow = H_3^{\uparrow}$, $\overline{Q}_3\!\downarrow = H_3^{\downarrow}$. Analogously $\overline{Q}_1$ is an hl-loop and $\overline{Q}_1\!\uparrow = H_1^{\uparrow}$, $\overline{Q}_1\!\downarrow = H_1^{\downarrow}$. We are going to show that $\overline{Q}_2$ is an hl-loop. As in the proof of 4.10 it can be seen that $\leqslant$ is a nontrivial partial order on $\overline{Q}_2$. For completing the proof we verify (ii)–(iv) from 2.2. Denote $B = A_2 \circ A_3$. Then $Q\!\uparrow = A_1 \circ B$. Any elements $x, y \in Q$ can be uniquely expressed as $x = a_x^{(1)} b_x \cdot r_x$ and $y = a_y^{(1)} b_y \cdot r_y$, where b_x, b_y are elements of B, which are uniquely determined by $b_x = a_x^{(2)} a_x^{(3)}$ and $b_y = a_y^{(2)} a_y^{(3)}$. Let $z \in Q$. The elements xz, yz can be uniquely written in the form

$$xz = a_{xz}^{(1)} a_{xz}^{(2)} a_{xz}^{(3)} \cdot r_{xz} = a_{xz}^{(1)} b_{xz} \cdot r_{xz},$$

where $a_{xz}^{(i)} \in A_i$, $r_{xz} \in R$, $b_{xz} = a_{xz}^{(2)} a_{xz}^{(3)} \in B$, and

$$yz = a_{yz}^{(1)} a_{yz}^{(2)} a_{yz}^{(3)} \cdot r_{yz} = a_{yz}^{(1)} b_{yz} \cdot r_{yz},$$

where $a_{yz}^{(i)} \in A_i$, $r_{yz} \in R$, $b_{yz} = a_{yz}^{(2)} a_{yz}^{(3)} \in B$. Clearly

$$\Theta_2[x] = \Theta_2[y] \Leftrightarrow \Theta_2[xz] = \Theta_2[yz]$$

and

$$\Theta_2[x] < \Theta_2[y] \Leftrightarrow r_x = r_y,\ a_x^{(2)} < a_y^{(2)} \Leftrightarrow r_x = r_y,\ a_x^{(2)} \neq a_y^{(2)},\ b_x < b_y. \tag{5}$$

Since $Q{\uparrow} = A_1 \circ B$, where A_1, B are normal subloops of Q, from (5) and 4.3 it follows that

$$\begin{aligned}\Theta_2[x] < \Theta_2[y] &\Leftrightarrow a^{(2)}_{xz} \neq a^{(2)}_{yz},\ r_{xz} = r_{yz},\ b_{xz} < b_{yz}\\ &\Leftrightarrow a^{(2)}_{xz} < a^{(2)}_{yz},\ r_{xz} = r_{yz} \Leftrightarrow \Theta_2[x]\Theta_2[z] < \Theta_2[y]\Theta_2[z],\end{aligned}$$

thus (ii) from 2.2 holds. Using similar methods as above we obtain (cf. also the proof of 4.10)

$$\overline{Q}_2{\uparrow} = H_2^{\uparrow} \text{ and } \overline{Q}_2{\downarrow} = H_2^{\downarrow},$$

which yields $\overline{Q}_2 = \overline{Q}_2{\uparrow} \cup \overline{Q}_2{\downarrow}$. Now, since $\overline{Q}_2{\uparrow}$ is a linearly ordered set, we can conclude that $\overline{Q}_2$ is an hl-loop. $\square$

5.5. Lemma. *Let Q be an hl-loop. Let $Q{\uparrow} = A_1 \circ A_2 \circ \ldots \circ A_n$, $n \neq 1$, where A_i $(i = 1, 2, \ldots, n)$ are nontrivial normal subloops of Q. Then for each $i = 1, 2, \ldots, n$ $\overline{Q}_i$ is an hl-loop and $\overline{Q}_i{\uparrow} = H_i^{\uparrow}$, $\overline{Q}_i{\downarrow} = H_i^{\downarrow}$.*

Proof. In view of 4.10 and 5.4 it suffices to consider the case $n \geqslant 4$. Let $i \neq 1$ and $i \neq n$. We denote $B_1 = A_1 \circ A_2 \circ \ldots \circ A_{i-1}$, $B_2 = A_{i+1} \circ \ldots \circ A_n$. Then $Q{\uparrow} = B_1 \circ A_i \circ B_2$, thus, by 5.4, $\overline{Q}_i$ is an hl-loop. For $i = 1$ and $i = n$ the assertion of the lemma follows from 5.4, where we set $Q{\uparrow} = A_1 \circ B \circ A_n$, $B = A_2 \circ \ldots \circ A_{n-1}$. $\square$

Let Q be an hl-loop. We assume that (1) holds, $n \neq 1$ and A_i, $i = 1, 2, \ldots, n$, are the nontrivial normal subloops of Q. For each $i = 1, 2, \ldots, n$ we set

$$\psi_i\colon\ \overline{Q}_1/\ \overline{Q}_1{\uparrow} \to \overline{Q}_i/\ \overline{Q}_i{\uparrow};\ T_{\Theta_1[x]} \to T_{\Theta_i[x]}. \tag{6}$$

Analogously as in Section 4 it can be shown that ψ_i is a loop isomorphism. Denote $\Psi = (\psi_i; i = 1, 2, \ldots, n)$ the system of isomorphisms from (6) (it is obvious that ψ_1 is the identity permutation of $\overline{Q}_1/\ \overline{Q}_1{\uparrow}$). Let

$$\alpha_1\colon\ Q \to (\Psi) \mathop{\Gamma}_{i=1}^{n} \overline{Q}_i;\ \alpha_1(x) = (\Theta_1[x], \Theta_2[x], \ldots, \Theta_n[x]).$$

Then α_1 is a Ψ-lexicographic product decomposition of Q (the proof is analogous with that of 4.12). Also, it is easy to see that the linearly ordered loops $\overline{Q}_i{\uparrow}$ and A_i are isomorphic. The decomposition α_1 will be called an extension of the decomposition (1).

Now, for each $i = 1, 2, \ldots, n$, $n \geqslant 2$, let G_i be an hl-loop and let

$$\gamma\colon\ Q \to (\Phi) \mathop{\Gamma}_{i=1}^{n} G_i \tag{7}$$

be a Φ-lexicographic product decomposition of an hl-loop Q. The component of $\gamma(x)$ in G_i will be denoted by $\gamma(x)_i$. For $i = 1, 2, \ldots, n$ we consider the relation Θ_i^* defined on Q by

$$x \Theta_i^* y \Leftrightarrow \gamma(x)_i = \gamma(y)_i.$$

Clearly, Θ_i^* is a normal congruence relation on Q. Under the relation

$$\Theta_i^*[x] \leqslant \Theta_i^*[y] \Leftrightarrow \gamma(x)_i \leqslant \gamma(y)_i,$$

Q/Θ_i^* is an hl-loop, $Q/\Theta_i^*\uparrow = \{\Theta_i^*[x];\ x \in Q\uparrow\}$, $Q/\Theta_i^*\downarrow = \{\Theta_i^*[x];\ x \in Q\downarrow\}$. It is routine to verify that G_i, Q/Θ_i^* are isomorphic hl-loops. For each $i = 1, 2, \ldots n$ let

$$B_i = \{x \in Q\uparrow\colon\ \gamma(x)_j = 1 \ \text{ for each } j \neq i\}.$$

Obviously B_i are normal, nontrivial subloops of Q and

$$Q\uparrow = B_1 \circ B_2 \circ \ldots \circ B_n. \tag{8}$$

Denote

$$\gamma_1\colon\ Q \to (\Psi) \mathop{\Gamma}_{i=1}^{n} \overline{Q}_i;\ \gamma_1(x) = (\Theta_1[x], \Theta_2[x], \ldots, \Theta_n[x]) \tag{9}$$

an extension of (8).

5.6. Lemma. *γ and γ_1 are isomorphic decompositions.*

Proof. From (8) it follows that elements $x, y \in Q$ can be uniquely written in the form $x = (b_x^{(1)} b_x^{(2)} \ldots b_x^{(n)}) \cdot r_x$, $y = (b_y^{(1)} b_y^{(2)} \ldots b_y^{(n)}) \cdot r_y$, where $b_x^{(i)}, b_y^{(i)} \in B_i$, $r_x, r_y \in R$. If $\Theta_i^*[x] \leqslant \Theta_i^*[y]$, then $r_x = r_y$. Indeed, from $\Theta_i^*[x] \leqslant \Theta_i^*[y]$ it follows that $T_{\gamma(x)_i} = T_{\gamma(y)_i}$, thus, by 3.2 and 3.3, $T_{\gamma(x)} = T_{\gamma(y)}$ and since γ is an isomorphism with respect to the partial order, we have $T_x = T_y$, i.e., $r_x = r_y$. Thus we can write

$$\begin{aligned}
\Theta_i^*[x] \leqslant \Theta_i^*[y] &\Leftrightarrow \gamma(x)_i \leqslant \gamma(y)_i \\
&\Leftrightarrow \gamma(b_x^{(1)} b_x^{(2)} \ldots b_x^{(n)} \cdot r_x)_i \leqslant \gamma(b_y^{(1)} b_y^{(2)} \ldots b_y^{(n)} \cdot r_y)_i \\
&\Leftrightarrow \gamma(b_x^{(i)})_i \leqslant \gamma(b_y^{(i)})_i \Leftrightarrow \gamma(b_x^{(i)}) \leqslant \gamma(b_y^{(i)}) \\
&\Leftrightarrow b_x^{(i)} \leqslant b_y^{(i)} \Leftrightarrow \Theta_i[x] \leqslant \Theta_i[y].
\end{aligned}$$

At the same time

$$\Theta_i^*[x] = \Theta_i^*[y] \Leftrightarrow \Theta_i[x] = \Theta_i[y].$$

Hence $Q/\Theta_i^* = \overline{Q}_i$, and since the hl-loops Q/Θ_i^* and G_i are isomorphic, we can conclude that $\overline{Q}_i$ and G_i are isomorphic hl-loops. $\square$

5.7. Lemma. *Let Q be an hl-loop. Let there exist a set of representatives R of Q such that R is a subgroupoid of Q. Then any two decompositions $Q\!\uparrow = A \circ B$ and $Q\!\uparrow = C \circ B$, where A, B, C are normal nontrivial subloops of Q, have isomorphic extensions.*

Proof. Each element $x \in Q$ can be uniquely written in the form $x = a_x b_x \cdot r_x$, where $a_x \in A$, $b_x \in B$, $r_x \in R$ and at the same time in the form $x = c_x d_x \cdot r_x$, where $c_x \in C$, $d_x \in B$. Let

$$\alpha_1\colon\ Q \to (\Phi)(Q/\Theta_1 \circ Q/\Theta_2) \tag{10}$$

be an extension of the decomposition $Q\!\uparrow = A \circ B$ and

$$\beta_1\colon\ Q \to (\Phi')(Q/\eta_1 \circ Q/\eta_2) \tag{11}$$

be an extension of the decomposition $Q\!\uparrow = C \circ B$. We are going to show that

$$\psi\colon\ Q/\Theta_1 \to Q/\eta_1;\ \psi(\Theta_1[x]) = \eta_1[x]$$

is an isomorphism of the hl-loop Q/Θ_1 onto Q/η_1. Let $x = a_x b_x \cdot r_x$, $y = a_y b_y \cdot r_y$. If $\Theta_1[x] = \Theta_1[y]$, then, by 5.3, $r_x = r_y, a_x = a_y$. There are unique $c \in C$, $d \in B$ such that $a_x = cd$. Hence $x = (cd \cdot b_x)r_x = (c \cdot db_x)r_x$ and $y = (cd \cdot b_y)r_x = (c \cdot db_y)r_x$. Thus $\eta_1[x] = \eta_1[y]$. Analogously, if $\eta_1[x] = \eta_1[y]$, then $\Theta_1[x] = \Theta_1[y]$. We see that ψ is an injective map. Obviously, ψ is a surjection which preserves the loop operation. Since ψ is an injection, we have that $r_x = r_y$ implies

$$a_x \neq a_y \Leftrightarrow c_x \neq c_y.$$

Thus, provided $r_x = r_y$ we obtain

$$\begin{aligned}\Theta_1[x] < \Theta_1[y] &\Leftrightarrow a_x < a_y \Leftrightarrow x < y,\ a_x \neq a_y\\ &\Leftrightarrow c_x < c_y \Leftrightarrow \eta_1[x] < \eta_1[y].\end{aligned}$$

Hence ψ is an isomorphism of the hl-loop Q/Θ_1 onto Q/η_1.

Now, we are going to show that Q/Θ_2 and Q/η_2 are isomorphic hl-loops. Consider

$$\xi\colon\ Q/\Theta_2 \to Q/\eta_2;\ \xi(\Theta_2[x]) = \eta_2[b_x r_x].$$

It is routine to verify that ξ is a bijection which preserves the partial order. Since B is a normal subloop of Q and R is a subgroupoid of Q, for each $b, d \in B$ and $r, s \in R$ we obtain $br \cdot ds = b_0 r_0$, where $b_0 \in B$, $r_0 = rs \in R$. Using this fact we get that ξ preserves the operation. Thus Q/Θ_2 and Q/η_2 are isomorphic hl-loops. □

6. Isomorphic refinements

Let Q be an hl-loop and let

$$\alpha\colon Q \to (\Phi) \mathop{\Gamma}_{i \in I} G_i, \ I = \{1, 2, \dots n\}, \tag{1}$$

$$\beta\colon Q \to (\Psi) \mathop{\Gamma}_{k \in K} H_k, \ K = \{1, 2, \dots m\} \tag{2}$$

be two lexicographic product decompositions of Q.

6.1. Definition (Cf. [11]). The lexicographic product decomposition β is said to be a refinement of α if for each $i \in I$ there exists a subset $K(i)$ of K and a lexicographic product decomposition

$$\alpha_i\colon G_i \to (\Phi_i) \mathop{\Gamma}_{k \in K(i)} H_k$$

such that, whenever $x \in Q$, $i \in I$ and $k \in K(i)$, then

$$\beta(x)_k = \alpha_i(\alpha(x)_i)_k.$$

We obviously have

6.2. Lemma. *Let α and β be isomorphic lexicographic product decompositions of Q and let α' be a refinement of α. Then there exists a refinement β' of β such that α' and β' are isomorphic.*

Let

$$Q{\uparrow} = A_1 \circ A_2 \circ \dots \circ A_n, \tag{3}$$

where $A_1, A_2, \dots A_n$ are normal subloops of Q. Suppose that for each $i = 1, 2, \dots, n$ there exists a lexicographic product decomposition

$$A_i = A_{i1} \circ A_{i2} \circ \dots \circ A_{ik(i)},$$

where A_{ij} are normal subloops of Q. Then (cf. [3])

$$Q{\uparrow} = A_{11} \circ A_{12} \circ \dots \circ A_{ij} \circ \dots \circ A_{nk(n)}. \tag{4}$$

Now, let

$$\alpha_1\colon Q \to (\Phi)(\overline{Q}_1 \circ \overline{Q}_2 \circ \dots \circ \overline{Q}_n)$$

be an extension of (3) and

$$\beta_1\colon Q \to (\Psi)(\overline{Q}_{11} \circ \overline{Q}_{12} \circ \dots \circ \overline{Q}_{12} \dots \circ \overline{Q}_{nk(n)})$$

be an extension of (4). From the construction of the extensions α_1 and β_1 we obtain

6.3. Lemma. *β_1 is a refinement of α_1.*

6.4. Theorem. *Let Q be an hl-loop and let there exist a set of representatives R of Q such that R is a subgroupoid of Q. Then any two lexicographic product decompositions of Q have isomorphic refinements.*

Proof. If $Q{\downarrow} = \emptyset$, then the assertion is valid in view of [3]. Suppose that $Q{\downarrow} \neq \emptyset$. Let

$$\alpha\colon\ Q \to (\Phi) \mathop{\Gamma}_{i=1}^{n} G_i,$$
$$\beta\colon\ Q \to (\Psi) \mathop{\Gamma}_{k=1}^{m} H_k$$

be two lexicographic product decompositions of Q. We prove the theorem by induction on $n+m$, $n+m \geqslant 2$. It is clear for $n+m = 2$. Let $n+m > 2$. The case $m = 1$ or $n = 1$ is trivial. Assume that $m, n \neq 1$. In the same way as we have constructed the decomposition (8) in Section 5 for γ and the extension γ_1 of γ, we can construct

$$Q{\uparrow} = A_1 \circ A_2 \circ \ldots \circ A_n \quad \text{for } \alpha, \tag{5}$$
$$Q{\uparrow} = B_1 \circ B_2 \circ \ldots \circ B_m \quad \text{for } \beta \tag{6}$$

and the extensions α_1 of (5) and β_1 of (6)

$$\alpha_1\colon\ Q \to (\Phi_1)(\overline{Q}_1 \circ \overline{Q}_2 \circ \ldots \circ \overline{Q}_n),$$
$$\beta_1\colon\ Q \to (\Psi_1)(\overline{G}_1 \circ \overline{G}_2 \circ \ldots \circ \overline{G}_m).$$

By 5.6, α, α_1 are isomorphic decompositions and also β, β_1 are isomorphic decompositions. According to [3; Lemma 4.7(i)] we can suppose without loss of generality that $A_n \subseteq B_m$. Hence, by (6) and [3; Lemma 4.7 (ii)], we have

$$Q{\uparrow} = B_1 \circ B_2 \circ \ldots \circ B_{m-1} \circ B_{m1} \circ B_{m2}, \tag{7}$$

where $B_{m1} = B_m \cap (A_1 \circ A_2 \circ \ldots \circ A_{n-1})$ and $B_{m2} = A_n$. From the construction of (5) and (6) it follows that the subloops A_i and B_j are normal in Q for each $i = 1, 2, \ldots n$, $j = 1, 2, \ldots m$. Since, according to 5.2, B_{m1} is an intersection of normal subloops of Q, B_{m1} is normal in Q. Thus there exists an extension β_2 of (7)

$$\beta_2\colon\ Q \to (\Psi_2)(\overline{G}_1 \circ \overline{G}_2 \circ \ldots \circ \overline{G}_{m-1} \circ \overline{G}_{m1} \circ \overline{G}_{m2}).$$

In view of 6.3 β_2 is a refinement of β_1. Denote

$$A = A_1 \circ A_2 \circ \ldots \circ A_{n-1}$$

and

$$B = B_1 \circ B_2 \circ \ldots \circ B_{m-1} \circ B_{m1}$$

(by 5.2, A, B are normal subloops of Q). Then

$$Q\!\uparrow = A \circ A_n \tag{8}$$

and at the same time

$$Q\!\uparrow = B \circ A_n. \tag{9}$$

Let $Q \to (\Phi')(\overline{Q}_A \circ \overline{Q}_{A_n})$ be an extension of (8) and $Q \to (\Psi')(\overline{G}_B \circ \overline{G}_{A_n})$ be an extension of (9). According to 5.7, $\overline{Q}_A$, $\overline{G}_B$ and also $\overline{Q}_{A_n}$, $\overline{G}_{A_n}$ are isomorphic hl-loops. Moreover, it can be verified that

$$\overline{Q}_{A_n} = \overline{Q}_n \text{ and } \overline{G}_{A_n} = \overline{G}_{m2}. \tag{10}$$

We denote by $\varphi_1, \varphi_2, \ldots, \varphi_n$ the isomorphisms from the system Φ_1 and by $\psi_1, \psi_2, \ldots, \psi_{m2}$ the isomorphisms from Ψ_2. Put $\Phi_1^* = \Phi_1 - \{\varphi_n\}$ and $\Psi_2^* = \Psi_2 - \{\psi_{m2}\}$. There exist decompositions

(I) $\overline{Q}_A \to (\Phi_1^*)(\overline{Q}_1 \circ \overline{Q}_2 \circ \ldots \circ \overline{Q}_{n-1})$;
(II) $\overline{G}_B \to (\Psi_2^*)(\overline{G}_1 \circ \overline{G}_2 \circ \ldots \circ \overline{G}_{m-1} \circ \overline{G}_{m1})$.

By the induction hypothesis there exist lexicographic product decompositions α_1', β_1' such that

- α_1' is a refinement of (I), β_1' is a refinement of (II)
- α_1', β_1' are isomorphic decompositions.

Hence according to (10), α_1 and β_2 have isomorphic refinements. Therefore, by 6.2, the lexicographic product decompositions α and β have isomorphic refinements. $\square$

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