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COHOMOLOGY OPERATIONS AND THE DELIGNE CONJECTURE

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Abstract. The aim of this note, which raises more questions than it answers, is to study natural operations acting on the cohomology of various types of algebras. It contains a lot of very surprising partial results and examples.

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Introduction

In this note, all algebraic objects will be defined over a fixed field $\mathbf{k}$ of characteristic zero. An *algebra* means an algebra over a quadratic Koszul operad $\mathcal{P}$ [26, II.3.3]. This generality covers all "reasonable" algebras—associative, Lie, commutative associative, Poisson, Gerstenhaber, Leibniz, &c.

By the *cohomology* of a $\mathcal{P}$-algebra A we mean the operadic cohomology $H^*_{\mathcal{P}}(A;A)$ of A with coefficients in itself [26, II.3.100], defined as the cohomology of the cochain complex $C^*_{\mathcal{P}}(A;A) = (C^*_{\mathcal{P}}(A;A), d_{\mathcal{P}})$ recalled in A.6 of the appendix to this note. The complex $C^*_{\mathcal{P}}(A;A)$ generalizes the "standard constructions" and $H^*_{\mathcal{P}}(A;A)$ the "classical" cohomology (Hochschild for associative algebras, Chevalley-Eilenberg for Lie algebras, Harrison for associative commutative algebras, &c.) In general, $H^*_{\mathcal{P}}(A;A)$ agrees with the triple cohomology [8, Proposition 8.6] and governs deformations of A in the category of $\mathcal{P}$-algebras.

By a *natural operation* we mean a multilinear operation on $H^*_{\mathcal{P}}(A;A)$ induced by a natural multilinear cochain operation on $C^*_{\mathcal{P}}(A;A)$. Naturality means being defined using data that do not depend on a concrete algebra A. An example is the

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classical cup product $f, g \mapsto f \cup g$ of Hochschild cochains, resp. the induced graded commutative associative multiplication on the Hochschild cohomology of associative algebras [9]. Our definition excludes some operations that are also "natural" in some sense, such as the degree zero unary operation defined as the projection $\pi_n\colon H^*_{\mathcal{P}}(A;A) \to H^n_{\mathcal{P}}(A;A)$, $n \geqslant 0$, because this operation is not induced by any natural cochain map. A precise definition of natural operations is given in Section 7. Our aim is to *describe the homotopy type* of the dg operad $\mathcal{B}_{\mathcal{P}} = \{\mathcal{B}_{\mathcal{P}}(n)\}_{n\geqslant 0}$ of all these natural operations, see Problem 1 and its baby version Problem 20. The reward would be an ultimate understanding of the structure of the cohomology of a given type of algebras.

Our original hope was that the homotopy type of $\mathcal{B}_{\mathcal{P}}$ would be that of another Koszul quadratic operad $\mathcal{Q}_{\mathcal{P}}$ determined by $\mathcal{P}$ in an explicit and simple manner. Examples we had in mind were $\mathcal{P} = \mathcal{A}ss$ for which probably $\mathcal{Q}_{\mathcal{P}} = \mathcal{G}er$, the operad for Gerstenhaber algebras, and $\mathcal{P} = \mathcal{L}ie$ for which probably $\mathcal{Q}_{\mathcal{P}} = \mathcal{L}ie$, the operad for Lie algebras. Calculations presented in this note however show that the homotopy type of $\mathcal{B}_{\mathcal{P}}$ is in general more complicated, therefore the property that makes the homotopy type of $\mathcal{B}_{\mathcal{P}}$ for $\mathcal{P} = \mathcal{A}ss$ or $\mathcal{P} = \mathcal{L}ie$ so nice must be finer than just the Koszulity of $\mathcal{P}$. We have no idea what this property is.

We feel that our formulations are somehow unsatisfactory—we would certainly prefer a concept that would not depend on a "representation" of the cohomology by a concrete cochain complex. In an ideal world, we should be working with natural operations in an appropriate "derived" category in which the cohomology is the hom-functor. The possibility of such a more conceptual approach for associative algebras and their Hochschild cohomology was indicated by [19], see also [15], [16].

Another possibility could be to consider $H^*_{\mathcal{P}}(A;A)$ as the cohomology of the cotangent complex of a suitable suitably derived stack of the variety of structure constants of $\mathcal{P}$-algebras, see [3], [4], [23], and study automorphisms of the point of this stack representing the algebra A. Our feeling is, however, that these fancier approaches, despite their beauty and generality, are still not developed enough to give *concrete* answers to *concrete* questions.

Let us explain the title of this note. In his famous letter [5], P. Deligne asked whether the Gerstenhaber algebra structure on the Hochschild cohomology of an associative algebra given by the cup product and the intrinsic bracket is induced by a natural action of singular chains on the little discs operad. There are several proofs of this so-called *Deligne conjecture* today [17], [20], [28], [29], [33], [14]. Assume one can prove that the operad of all natural operations on the Hochschild complex (that is, $\mathcal{B}_{\mathcal{A}ss}$ in our notation) has the homotopy type of the operad for Gerstenhaber algebras. The formality of the little discs operad [34] would then immediately imply the Deligne conjecture by simple homological considerations.

In fact, most of the proofs of the Deligne conjecture we are aware of [17], [20], [28], [29], involve a conveniently chosen suboperad of $\mathcal{B}_{\mathcal{A}ss}$ whose homotopy type is detected by Fiedorowicz' recognition principle for E_2-operads [7]. We will discuss these proofs in Section 6. Other proofs based on the Etingof-Kazhdan (de)quantization were given in [33], [14]. Several attempts have also been made to find a suitable filtration of the Fulton-MacPherson compactification of the configuration space of points in the plane to prove the conjecture [12], [36]. The Deligne conjecture has surprising implications for the existence of the deformation quantization of Poisson manifolds [14], [33].

1. Formulation of the problem

In this section we state the problems sketched out in the introduction more concretely and formulate also some conjectures. Let $\mathcal{B}_{\mathcal{P}} = (\mathcal{B}_{\mathcal{P}}, \delta_{\mathcal{P}})$ be the dg-operad of all natural multilinear operations on the cochain complex $C^*_{\mathcal{P}}(A;A) = (C^*_{\mathcal{P}}(A;A), d_{\mathcal{P}})$. The nth component $\mathcal{B}_{\mathcal{P}}(n)$ of $\mathcal{B}_{\mathcal{P}}$ is the space of all n-linear natural operations $C^*_{\mathcal{P}}(A;A)^{\otimes n} \to C^*_{\mathcal{P}}(A;A)$ with the grading induced by the grading of $C^*_{\mathcal{P}}(A;A)$: $U \in \mathcal{B}_{\mathcal{P}}(n)$ has degree d if

$$U(f_1, \dots, f_n) \in \mathcal{C}^{m_1+\dots+m_n+d}_{\mathcal{P}}(A;A),$$

whenever $f_i \in \mathcal{C}^{m_i}_{\mathcal{P}}(A;A)$ for $1 \leqslant i \leqslant n$. In this case we write $U \in \mathcal{B}^d_{\mathcal{P}}(n)$. Each $\mathcal{B}_{\mathcal{P}}(n)$ is equipped with the degree $+1$ differential $\delta_{\mathcal{P}}$ induced by the differential $d_{\mathcal{P}}$ of $C^*_{\mathcal{P}}(A;A)$ in the usual way.

A precise definition of the operad $\mathcal{B}_{\mathcal{P}}$ is given in Section 7. Here we emphasize only that $\mathcal{B}^d_{\mathcal{P}}(n) = 0$ for $d < 0$ and that $\mathcal{B}_{\mathcal{P}}(0) \neq 0$ for any nontrivial $\mathcal{P}$. The central problem of the paper reads:

Problem 1. Describe the homotopy type (in the non-abelian derived category) of the dg operad $\mathcal{B}_{\mathcal{P}}$. In particular, calculate the cohomology of $\mathcal{B}_{\mathcal{P}}$.

A baby-version of this problem is Problem 20 of Section 3. Closely related is:

Problem 2. Find a property characterizing operads $\mathcal{P}$ for which $\mathcal{B}_{\mathcal{P}}$ is formal and has the homotopy type of some Koszul quadratic operad.

We will see, in Example 15, a simple quadratic Koszul operad $\mathcal{D}$ such that $H^*(\mathcal{B}_{\mathcal{D}}(0), \delta_{\mathcal{D}}) \neq 0$. This clearly means that $\mathcal{B}_{\mathcal{D}}$ does not have the homotopy type of a quadratic Koszul operad, therefore the property answering Problem 2 must be stronger than Koszulness of $\mathcal{P}$.

Suppose that $\mathcal{P}$ is the symmetrization of a non-Σ operad $\underline{\mathcal{P}}$ [26, Remark II.1.15]. In this case there exists a dg-suboperad $\mathcal{B}_{\underline{\mathcal{P}}}$ of $\mathcal{B}_{\mathcal{P}}$ consisting of natural operations that preserve the order of inputs of $\mathcal{P}$-cochains. For example, the classical cup product $f \cup g \in C^1_{\mathcal{A}ss}(A;A) \cong Lin(A^{\otimes 2}, A)$ of Hochschild cochains $f, g \in C^0_{\mathcal{A}ss}(A;A) \cong Lin(A,A)$ defined as

$$(f \cup g)(a \otimes b) := f(a) \cdot g(b) \quad \text{for } a \otimes b \in A \otimes A,$$

with $\cdot$ denoting the associative multiplication of A, belongs to $\mathcal{B}_{\underline{\mathcal{A}ss}}$, while the operation

$$U(f,g)(a \otimes b) := f(b) \cdot g(a) \quad \text{for } a \otimes b \in A \otimes A,$$

does not, see Definition 34 of Section 7 for details.

Since $\mathcal{B}_{\underline{\mathcal{P}}}(n)$ is a Σ_n-closed subspace of $\mathcal{B}_{\mathcal{P}}(n)$, $n \geqslant 0$, $\mathcal{B}_{\underline{\mathcal{P}}}$ is a usual, not only a non-Σ, operad. We will see in Example 17 that, surprisingly, the homotopy type of $\mathcal{B}_{\underline{\mathcal{P}}}$ in general *differs* from the homotopy type of $\mathcal{B}_{\mathcal{P}}$. We therefore formulate:

Problem 3. Let $\underline{\mathcal{P}}$ be a non-Σ quadratic Koszul operad. Describe the homotopy type of the dg operad $\mathcal{B}_{\underline{\mathcal{P}}}$. In particular, calculate the cohomology of $\mathcal{B}_{\underline{\mathcal{P}}}$.

In Section 6 (i) we give some indications that the operad $\mathcal{B}_{\underline{\mathcal{A}ss}}$ has the homotopy type of the operad $\mathcal{G}er$ for Gerstenhaber algebras, see A.4 for a definition of $\mathcal{G}er$.

One may consider also *strongly homotopy versions* of the above problems. Recall that a strongly homotopy $\mathcal{P}$-algebra is, by [24], an algebra over the minimal model $sh\mathcal{P}$ of the operad $\mathcal{P}$. Let us denote by $sh\mathcal{B}_{\mathcal{P}} = \mathcal{B}_{sh\mathcal{P}}$ the dg-operad of natural operations on the cochain complex $C^*_{sh\mathcal{P}}(A;A)$ for the cohomology of a strongly homotopy algebra A with coefficients in itself. An example of this type of operad is the "minimal operad" M considered in [20], which is a certain suboperad of $\mathcal{B}_{sh\mathcal{A}ss}$, see Section 6 (iii).

It is clear that there exists a canonical map $\mathcal{B}_{sh\mathcal{P}} \to \mathcal{B}_{\mathcal{P}}$, but simple examples show that, again rather surprisingly, this map is in general not a homotopy equivalence. Let us formulate:

Problem 4. Describe the homotopy type of the dg-operad $\mathcal{B}_{sh\mathcal{P}}$ of natural operations on the cohomology of strongly homotopy $\mathcal{P}$-algebras.

Other problems formulated in this paper are Problem 16 of Section 2 and Problems 20, 21 of Section 3.

Let us finally formulate also some conjectures. Although the operads $\mathcal{B}_{\mathcal{P}}$ and $\mathcal{B}_{\mathcal{P}^!}$ are not isomorphic (see Section 7), computational evidences together with an equivalence between the derived category of $\mathcal{P}$ algebras and the derived category of $\mathcal{P}^!$-algebras lead us to believe in:

Conjecture 5. *The homotopy type of the operad $\mathcal{B}_{\mathcal{P}}$ is the same as the homotopy type of $\mathcal{B}_{\mathcal{P}^!}$.*

The following two conjectures concern the homotopy type of $\mathcal{B}_{\mathcal{P}}$ for $\mathcal{P} = \mathcal{A}ss$ and $\mathcal{P} = \mathcal{L}ie$.

Conjecture 6. *The operad $\mathcal{B}_{\mathcal{A}ss}$ has the homotopy type of the operad $\mathcal{G}er$ for Gerstenhaber algebras.*

Some results which may be helpful in the proof of the above conjecture are recalled in Section 6.

Conjecture 7. *The operad $\mathcal{B}_{\mathcal{L}ie}$ has the homotopy type of the operad $\mathcal{L}ie$.*

According to a formality theorem [24, Proposition 3.4], it is enough to prove that

$$H^*(\mathcal{B}_{\mathcal{L}ie}, \delta_{\mathcal{L}ie}) \cong \mathcal{L}ie.$$

Since $H^0(\mathcal{B}_{\mathcal{L}ie}, \delta_{\mathcal{L}ie}) \cong \mathcal{L}ie$ (see Section 4), Conjecture 7 is equivalent to the acyclicity of $\mathcal{B}_{\mathcal{L}ie}$ in positive degrees. Another conjecture, Conjecture 22, is given in Section 4.

Let us finish this section with one exceptional example. The trivial operad $\mathbf{1}$ is a Koszul quadratic self-dual operad. A $\mathbf{1}$-algebra is a vector space A with no operations. Clearly $C^*_{\mathbf{1}}(A; A)$ is just the space $Lin(A, A)$ of linear maps $f\colon A \to A$ concentrated in degree zero with trivial differential, thus $H^*_{\mathbf{1}}(A; A) = Lin(A, A)$. It is also clear that all natural operations on $Lin(A, A)$ are the identity $\mathbb{1}_A \in Lin(A, A)$ considered as a degree zero constant, and iterated compositions

$$Lin(A, A) \ni f_1, f_2, \ldots, f_n \mapsto f_1 \circ f_2 \circ \ldots \circ f_n \in Lin(A, A), \quad n \geqslant 1.$$

Therefore

$$\mathcal{B}_{\mathbf{1}} \cong U\mathcal{A}ss,$$

the operad for unital associative algebras. This example is pathological in that the canonical element introduced in Definition 8 equals zero. Therefore, from now on all quadratic Koszul operads in this note will be *nontrivial* in the sense that $\mathcal{P} \neq \mathbf{1}$.

2. THE CONSTANTS $\mathcal{B}_{\mathcal{P}}(0)$—SOUL WITHOUT BODY

This section, as well as the rest of the paper, relies on terminology and notation recalled in the Appendix. The main result of this part is Proposition 9 which describes the dg-vector space $\mathcal{B}_{\mathcal{P}}(0) = (\mathcal{B}_{\mathcal{P}}(0), \delta_{\mathcal{P}})$ of "constants." It is not difficult to see (compare also Example 35 of Section 7) that

$$\mathcal{B}_{\mathcal{P}}^{m-1}(0) \cong \mathbf{s}\,(\mathcal{P}(m) \otimes \mathcal{P}^!(m))^{\Sigma_m}, \quad m \geqslant 1,$$

with the action $\mathcal{B}_{\mathcal{P}}^{m-1}(0) \to C_{\mathcal{P}}^{m-1}(A;A)$ given as the composition

$$\begin{aligned}(1) \quad & \mathbf{s}\,(\mathcal{P}(m) \otimes \mathcal{P}^!(m))^{\Sigma_m} \xrightarrow{\cong} (\mathbf{s}\mathcal{P}(m) \otimes \mathcal{P}^!(m))^{\Sigma_m} \xrightarrow{\mathbf{s}\alpha\otimes\mathbb{1}} (\mathbf{s}\mathcal{E}nd_A(m) \otimes \mathcal{P}^!(m))^{\Sigma_m} \\ & \cong (\mathcal{E}nd_{\downarrow A}(m) \otimes \mathcal{P}^!(m))^{\Sigma_m} = [Lin((\downarrow A)^{\otimes m}, \downarrow A) \otimes \mathcal{P}^!(m)]^{\Sigma_m} = C_{\mathcal{P}}^{m-1}(A;A).\end{aligned}$$

Since composition (1) is monic for all "generic" $\mathcal{P}$-algebras A, $(\mathcal{B}_{\mathcal{P}}^*(0), \delta_{\mathcal{P}})$ is "morally" the subcomplex of natural elements in $(C_{\mathcal{P}}^*(A;A), d_{\mathcal{P}})$. Before going further, we must recall the following general construction. Let $\mathcal{T}$ be an operad. It is well-known that the formula

$$[f, g] := f \circ g - (-1)^{(m-1)(n-1)} g \circ f,$$

where $f \circ g$ is, for $f \in \mathcal{T}(m)$ and $g \in \mathcal{T}(n)$, defined by

$$f \circ g := \sum_{1 \leqslant i \leqslant m} (-1)^{(n-1)(i-1)} f \circ_i g,$$

makes the direct sum $\mathcal{T}_* = \bigoplus_{m \geqslant 0} \mathcal{T}_*$, with

$$\mathcal{T}_{m-1} := \uparrow^{m-1}\mathcal{T}(m) = \mathbf{s}\mathcal{T}(m),$$

a graded Lie algebra. Another standard fact is that each element $\omega \in \mathcal{T}_1 = \mathbf{s}\mathcal{T}(2)$ satisfying $[\omega, \omega] = 0$ defines a degree $+1$ differential $\delta_\omega\colon \mathcal{T}_* \to \mathcal{T}_{*+1}$ by

$$\delta_\omega(t) := [t, \omega], \quad \text{for } t \in \mathcal{T}_*.$$

It is helpful to observe that the condition $[\omega, \omega] = 0$ means the associativity:

$$(2) \qquad \omega \circ_1 \omega = \omega \circ_2 \omega$$

and that the differential δ_ω in terms of $\circ_i$-operations equals

$$\delta_\omega(t) = t \circ_1 \omega - t \circ_2 \omega + \ldots - (-1)^m t \circ_m \omega + (-1)^m \omega \circ_1 t - \omega \circ_2 t, \quad \text{for } t \in \mathcal{T}(m).$$

As proved in [35], the graded Lie algebra structure $(\mathcal{T}_*, [-,-])$ descends to the space of coinvariants, therefore it induces, via the canonical isomorphism between invariants and coinvariants, a Lie bracket, denoted again $[-,-]$, on the graded vector space $\mathcal{T}^\Sigma_* = \bigoplus_{m\geqslant 0} \mathcal{T}^\Sigma_m$ with pieces

$$\mathcal{T}^\Sigma_{m-1} := \uparrow^{m-1} (\mathcal{T}(m) \otimes sgn_m)^{\Sigma_m} = \mathbf{s}\mathcal{T}(m)^{\Sigma_m}.$$

As usual, an element $\phi \in \mathcal{T}^\Sigma_1 = \mathbf{s}\mathcal{T}(2)^{\Sigma_2}$ satisfying $[\phi,\phi] = 0$ induces a degree $+1$ differential $\delta^\Sigma_\phi \colon \mathcal{T}^\Sigma_* \to \mathcal{T}^\Sigma_{*+1}$ by

$$\delta^\Sigma_\phi t := [\phi, t], \quad \text{for } t \in \mathcal{T}^\Sigma_*. \tag{3}$$

In Proposition 9 below we put $\mathcal{T} := (\mathcal{P} \otimes \mathcal{P}^!)$ and define the differential (3) by taking as ϕ the canonical element χ introduced in the following definition in which # denotes the linear dual.

Definition 8. Let $\mathcal{P}$ be a quadratic Koszul operad. The *canonical element* χ is the element of $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)(2)^{\Sigma_2}$ corresponding, under the standard identification

$$\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)(2) \cong \uparrow(\mathcal{P} \otimes \mathcal{P}^{\#})(2) \cong \uparrow(\mathcal{P}(2) \otimes \mathcal{P}(2)^{\#}) \cong \uparrow Lin(\mathcal{P}(2), \mathcal{P}(2)),$$

to the suspension of the identity map $\uparrow \mathbb{1}_{\mathcal{P}(2)} \in \uparrow Lin(\mathcal{P}(2), \mathcal{P}(2))$.

Observe that χ is symmetric,

$$\chi\tau = \chi \quad \text{for } \tau \in \Sigma_2, \tag{4}$$

therefore indeed $\chi \in \mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)(2)^{\Sigma_2}$. The condition $[\chi, \chi] = 0$ is equivalent to the Jacobi identity (17) for χ which follows from [13, Corollary 2.2.9 (b)], see also the proof of Proposition 26.

Proposition 9. *There is a natural isomorphism of cochain complexes*

$$(\mathcal{B}^*_{\mathcal{P}}(0), \delta_{\mathcal{P}}) \cong ((\mathcal{P} \otimes \mathcal{P}^!)^\Sigma_*, \delta^\Sigma_\chi).$$

If $\mathcal{P}$ is the symmetrization of a non-Σ operad $\underline{\mathcal{P}}$ [26, Remark II.1.15], then there is a similar description of the chain complex $(\mathcal{B}^*_{\underline{\mathcal{P}}}(0), \delta_{\underline{\mathcal{P}}})$ obtained as follows. The definition of the graded Lie algebra $(\mathcal{T}_*, [-,-])$ given above clearly makes sense also when $\mathcal{T}$ is a non-Σ operad. Observe also that there exists the non-Σ quadratic dual $\underline{\mathcal{P}}^!$ of $\underline{\mathcal{P}}$ and that one may introduce the *non-Σ canonical element* $\underline{\chi} \in \mathbf{s}(\underline{\mathcal{P}} \otimes \underline{\mathcal{P}}^!)(2)$ in exactly the same manner as its symmetric version. The element $\underline{\chi}$ obviously satisfies the associativity condition (2):

$$\underline{\chi} \circ_1 \underline{\chi} = \underline{\chi} \circ_2 \underline{\chi}.$$

Our non-Σ version of Proposition 9 reads:

Proposition 10. *Let $\mathcal{P}$ be the symmetrization of a quadratic Koszul non-Σ operad $\underline{\mathcal{P}}$. Then*

$$(\mathcal{B}^*_{\underline{\mathcal{P}}}(0), \delta_{\underline{\mathcal{P}}}) \cong ((\underline{\mathcal{P}} \otimes \underline{\mathcal{P}}^!)_*, \delta_{\underline{\chi}}).$$

Let us make a comment on the meaning of the cohomology $H^*(\mathcal{B}_{\mathcal{P}}(0), \delta_{\mathcal{P}})$. The natural morphism

$$M\colon\ H^*(\mathcal{B}_{\mathcal{P}}(0), \delta_{\mathcal{P}}) \to H^*_{\mathcal{P}}(A; A)$$

induced by action (1) is monic for any "generic" $\mathcal{P}$-algebra A, therefore elements $H^*(\mathcal{B}_{\mathcal{P}}(0), \delta_{\mathcal{P}})$ represent *natural generically nontrivial homology classes* in the cohomology of $\mathcal{P}$-algebras. This leads one to believe that $H^*(\mathcal{B}_{\mathcal{P}}(0), \delta_{\mathcal{P}}) = 0$ for all well-behaved operads, since otherwise people would stumble upon nontrivial natural classes—compare the existence of the invariant non-degenerate symmetric bilinear Killing-Cartan form on any simple Lie algebra, generating a 3-cocycle via $X, Y, Z \mapsto B([X, Y], Z)$. Example 15 however contradicts this reasonable assumption. We believe that $H^*(\mathcal{B}_{\mathcal{P}}(0), \delta_{\mathcal{P}})$ is an important invariant of the operad $\mathcal{P}$ that deserves its own name:

Definition 11. We call the graded vector space $H^*(\mathcal{B}_{\mathcal{P}}(0), \delta_{\mathcal{P}})$ described in Proposition 9 the *soul of the cohomology* of $\mathcal{P}$-algebras.

It is easy to prove that $H^0(\mathcal{B}_{\mathcal{P}}(0), \delta_{\mathcal{P}})$ is always trivial.

Example 12. Let us describe the complex calculating the soul $H^*(\mathcal{B}_{\mathcal{A}ss}(0), \delta_{\mathcal{A}ss})$ of the Hochschild cohomology. Clearly

$$(\mathcal{P} \otimes \mathcal{P}^!)^{\Sigma}_{m-1} = (\mathcal{A}ss \otimes \mathcal{A}ss)^{\Sigma}_{m-1} \cong \mathbf{s}(\mathcal{A}ss \otimes \mathcal{A}ss)(m)^{\Sigma_m} \cong \mathbf{s}\mathcal{A}ss(m),$$

therefore the complex $((\mathcal{A}ss \otimes \mathcal{A}ss)^{\Sigma}, \delta^{\Sigma}_{\chi})$ has the form

$$\mathbf{k} \xrightarrow{\delta^{\Sigma}_{\chi}} \mathbf{k}[\Sigma_2] \xrightarrow{\delta^{\Sigma}_{\chi}} \mathbf{k}[\Sigma_3] \xrightarrow{\delta^{\Sigma}_{\chi}} \mathbf{k}[\Sigma_4] \xrightarrow{\delta^{\Sigma}_{\chi}} \dots \tag{5}$$

It is also easy to describe the differential δ^{Σ}_{χ}; on a permutation $\sigma \in \Sigma_m$ it acts as

$$\delta^{\Sigma}_{\chi}(\sigma) := d_0(\sigma) - d_1(\sigma) + d_2(\sigma) - \dots + (-1)^{m+1} d_{m+1}(\sigma) \in \mathbf{k}[\Sigma_{m+1}],$$

where $d_0(\sigma) := \mathbb{1} \times \sigma$, $d_{m+1}(\sigma) := \sigma \times \mathbb{1}$ and $d_i(\sigma) \in \Sigma_{m+1}$ is the permutation obtained by doubling the ith input of σ. In Theorem 13 below we prove that (5) is acyclic.

Since $\mathcal{A}ss$ is the symmetrization of the non-Σ operad $\underline{\mathcal{A}ss}$, it makes sense to consider also the subcomplex $(\mathcal{B}^*_{\underline{\mathcal{A}ss}}(0), \delta_{\underline{\mathcal{A}ss}})$ of $(\mathcal{B}^*_{\mathcal{A}ss}(0), \delta_{\mathcal{A}ss})$ described in Proposition 10. This subcomplex is obviously isomorphic to the acyclic complex

$$\mathbf{k} \xrightarrow{d_0} \mathbf{k} \xrightarrow{d_1} \mathbf{k} \xrightarrow{d_2} \mathbf{k} \xrightarrow{d_3} \ldots \tag{6}$$

in which $d_{2i} = \mathbb{1}_{\mathbf{k}}$ and $d_{2i+1} = 0$, $i \geqslant 0$. The inclusion $(\mathcal{B}^*_{\underline{\mathcal{A}ss}}(0), \delta_{\underline{\mathcal{A}ss}}) \hookrightarrow (\mathcal{B}^*_{\mathcal{A}ss}(0), \delta_{\mathcal{A}ss})$ sends the generator $1 \in \mathbf{k}$ of the nth piece of (6) into the identity permutation $\mathbb{1}_{n-1} \in \mathbf{k}[\Sigma_{n-1}]$ in the complex (5).

Theorem 13. *The soul $H^*(\mathcal{B}_{\mathcal{A}ss}(0), \delta_{\mathcal{A}ss})$ of the Hochschild cohomology is trivial.*

P r o o f. We must prove that (5) is an acyclic complex. The idea will be to show that it decomposes into a direct sum of acyclic subcomplexes indexed by simple, in the sense introduced below, permutations.

We define first, for each $\sigma \in \Sigma_n$, a natural number $g(\sigma)$, $0 \leqslant g(\sigma) \leqslant n$, which we call the *grade* of σ, as follows. The grade of the unit $\mathbb{1}_n \in \Sigma_n$ is $n-1$, $g(\mathbb{1}_n) := n-1$. For $\sigma \neq \mathbb{1}_n$, let

$$\begin{aligned} a(\sigma) &:= \max\{i;\ \sigma = \mathbb{1}_i \times \tau \text{ for some } \tau \in \Sigma_{n-i}\}, \text{ and} \\ c(\sigma) &:= \max\{j;\ \sigma = \lambda \times \mathbb{1}_j \text{ for some } \lambda \in \Sigma_{n-i}\}. \end{aligned}$$

There clearly exists a unique $\omega = \omega(\sigma) \in \Sigma_{n-a(\sigma)-c(\sigma)}$ such that $\sigma = \mathbb{1}_{a(\sigma)} \times \omega(\sigma) \times \mathbb{1}_{c(\sigma)}$. Let, finally, $b(\sigma)$ be the number of "doubled strings" in $\omega(\sigma)$,

$$b(\sigma) := \{1 \leqslant s < k;\ \omega(s+1) = \omega(s) + 1\}.$$

The grade of σ is then defined by

$$g(\sigma) := a(\sigma) + b(\sigma) + c(\sigma),$$

see Fig. 1 for examples. Observe that the differential δ^{Σ}_{χ} of (5) raises the grade by $+1$.

Let us call $\chi \in \Sigma_k$, $k \geqslant 1$, *simple* if $g(\sigma) = 0$. Observe that, according to our definitions, $\mathbb{1}_n \in \Sigma_n$ is simple if and only if $n = 1$. For each $\sigma \in \Sigma_n$, $\sigma \neq \mathbb{1}_n$, we define a unique simple $\kappa = \kappa(\sigma) \in \Sigma_k$, $k = n - g(\sigma)$, by contracting all "multiple strings" of $\omega(\sigma)$ into "simple" ones, see Fig. 1. We put $\chi(\mathbb{1}_n) := \mathbb{1}_1$.

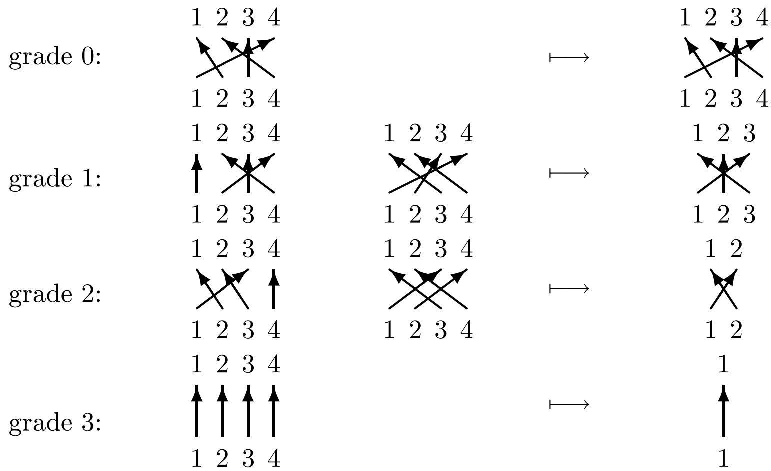


Figure 1. Left: Examples of elements of Σ_4 of grade 0 (first line), grade 1 (second line), grade 2 (third line) and grade 3 (fourth line). Right: the corresponding simple elements.

For a simple χ, let $P^*(\chi)$ be the graded subspace of (5) spanned by all permutations σ with $\chi = \kappa(\sigma)$. The following statements can easily be verified:

(i) Each $P^*(\chi)$ is a subcomplex of (5).

(ii) The complex (5) decomposes as the summation $\bigoplus\limits_{\chi} P^*(\chi)$ over all simple permutations χ.

(iii) For each simple $\chi \in \Sigma_n$,

$$P^*(\chi) \cong P^*(\mathbb{1}_1) \otimes \ldots \otimes P^*(\mathbb{1}_1) \quad ((n+2)\text{times}).$$

The proof is finished by observing that $P^*(\mathbb{1}_1)$ is isomorphic to the acyclic complex (6) and applying the Künneth formula. □

Example 14. In this example we describe the soul of the Chevalley-Eilenberg cohomology of Lie algebras which is, due to the obvious self-duality of Proposition 9, the same as the soul of the Harrison cohomology of commutative associative algebras. In both cases

$$(\mathcal{P} \otimes \mathcal{P}^!)^{\Sigma}_{m-1} = (\mathcal{C}om \otimes \mathcal{L}ie)^{\Sigma}_{m-1} \cong \mathbf{s}\mathcal{L}ie(m)^{\Sigma_m} = \uparrow^{m-1}(\mathcal{L}ie(m) \otimes sgn_m)^{\Sigma_m} \cong \mathbf{k}$$

(see [18]) and one may identify $((\mathcal{C}om \otimes \mathcal{L}ie)^{\Sigma}, \delta^{\Sigma}_{\chi})$ with the acyclic complex (6). Therefore the souls of both the Chevalley-Eilenberg cohomology and the Harrison cohomology are trivial.

Example 15. This example presents a Koszul quadratic operad with a nontrivial soul. Let $\mathcal{D} := \mathcal{A}ss * \mathcal{A}ss$ be the free product of two copies of the associative operad.

$$\begin{tikzcd}
\vdots \\
Lin(A^{\otimes 4}, A) \\
Lin(A^{\otimes 3}, A) \arrow[u, "d_2"] \\
Lin(A^{\otimes 2}, A) \arrow[u, "d_2"] \\
Lin(A, A) \arrow[u, "d_2"] \arrow[r, "d_1"] & Lin(A^{\otimes 2}, A) \arrow[r, "d_1"] & Lin(A^{\otimes 3}, A) \arrow[r, "d_1"] & Lin(A^{\otimes 4}, A) & \cdots
\end{tikzcd}$$

Figure 2. The "meager" bicomplex describing the cohomology of $\mathcal{D}$-algebras.

The operad $\mathcal{D}$ is a Koszul quadratic operad, whose quadratic dual $\mathcal{D}^!$ equals the coproduct $\mathcal{A}ss \vee \mathcal{A}ss$ defined by

$$(\mathcal{A}ss \vee \mathcal{A}ss)(m) := \begin{cases} \mathbf{k}, & \text{if } m = 1 \text{ and} \\ \mathcal{A}ss(m) \oplus \mathcal{A}ss(m), & \text{if } m \geqslant 2. \end{cases}$$

Obviously, $\mathcal{D}$-algebras are triples $A = (A, \mu_1, \mu_2)$ consisting of two independent associative multiplications $\mu_1, \mu_2 \colon A \otimes A \to A$. The cohomology of these algebras is the cohomology of the total complex $(C^*_{\mathcal{D}}(A; A), d_{\mathcal{D}})$ of the "meager" bicomplex in Fig. 2. The horizontal line is the Hochschild cochain complex of the associative algebra $A_1 := (A, \mu_1)$, the vertical line the Hochschild complex of $A_2 := (A, \mu_2)$.

Let e denote the identity $\mathbb{1} \in Lin(A, A)$ considered as a natural element of $C^0_{\mathcal{D}}(A; A)$. Clearly

$$d_{\mathcal{D}}(d_1 e) = d_{\mathcal{D}}(d_2 e) = 0$$

therefore both $d_1 e$ and $d_2 e$ are natural cochains in $C^1_{\mathcal{D}}(A; A)$ thus representing $\delta_{\mathcal{D}}$-cochains in $\mathcal{B}^1_{\mathcal{D}}(0)$. The equality

$$d_{\mathcal{D}} e = d_1 e + d_2 e$$

implies that $d_1 e + d_2 e$ is $\delta_{\mathcal{D}}$-homologous to zero in $\mathcal{B}^1_{\mathcal{D}}(0)$. We conclude that

$$H^1(\mathcal{B}_{\mathcal{D}}(0), \delta_{\mathcal{D}}) \cong \operatorname{Span}([d_1 e]) \cong \operatorname{Span}([d_2 e]) \cong \mathbf{k}.$$

We saw that the souls of the Hochschild ($\mathcal{P} = \mathcal{A}ss$), Chevalley-Eilenberg ($\mathcal{P} = \mathcal{L}ie$) and Harrison ($\mathcal{P} = \mathcal{C}om$) cohomologies were trivial, while the soul of the cohomology for $\mathcal{D}$-algebras analyzed in Example 15 was not. This leads us to formulate:

Problem 16. Which property of a quadratic Koszul operad $\mathcal{P}$ implies the triviality of the soul $H^*((\mathcal{P} \otimes \mathcal{P}^!)^\Sigma, \delta_\chi^\Sigma)$ of the $\mathcal{P}$-cohomology?

Example 17. In this example we describe a non-Σ quadratic Koszul operad with the property that $(\mathcal{B}^*_{\underline{\mathcal{P}}}(0), \delta_{\underline{\mathcal{P}}})$ is acyclic but the soul $(\mathcal{B}^*_{\mathcal{P}}(0), \delta_{\mathcal{P}})$ is not. This shows that the homotopy type of $\mathcal{B}_{\mathcal{P}}$ is in general different from the homotopy type of $\mathcal{B}_{\underline{\mathcal{P}}}$.

Let $\underline{\mathcal{M}ag}$ be the free non-Σ operad generated by one bilinear operation, $\underline{\mathcal{M}ag} := \underline{\Gamma}(\underline{\mu})$, and $\mathcal{M}ag$ its symmetrization. The corresponding cochain complex $C^*_{\mathcal{M}ag}(A; A)$ is the truncation

$$Lin(A, A) \xrightarrow{d} Lin(A^{\otimes 2}, A)$$

of the Hochschild complex. The complex $(\mathcal{B}^*_{\mathcal{M}ag}(0), \delta_{\mathcal{M}ag})$ defining the soul of $\mathcal{M}ag$ is the truncation

$$\mathbf{k} \xrightarrow{\delta_\chi^\Sigma} \mathbf{k}[\Sigma_2]$$

of (5), and is manifestly not acyclic. On the other hand, $(\mathcal{B}^*_{\underline{\mathcal{M}ag}}(0), \delta_{\underline{\mathcal{M}ag}})$ is acyclic, isomorphic to the truncation $\mathbf{k} \xrightarrow{d_0} \mathbf{k}$ of (6). We conclude that $H^*(\mathcal{B}^*_{\underline{\mathcal{M}ag}}(0), \delta_{\underline{\mathcal{M}ag}}) = 0$ while

$$H^*(\mathcal{B}^*_{\mathcal{M}ag}(0), \delta_{\mathcal{M}ag}) = H^1(\mathcal{B}^*_{\mathcal{M}ag}(0), \delta_{\mathcal{M}ag}) \cong \mathbf{k}.$$

3. Homotopy type of $\mathcal{B}(1)$—surprises continue

In this section we study, as a next step toward the understanding of $\mathcal{B}_{\mathcal{P}}$, the homotopy type of the associative dg-algebra $\mathcal{B}_{\mathcal{P}}(1) = (\mathcal{B}^*_{\mathcal{P}}(1), \delta_{\mathcal{P}})$. Since the operad $\mathcal{P}^!$ is a module, in the sense of [24], over itself, it makes sense to consider the space $\mathcal{E}nd_{\mathcal{P}^!}(\mathcal{P}^!)$ of all $\mathcal{P}^!$-module endomorphisms $\alpha \colon \mathcal{P}^! \to \mathcal{P}^!$. Very crucially,

$$End_{\mathcal{P}^!}(\mathcal{P}^!) \cong \mathbf{k}, \tag{7}$$

because each $\alpha \in End_{\mathcal{P}^!}(\mathcal{P}^!)$ is uniquely determined by the value $\alpha_1(1) \in \mathcal{P}^!(1) \cong \mathbf{k}$ and, conversely, for each $\varphi \in \mathbf{k}$ the homomorphism $\alpha := \varphi \cdot \mathbb{1}_{\mathcal{P}^!}$ is such that $\alpha_1(1) = \varphi$.

Proposition 18. *There is a canonical identification of associative unital algebras*

$$H^0(\mathcal{B}^*_{\mathcal{P}}(1), \delta_{\mathcal{P}}) \cong End_{\mathcal{P}^!}(\mathcal{P}^!) \cong \mathbf{k}. \tag{8}$$

P r o o f. Since there are no elements in negative degrees,

$$H^0(\mathcal{B}_{\mathcal{P}}(1)) = \mathrm{Ker}(\delta \colon \mathcal{B}^0_{\mathcal{P}}(1) \to \mathcal{B}^1_{\mathcal{P}}(1)).$$

By definition, elements of the kernel $\mathrm{Ker}(\delta)$ are natural chain maps

$$\{\varphi_m\colon\ C^m_{\mathcal{P}}(A;A) \to C^m_{\mathcal{P}}(A;A)\}_{m\geqslant 0}.$$

As explained in Example 36, the naturality of φ_m means that it is induced by a Σ_{m+1}-equivariant map $\alpha_{m+1}\colon\ \mathcal{P}^!(m+1) \to \mathcal{P}^!(m+1)$. It is easy to verify that the collection $\{\alpha_m\}$ determines a chain map if and only if it assembles into a $\mathcal{P}^!$-module endomorphism $\alpha\colon\ \mathcal{P}^! \to \mathcal{P}^!$. □

The following example shows that the dg-associative algebra $\mathcal{B}_{\mathcal{P}}(1)$ might in general have nontrivial cohomology in positive degrees.

Example 19. Let $\mathcal{S}ym$ be the operad describing algebras with one commutative bilinear multiplication and no axioms. Explicitly, $\mathcal{S}ym$ is the free operad generated by the trivial representation of Σ_2 placed in arity two. It is a Koszul quadratic operad whose quadratic dual $\mathcal{S}ym^!$ is given by $\mathcal{S}ym^!(1) = \mathbf{k}$, $\mathcal{S}ym^!(2) = sgn_2$ (the signum representation of Σ_2), and $\mathcal{S}ym^!(m) = 0$ for $m \geqslant 3$.

The cohomology of a $\mathcal{S}ym$-algebra $A = (A,\ \cdot\ )$ is the cohomology of the two-term complex (which should be interpreted as a truncation of the Harrison complex)

$$Lin(A,A) \xrightarrow{d} Lin(S^2A, A),$$

where S^2A is the second symmetric power of A. The differential d is given by the usual formula

$$(d\phi)(a,b) := a\cdot\phi(b) - \phi(a\cdot b) + \phi(b)\cdot a,$$

for $\phi \in Lin(A,A)$ and $a, b \in A$.

We are going to describe the dg-algebra $\mathcal{B}^*_{\mathcal{S}ym}(1)$. Let α be the projection of $Lin(A,A) \oplus Lin(S^2A,A)$ onto $Lin(A,A)$ and β the projection onto $Lin(S^2A,A)$. Let u and v be degree $+1$ operations given by

$$u(\phi)(a,b) := a\cdot\phi(b) + \phi(a)\cdot b \quad\text{and}\quad v(\phi)(a,b) := \phi(a\cdot b),$$

for $\phi \in Lin(A,A)$ and $a, b \in A$. Then clearly $\mathcal{B}^0_{\mathcal{S}ym}(1)$ is the semisimple algebra $\mathbf{k}\oplus\mathbf{k}$ spanned by α and β, and the space $\mathcal{B}^1_{\mathcal{S}ym}(1)$ is two-dimensional, spanned by u and v. The higher $\mathcal{B}^i_{\mathcal{S}ym}(1)$ are, for $i \geqslant 2$, trivial. To describe the multiplication in $\mathcal{B}^*_{\mathcal{S}ym}(1)$, it is enough to specify how $\mathcal{B}^0_{\mathcal{S}ym}(1)$ acts on $\mathcal{B}^1_{\mathcal{S}ym}(1)$. This action is given by

$$\alpha b = 0 = b\beta \quad\text{and}\quad b\alpha = b = \beta b, \quad \text{for } b \in \mathcal{B}^1_{\mathcal{S}ym}(1).$$

The differential $\delta_{\mathcal{S}ym}$ of $\mathcal{B}^*_{\mathcal{S}ym}(1)$ acts by

$$\delta\alpha = -\delta\beta = u - v, \quad \delta u = \delta v = 0.$$

The cohomology of $(\mathcal{B}^*_{\mathcal{S}ym}(1), \delta_{\mathcal{S}ym})$ can be easily calculated,

$$H^*(\mathcal{B}^*_{\mathcal{S}ym}(1)) \cong \mathbf{k} \oplus W,$$

where W is the vector space spanned by the class $[u]$. We leave as a simple exercise to prove that there exists a quasi-isomorphism $H^*(\mathcal{B}^*_{\mathcal{S}ym}(1)) \to \mathcal{B}^*_{\mathcal{S}ym}(1)$. The dg-associative algebra $\mathcal{B}_{\mathcal{S}ym}(1)$ is therefore *formal*.

Here is a baby version of Problem 1:

Problem 20. Describe the homotopy type of the unital differential graded associative algebra $\mathcal{B}_{\mathcal{P}}(1) = (\mathcal{B}^*_{\mathcal{P}}(1), \delta_{\mathcal{P}})$. In particular, calculate the cohomology of $\mathcal{B}_{\mathcal{P}}(1)$.

We expect that the homotopy type of $\mathcal{B}_{\mathcal{P}}(1)$ is that of $\mathbf{k}$ for all "reasonable" operads, though we do not know what "reasonable" means—the operad $\mathcal{S}ym$ of Example 19 seems reasonable enough, yet the homotopy type of $\mathcal{B}_{\mathcal{S}ym}(1)$ is nontrivial. Let us close this section by formulating:

Problem 21. Which property of the operad $\mathcal{P}$ implies that the dg associative algebra $(\mathcal{B}^*_{\mathcal{P}}(1), \delta_{\mathcal{P}})$ has the homotopy type of $\mathbf{k}$?

4. The operad $H^0(\mathcal{B}_{\mathcal{P}})$ and the intrinsic bracket

It is well-known [25], [31] that the chain complex $C^*_{\mathcal{P}}(A; A)$ always carries a natural dg Lie algebra structure given by the *intrinsic bracket*. The easiest way to construct such a bracket is to identify $C^*_{\mathcal{P}}(A; A)$ with the dg Lie algebra $Coder^*(\mathcal{F}^c_{\mathcal{P}^!}(\downarrow A))$ of coderivations of the cofree nilpotent $\mathcal{P}^!$-coalgebra cogenerated by the desuspension $\downarrow A$ as it was done in [26, Definition II.3.99]. In this way we obtain a natural homomorphism

$$I\colon\ (\mathcal{L}ie, 0) \to (\mathcal{B}_{\mathcal{P}}, \delta_{\mathcal{P}}) \tag{9}$$

of dg operads. If $\mathcal{P}$ is the symmetrization of a non-Σ operad $\underline{\mathcal{P}}$, then $Im(I) \subset \mathcal{B}_{\underline{\mathcal{P}}}$, therefore the map I of (9) factorizes through the natural inclusion $(\mathcal{B}_{\underline{\mathcal{P}}}, \delta_{\underline{\mathcal{P}}}) \hookrightarrow (\mathcal{B}_{\mathcal{P}}, \delta_{\mathcal{P}})$. Computational evidences lead us to:

Conjecture 22. *The natural homomorphism* $I\colon (\mathcal{L}ie, 0) \to (\mathcal{B}_{\mathcal{P}}, \delta_{\mathcal{P}})$ *induces an isomorphism of operads*

$$H^0(\mathcal{B}_{\mathcal{P}}) \cong \mathcal{L}ie,$$

for an arbitrary nontrivial quadratic Koszul $\mathcal{P}$.

We were able to verify Conjecture 22 for $\mathcal{P} = \mathcal{L}ie$, that is, to prove

$$H^0(\mathcal{B}_{\mathcal{L}ie}) \cong \mathcal{L}ie. \tag{10}$$

This isomorphism turned out to be related to a certain characterization of free Lie algebras inside free pre-Lie algebras. More precisely, let $pre\mathbb{L}(X)$ denote the free pre-Lie algebra generated by a set X. The commutator of the pre-Lie product makes $pre\mathbb{L}(X)$ a Lie algebra. Let $\mathbb{L}(X) \subset pre\mathbb{L}(X)$ be the Lie algebra generated by X in $pre\mathbb{L}(X)$. It is not hard to see that $\mathbb{L}(X)$ is in fact isomorphic to the free Lie algebra on X, see also [6]. Then (10) is implied by a very explicit characterization of the subspace $\mathbb{L}(X)$ of $pre\mathbb{L}(X)$.

Similarly, the conjectured isomorphism $H^0(\mathcal{B}_{\underline{\mathcal{A}ss}}) \cong \mathcal{L}ie$ can be translated into a certain characterization of free Lie algebras inside free brace algebras. We were also able to prove that, for an arbitrary quadratic Koszul operad,

$$H^0(\mathcal{B}_{\mathcal{P}}(2)) \cong sgn_2, \tag{11}$$

the signum representation of Σ_2, by describing $H^0(\mathcal{B}_{\mathcal{P}}(2))$ in terms of suitably defined *pairings* $\mathcal{P}^! \otimes \mathcal{P}^! \to \mathcal{P}^!$.

Let us close this section by a couple of remarks which will be useful in the proof of Proposition 26. As we recalled at the beginning of this section, there is a canonical isomorphism $C^*_{\mathcal{P}}(A;A) \cong Coder^*(\mathcal{F}^c_{\mathcal{P}^!}(\downarrow A))$. It is well-known that coderivations of a cofree nilpotent algebra form a natural pre-Lie algebra [26, II.3.9], therefore one has a natural homomorphism of non-dg operads

$$preI\colon\ pre\mathcal{L}ie \to \mathcal{B}_{\mathcal{P}}. \tag{12}$$

The map (9) is then the composition

$$\mathcal{L}ie \longrightarrow pre\mathcal{L}ie \xrightarrow{preI} \mathcal{B}_{\mathcal{P}}$$

of $preI$ with the anti-symmetrization map $\mathcal{L}ie \to pre\mathcal{L}ie$.

5. The cup products

The central statement of this section is Theorem 23 that claims that the suspension $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)$ (see A.3) of the operad $(\mathcal{P} \otimes \mathcal{P}^!)$ acts on $C^*_{\mathcal{P}}(A;A)$, and Theorem 24 that characterizes which elements of $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)$ act via chain maps. Observe that the operad $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)$ need not be quadratic even when $\mathcal{P}$ is.

Theorem 23. *There is a canonical action of the operad* $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)$ *on the graded vector space* $C^*_{\mathcal{P}}(A;A)$*, via natural operations. This action can be interpreted as an inclusion of non-differential graded operads*

$$cup\colon\ \mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!) \hookrightarrow \mathcal{B}_{\mathcal{P}}. \tag{13}$$

P r o o f. The proof relies on the notation introduced/recalled in A.6 and A.5. The "tautological" action of the endomorphism operad $\mathcal{E}nd_A$ on A tensored with the action of $\mathcal{P}^!$ on itself makes the graded vector space $\widetilde{C}^*_{\mathcal{P}}(A;A) = \bigoplus_{m \geqslant 0} \widetilde{C}^m_{\mathcal{P}}(A;A)$ an $\mathbf{s}(\mathcal{E}nd_A \otimes \mathcal{P}^!)$-algebra. It is straightforward to prove that this action induces, via

$$t(f_1,\dots,f_n) := Aver\big(t(\iota(f_1),\dots,\iota(f_n))\big), \tag{14}$$

for $t \in \mathbf{s}(\mathcal{E}nd_A \otimes \mathcal{P}^!)(n)$ and $f_1,\dots,f_n \in C^*_{\mathcal{P}}(A;A)$, an action of $\mathbf{s}(\mathcal{E}nd_A \otimes \mathcal{P}^!)$ on the graded vector space $C^*_{\mathcal{P}}(A;A)$. Suppose that A is a $\mathcal{P}$-algebra, with the structure given by $\alpha\colon \mathcal{P} \to \mathcal{E}nd_A$. The action (13) is obtained by composing the action (14) with the homomorphism $\mathbf{s}(\alpha \otimes \mathbb{1})\colon \mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!) \to \mathbf{s}(\mathcal{E}nd_A \otimes \mathcal{P}^!)$. An alternative description of (13) is given in Example 38 of Section 7. □

We use the inclusion (13) to view $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)$ as a suboperad of $\mathcal{B}_{\mathcal{P}}$. Elements of $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)$ need not be $\delta_{\mathcal{P}}$-closed in $\mathcal{B}_{\mathcal{P}}$; let $\mathcal{Z}_{\mathcal{P}} \subset \mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)$ denote the suboperad of $\delta_{\mathcal{P}}$-cocycles. In Theorem 24, which describes $\mathcal{Z}_{\mathcal{P}}$ explicitly, we use the canonical element χ introduced in Definition 8.

Theorem 24. *The suboperad* $\mathcal{Z}_{\mathcal{P}}$ *of* $\delta_{\mathcal{P}}$*-closed elements in* $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)$ *is characterized as follows:* $t \in \mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)(n)$ *belongs to* $\mathcal{Z}_{\mathcal{P}}(n)$ *if and only if*

$$\chi \circ_2 t + t \circ_1 \chi + (t \circ_2 \chi)(12) + (t \circ_3 \chi)(123) + \ldots + (t \circ_n \chi)(123\ldots n) = 0, \tag{15}$$

where $(123\ldots k) \in \Sigma_{n+1}$ *is the cycle*

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & k & k+1 & \ldots & n+1 \\ 2 & 3 & 4 & \ldots & 1 & k+1 & \ldots & n+1 \end{pmatrix}.$$

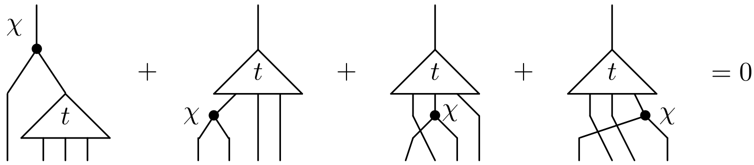


Figure 3. Equation (15) for $n = 3$.

The proof is a completely straightforward calculation. We recommend as an exercise to verify that solutions of (15) are indeed closed under operadic composition. The meaning of the equation (15) should be clear from Fig. 3. The importance of the operad $\mathcal{Z}_{\mathcal{P}}$ is explained by

Corollary 25. *The map cup of* (13) *induces a canonical map (denoted again cup)*

$$\mathit{cup}\colon\ \mathcal{Z}_{\mathcal{P}} \to H^*(\mathcal{B}_{\mathcal{P}}, \delta_{\mathcal{P}}), \tag{16}$$

therefore $H^*_{\mathcal{P}}(A; A)$ is a natural $\mathcal{Z}_{\mathcal{P}}$-algebra.

From reasons which become clear later we call operations induced by elements of $\mathcal{Z}_{\mathcal{P}}$ the *cup products*. The following proposition in which $\mathcal{L}ie$ is the operad for Lie algebras (see A.2) shows that the operad $\mathcal{Z}_{\mathcal{P}}$ is *always* nontrivial (provided $\mathcal{P} \neq \mathbf{1}$) while the map (16) is *never* monic.

Proposition 26. *The operad $\mathcal{Z}_{\mathcal{P}}$ contains the canonical element χ. There exists a unique map $L\colon\ \mathbf{s}\mathcal{L}ie \to \mathcal{Z}_{\mathcal{P}}$ that sends the generator $\mathbf{s}\lambda \in \mathbf{s}\mathcal{L}ie(2)$ into $\chi \in \mathcal{Z}_{\mathcal{P}}(2)$. All elements in the image of this map are $\delta_{\mathcal{P}}$-cohomologous to zero in $\mathcal{B}_{\mathcal{P}}$.*

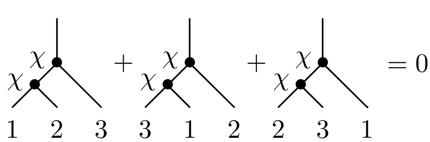
P r o o f. Recall [13, Corollary 2.2.9 (b)] that, for each quadratic operad $\mathcal{P}$, there exits a morphism of operads $\mathcal{L}ie \to \mathcal{P} \otimes \mathcal{P}^!$ that takes the generator $\lambda \in \mathcal{L}ie(2)$ into the identity operator in $\mathcal{P}(2) \otimes \mathcal{P}(2)^{\#} \cong \mathcal{P}(2) \otimes \mathcal{P}^!(2)$. Let $L\colon\ \mathbf{s}\mathcal{L}ie \to \mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)$ be the suspension of this morphism. Let us prove, using Theorem 24, that $\chi \in \mathcal{Z}_{\mathcal{P}}(2)$. Equation (15) for $t = \chi$ reads

$$\chi \circ_2 \chi + \chi \circ_1 \chi + (\chi \circ_2 \chi)(12) = 0,$$

which can be written, due to the symmetry (4) of χ, as the Jacobi identity for a degree 1 "multiplication" χ:

$$\chi \circ_1 \chi + (\chi \circ_1 \chi)(123) + (\chi \circ_1 \chi)(132) = 0, \tag{17}$$

or, pictorially,



But (17) is satisfied, because $\chi = L(\mathbf{s}\lambda)$ by definition, and $\mathbf{s}\lambda \in \mathbf{s}\mathcal{L}ie(2)$ satisfies the same condition in $\mathbf{s}\mathcal{L}ie$. The inclusion $Im(L) \subset \mathcal{Z}_{\mathcal{P}}$ follows from the fact that $Im(L)$ is generated by χ and that $\mathcal{Z}_{\mathcal{P}}$ is a suboperad of $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)$.

Let us prove that all elements in the image of L are $\delta_{\mathcal{P}}$-cohomologous to zero. Let $\ell \in pre\mathcal{L}ie(2)$ be the generator of the quadratic operad $pre\mathcal{L}ie$ for pre-Lie algebras and let $\circ := preI(\ell) \in \mathcal{B}^0_{\mathcal{P}}(2)$, where $preI\colon pre\mathcal{L}ie \to \mathcal{B}_{\mathcal{P}}$ is the map considered in (12) at the end of Section 4. It is easy to verify that then $\chi = \delta_{\mathcal{P}}(\circ)$. This finishes the proof of Proposition 26, because $Im(L)$ is generated by χ. $\square$

Suppose that $\mathcal{P}$ is the symmetrization of a non-Σ operad $\underline{\mathcal{P}}$. Given $t \in \mathbf{s}(\mathcal{P}\otimes\mathcal{P}^!)(n)$ as in Theorem 24, $cup(t) \in \mathcal{B}_{\underline{\mathcal{P}}}(n)$ if and only if t belongs to the Σ_n-closure of $\mathbf{s}(\underline{\mathcal{P}}\otimes\underline{\mathcal{P}}^!)(n)$ in $\mathbf{s}(\mathcal{P}\otimes\mathcal{P}^!)(n)$, that is, if $t = \underline{t}\sigma$ for some $\underline{t} \in \mathbf{s}(\underline{\mathcal{P}}\otimes\underline{\mathcal{P}}^!)(n)$ and $\sigma \in \Sigma_n$. In the following non-Σ version of Theorem 24, $\underline{\chi} \in \mathbf{s}(\underline{\mathcal{P}}\otimes\underline{\mathcal{P}}^!)(2)$ is the non-Σ canonical element introduced in Section 2.

Theorem 27. *An element $\underline{t} \in \mathbf{s}(\underline{\mathcal{P}} \otimes \underline{\mathcal{P}}^!)(n) \subset \mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)(n)$ belongs to $\mathcal{Z}_{\mathcal{P}}(n)$ if and only if*

$$\underline{\chi} \circ_2 \underline{t} = \underline{t} \circ_1 \underline{\chi} = \underline{t} \circ_2 \underline{\chi} = \ldots = \underline{t} \circ_n \underline{\chi} = \underline{\chi} \circ_1 \underline{t}, \tag{18}$$

see Fig. 4.

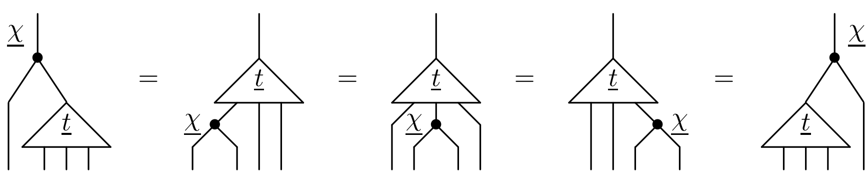


Figure 4. Equation (18) for $n = 3$.

The proof of Theorem 27 is a straightforward verification. The proof of the following proposition is similar to that of Proposition 26.

Proposition 28. *Let $\mathcal{P}$ be the symmetrization of a non-Σ operad $\underline{\mathcal{P}}$. Then $\underline{\chi} \in \mathcal{Z}_{\mathcal{P}}$ and there exists a unique map $A\colon \mathbf{s}\mathcal{A}ss \to \mathcal{Z}_{\mathcal{P}}$ defined by $A(\mathbf{s}\mu) := \underline{\chi}$, where $\mathbf{s}\mu \in \mathbf{s}\mathcal{A}ss(2)$ is the suspension of the generator μ (see* A.2*). Moreover, the diagram*

$$\begin{tikzcd} \mathbf{s}\mathcal{L}ie \arrow[r, "L"] \arrow[d] & \mathcal{Z}_{\mathcal{P}} \\ \mathbf{s}\mathcal{A}ss \arrow[ur, "A"'] & \end{tikzcd}$$

where L is as in Proposition 26, *with the vertical map given by the anti-commutator of the associative product, commutes.*

Example 29 (Hochschild cohomology)**.** Let $\mathcal{P} = \mathcal{A}ss$ be the operad for associative algebras. Then $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!) = \mathbf{s}(\mathcal{A}ss \otimes \mathcal{A}ss)$ and a simple calculation reveals that the map A of Proposition 28 is the suspended diagonal $\mathbf{s}\Delta\colon \mathbf{s}\mathcal{A}ss \to \mathbf{s}(\mathcal{A}ss \otimes \mathcal{A}ss)$ and that $\mathcal{Z}_{\mathcal{A}ss} = Im(A)$. Therefore

$$\mathcal{Z}_{\mathcal{A}ss} \cong \mathbf{s}\mathcal{A}ss.$$

The generator $\mathbf{s}\mu \in \mathbf{s}\mathcal{A}ss(2)$ is mapped to the "classical" cup product $f, g \mapsto f \cup g$ of Hochschild cochains [9], and the generator $\mathbf{s}\lambda \in \mathbf{s}\mathcal{L}ie(2)$ to the anti-commutator of this cup product:

$$f, g \mapsto f \cup g + (-1)^{|f||g|} g \cup f$$

which is cohomologous to zero, because the cup product of Hochschild cochains is homotopy commutative [9, Theorem 3].

Example 30 (Chevalley-Eilenberg cohomology)**.** If $\mathcal{P} = \mathcal{L}ie$ is the operad for Lie algebras, then $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!) = \mathbf{s}(\mathcal{L}ie \otimes \mathcal{C}om) \cong \mathbf{s}\mathcal{L}ie$ and we see immediately that

$$\mathcal{Z}_{\mathcal{L}ie} \cong \mathbf{s}\mathcal{L}ie = Im(L). \tag{19}$$

The generator $\mathbf{s}\lambda \in \mathcal{L}ie(2)$ is mapped to the product $f, g \mapsto \{f, g\}$, which is cohomologous to zero, see [22, Exercise 7].

Example 31 (Harrison cohomology)**.** Here $\mathcal{P} = \mathcal{C}om$ is the operad for commutative associative algebras and $\mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!) = \mathbf{s}(\mathcal{C}om \otimes \mathcal{L}ie) \cong \mathbf{s}\mathcal{L}ie$, therefore, as in Example 30,

$$\mathcal{Z}_{\mathcal{C}om} \cong \mathbf{s}\mathcal{L}ie = Im(L). \tag{20}$$

Equations (19) and (20) illustrate the obvious self-duality of the space of cup products:

$$\mathcal{Z}_{\mathcal{P}^!} \cong \mathcal{Z}_{\mathcal{P}},$$

compare Conjecture 5.

Example 32. If $\mathcal{D} = \mathcal{A}ss * \mathcal{A}ss$ is as in Example 15, then

$$\mathcal{Z}_{\mathcal{D}} = \mathbf{s}\mathcal{A}ss \vee \mathbf{s}\mathcal{A}ss.$$

Let us describe products corresponding to the generators of the 4-dimensional vector space

$$(\mathbf{s}\mathcal{A}ss \vee \mathbf{s}\mathcal{A}ss)(2) = \mathbf{s}\mathcal{A}ss(2) \oplus \mathbf{s}\mathcal{A}ss(2) \cong \uparrow \mathbf{k}[\Sigma_2] \oplus \uparrow \mathbf{k}[\Sigma_2].$$

Recall that $C^*_{\mathcal{D}}(A;A)$ is the total complex of the meager bicomplex in Fig. 2. Let $\cup^1$ (resp. $\cup^2$) be the cup product in the horizontal (resp. vertical) subcomplex in Fig. 2. Let π_1 (resp. π_2) be the projection of $C^*_{\mathcal{D}}(A;A)$ onto the horizontal (vertical) subcomplex. Likewise, let ι_1 (resp. ι_2) be the inclusion. Although neither π_i, ι_i nor $\cup^i$ are chain maps ($i = 1,2$), the compositions

$$f \cup_1 g := \iota_1(\pi_1 f \cup^1 \pi_1 g) \quad \text{and} \quad f \cup_2 g := \iota_2(\pi_2 f \cup^2 \pi_2 g)$$

are chain operations. The generators of $\mathbf{s}\mathcal{A}ss(2) \oplus \mathbf{s}\mathcal{A}ss(2)$ then correspond to the four operations

$$f, g \mapsto f \cup_1 g, \quad f, g \mapsto f \cup_2 g, \quad f, g \mapsto g \cup_1 f \quad \text{and} \quad f, g \mapsto g \cup_2 f.$$

The combination

$$(f \cup_1 g + f \cup_2 g) + (-1)^{|f||g|}(g \cup_1 f + g \cup_2 f)$$

is cohomologous to zero and the image $T(\mathcal{Z}_{\mathcal{D}}(2))$ of $\mathcal{Z}_{\mathcal{D}}$ in $H^1(\mathcal{B}_{\mathcal{D}}(2))$ is easily seen to be 3-dimensional.

6. Operad $\mathcal{B}_{\mathcal{A}ss}$ and the Deligne conjecture

In this section we recall some results related to $\mathcal{B}_{\mathcal{A}ss}$ and the Deligne conjecture. Let us make a necessary comment about our degree convention. We use the grading such that the intrinsic bracket of Section 4 has degree 0 in $\mathcal{B}^*_{\mathcal{P}}(2)$, while the n-fold cup products of Section 5 are of degree $n-1$ in $\mathcal{B}^*_{\mathcal{P}}(n)$. In the literature related to the Deligne conjecture, the convention under which the intrinsic bracket has degree 1 and the n-fold cup products are of degree 0 is often used. These two conventions are tied by the following regrading operator:

$$Reg(\mathcal{B}^*_{\mathcal{P}}(n)) := \mathcal{B}^{n-1-*}_{\mathcal{P}}(n).$$

In what follows we identify operads that differ only by the above regrading. In particular, the operad $\mathcal{G}er$ for Gerstenhaber algebras becomes identified with the operad $\mathcal{B}raid$ for braid algebras (also called 1-algebras), see A.4.

Let us recall that a topological operad $\mathcal{A}$ is an *E_2-operad* if it has the homotopy type of the little discs operad $\mathcal{D}_2$ [27]. According to the Formality Theorem [34], the operad $S_*(\mathcal{A})$ of singular chains on such an operad has the homotopy type of the operad $\mathcal{B}raid$ for braid algebras.

(i) D. Tamarkin and B. Tsygan studied in [32, Section 3] a certain operad $F = \{F(n)\}_{n\geqslant 1}$ of natural operations on the cosimplicial Hochschild complex $C^\bullet(X,X)$ of a topological unital monoid X. The nth space of this cosimplicial set is the space $Cont(X^{\times n}, X)$ of continuous maps from the nth Cartesian power of X to X. For each $n \geqslant 1$, $F(n)$ is a functor $(\Delta^{op})^n \times \Delta \to \mathsf{Sets}$. They then considered a topological operad $E = \{E(n)\}_{n\geqslant 1}$ whose pieces are the topological realizations of these functors and claimed that E is an E_2-operad.

It is not difficult to see that the operad $CN_*(F)$ of normalized chains of F coincides with the operad $\{\mathcal{B}_{\underline{\mathcal{A}ss}}(n)\}_{n\geqslant 1}$ (our $\mathcal{B}_{\underline{\mathcal{A}ss}}$ without constants). Since $(\mathcal{B}^*_{\underline{\mathcal{A}ss}}(0), \delta_{\underline{\mathcal{A}ss}})$ is acyclic (see Example 12), we could conclude that $\mathcal{B}_{\underline{\mathcal{A}ss}}$ has the homotopy type of $\mathcal{G}er$, but we must bear in mind that the arguments in [32] were merely sketched.

(ii) J. E. McClure and J. H. Smith considered in [29] a dg-suboperad $\mathcal{S}_2$ of their "sequence" operad $\mathcal{S}$ and proved that $\mathcal{S}_2$ naturally acts on the Hochschild cochain complex of an associative algebra. In our terminology this means that they constructed a canonical map $\mathcal{S}_2 \to \mathcal{B}_{\mathcal{A}ss}$. They then verified the Deligne conjecture by showing, using a result of [1], that $\mathcal{S}_2$ has the homotopy type of the singular chain complex $S_*(\mathcal{D}_2)$ of the little discs operad. Their proof is a very reliable one.

(iii) M. Kontsevich and Y. Soibelman [20] introduced a "minimal operad" M naturally acting on the Hochschild cochain complex of an A_∞-algebra. In our terminology, M was a suboperad of $\mathcal{B}_{sh\mathcal{A}ss}$ generated by braces and cup products. They then argued that M has the homotopy type of the operad of suitably defined piecewise algebraic chains on the operad FM_2 of the Fulton-MacPherson compactification of the configuration space of points in $\mathbb{R}^2$. Since FM_2 is, by [30, Proposition 3.9], an E_2-operad, they concluded that M has the homotopy type of $\mathcal{G}er$.

(iv) R. M. Kaufmann realized in [17] that the cellular chains $CC_*(\mathcal{C}act^1)$ on his operad-up-to-homotopy $\mathcal{C}act^1$ of spineless normalized cacti is an honest operad which naturally acts on the Hochschild cochain complex, via braces and cup products. By comparing $\mathcal{C}act^1$ to the operad $\mathcal{C}act$ of spineless (non-normalized) cacti, he concluded that $CC_*(\mathcal{C}act^1)$ is a model for chains on the little discs operad $\mathcal{D}_2$.

All the proofs of the Deligne conjecture mentioned above use some special features of associative algebras and E_2-operads, such as the cosimplicial structure of the Hochschild cochain complex, Fiedorowicz' detection principle, or a relation to the Fulton-MacPherson and cacti operads. None of these features are available for a general operad $\mathcal{P}$; we therefore think that the analysis of the homotopy type of $\mathcal{B}_\mathcal{P}$ for a general Koszul quadratic $\mathcal{P}$ is substantially more difficult than the analysis of $\mathcal{B}_{\mathcal{A}ss}$.

Let us mention that there are other approaches to the Deligne conjecture, as D. Tamarkin's proofs that use the Etingof-Kazhdan quantization [14], [33], or those based on a suitable filtration of the Fulton-MacPherson compactification FM_2, see E. Getzler and J. D. S. Jones [12] or A. A. Voronov [36].

7. Natural operations

Let us recall the following definitions which can be found for example in [26, Section II.1.5]. By a *tree* we mean a connected graph T without loops. A *valence* of a vertex v of T is the number of edges adjacent to v. A *leg* or *leaf* of T is an edge adjacent to a vertex of valence one, other edges of T are *interior.* We in fact discard vertices of valence one at the endpoints of the legs, therefore the legs become "half-edges" having only one vertex. By a *rooted* or *directed* tree we mean a tree with a distinguished *output* leg called the *root.* The remaining legs are called the *input legs* of the tree. A tree with a input legs labelled by elements of the set $\{1, 2, \ldots, a\}$ is called an *a-tree.* A rooted tree is automatically *oriented*, each edge pointing towards the root. The edges pointing towards a given vertex v are called the *input edges* of v, the number of these input edges is then the *arity* of v denoted $ar(v)$. Vertices of arity one are called unary, vertices of arity two binary, vertices of arity three ternary, etc.

Notation. Let n, m and $m_1, \ldots, m_n$ be non-negative integers. In the rest of this section, i will always denote an integer between 0 and n, $a := m+1$ and $a_i := m_i + 1$. We will also assume the notation introduced in A.6.

An *n-linear natural operation*

$$U\colon\ C^{m_1}_{\mathcal{P}}(A;A) \otimes \ldots \otimes C^{m_n}_{\mathcal{P}}(A;A) \to C^{m}_{\mathcal{P}}(A;A)$$

is given by the following data.

(i) A rooted a-tree T with n white vertices $w_1, \ldots, w_n$ of arities $a_1, \ldots, a_n$, and k at least binary black vertices, $k \geqslant 0$.

(ii) A linear order on the set of input edges of each white vertex of T.

(iii) A decoration of black vertices of T by elements of $\mathcal{P}$.

(iv) A linear map $\Phi\colon\ \mathbf{s}\mathcal{P}^!(a_1) \otimes \ldots \otimes \mathbf{s}\mathcal{P}^!(a_n) \to \mathbf{s}\mathcal{P}^!(a)$.

Given the above data and $f_i \in C^{m_i}_{\mathcal{P}}(A;A)$, the value $U(f_1, \ldots, f_n) \in C^{m}_{\mathcal{P}}(A;A)$ is defined as follows. Let us decompose

$$f_i = \sum_{\kappa_i} \phi_i^{\kappa_i} \otimes q^i_{\kappa_i} \in [Lin(A^{\otimes a_i}, A) \otimes \mathbf{s}\mathcal{P}^!(a_i)]^{\Sigma_{a_i}} \cong C^{m_i}_{\mathcal{P}}(A;A),$$

where $\phi_i^{\kappa_i} \in Lin(A^{\otimes a_i}, A)$, $q^i_{\kappa_i} \in \mathbf{s}\mathcal{P}^!(a_i)$ and κ_i is a summation index. Since the inputs of white vertices are linearly ordered, each $\phi_i^{\kappa_i}$ determines a decoration of the white vertex w_i by an element of $Lin(A^{\otimes a_i}, A) = \mathcal{E}nd_A(a_i)$. Recall that A is a $\mathcal{P}$-algebra with the structure homomorphism $\alpha\colon\ \mathcal{P} \to \mathcal{E}nd_A$. Applying α to the decorations of the black vertices we decorate also black vertices with elements of $\mathcal{E}nd_A$. So T is now a tree with all vertices decorated by $\mathcal{E}nd_A$. The composition in the operad $\mathcal{E}nd_V$ along T [13] determines, for each $k_1, \ldots, k_n$, the element

$$T(\phi_1^{\kappa_1}, \ldots, \phi_n^{\kappa_n}) \in Lin(A^{\otimes a}, A).$$

Let

$$\begin{aligned}\widetilde{U}(f_1, \ldots, f_n) := \sum_{\kappa_1, \ldots, \kappa_n} T(\phi_1^{\kappa_1}, \ldots, \phi_n^{\kappa_n}) \otimes \Phi(q^1_{\kappa_1}, \ldots, q^n_{\kappa_n}) \in Lin(A^{\otimes a}, A) \otimes \mathbf{s}\mathcal{P}^!(a)\\ \cong \widetilde{C}^m_{\mathcal{P}}(A; A).\end{aligned}$$

Finally, let $U(f_1, \ldots, f_n) := Aver(\widetilde{U}(f_1, \ldots, f_n)) \in C^m_{\mathcal{P}}(A; A)$. It follows from an elementary combinatorics of trees that

$$\deg(U) = ar(b_1) + \ldots + ar(b_k) - k,$$

therefore $\deg(U)$ is *always non-negative* and $\deg(U) = 0$ if and only if T has no black vertex.

Definition 33. Let $\mathcal{B}_{\mathcal{P}} := \{\mathcal{B}_{\mathcal{P}}(n)\}_{n \geqslant 0}$ be the operad spanned by all natural operations $U = U_{(T,\Phi)}$ in the above sense. Since the differential $d_{\mathcal{P}}$ of $C^*_{\mathcal{P}}(A; A)$ is itself a natural operation living in $\mathcal{B}^1_{\mathcal{P}}(1)$, it induces a differential $\delta_{\mathcal{P}}$ on $\mathcal{B}_{\mathcal{P}}$ by the standard formula

$$\begin{aligned}\delta_{\mathcal{P}}(U)(f_1, \ldots, f_n) &:= d_{\mathcal{P}} U(f_1, \ldots, f_n)\\ &\quad - (-1)^{|U|} \sum_{1 \leqslant i \leqslant n} (-1)^{|f_1| + \ldots + |f_{i-1}|} U(f_1, \ldots, d_{\mathcal{P}} f_i, \ldots, f_n),\end{aligned}$$

making $\mathcal{B}_{\mathcal{P}} = (\mathcal{B}^*_{\mathcal{P}}, \delta_{\mathcal{P}})$ a dg-operad.

Heuristically, the value $U_{(T,\Phi)}(f_1, \ldots, f_n)$ is given by inserting f_i at the vertex w_i of T, $1 \leqslant i \leqslant n$, and then performing the composition along Φ. The operadic composition of $\mathcal{B}_{\mathcal{P}}$ is the vertex insertion similar to that of [2] and the symmetric group permutes the labels of white vertices. In the following definition we introduce a non-Σ version of $\mathcal{B}_{\mathcal{P}}$.

Definition 34. Suppose $\mathcal{P}$ is the symmetrization of a non-Σ operad $\underline{\mathcal{P}}$. Let $\mathcal{B}_{\underline{\mathcal{P}}}$ be the dg-suboperad of $\mathcal{B}_{\mathcal{P}}$ spanned by natural operations $U_{(T,\Phi)}$ as in Definition 33 such that the tree T is planar, with black vertices decorated by elements of $\underline{\mathcal{P}}$, and the map Φ such that

$$\Phi(\mathbf{s}\underline{\mathcal{P}}^!(a_1) \otimes \ldots \otimes \mathbf{s}\underline{\mathcal{P}}^!(a_n)) \subset \mathbf{s}\underline{\mathcal{P}}^!(a).$$

Example 35 (Constants)**.** Let us see what happens if T is the a-corolla with one black vertex decorated by $p \in \mathcal{P}(a)$ and no white vertices as in Fig. 5. The map $\Phi\colon \mathbf{k} \to \mathbf{s}\mathcal{P}^!(a)$ is given by specifying an element $\varphi := \Phi(1) \in \mathbf{s}\mathcal{P}^!(a)$ and $\widetilde{U}$ determined by this Φ equals $\alpha(p) \otimes \varphi \in \widetilde{C}^m_{\mathcal{P}}(A; A)$. Since α is equivariant,

$$Aver(\alpha(p) \otimes \varphi) = (\alpha \otimes \mathbb{1})(Aver(p \otimes \varphi))$$

therefore $U := Aver(\alpha(p) \otimes \varphi) \in C^m_{\mathcal{P}}(A; A)$ is parametrized by an element in the image of the averaging map

$$Aver\colon\ \mathcal{P}(a) \otimes \mathbf{s}\mathcal{P}^!(a) \to (\mathcal{P}(a) \otimes \mathbf{s}\mathcal{P}^!(a))^{\Sigma_a},$$

in other words,

$$\mathcal{B}^m_{\mathcal{P}}(0) \cong \mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)(a)^{\Sigma_a}, \quad m \geqslant 0.$$

It is equally easy to see that, for a quadratic Koszul non-Σ operad $\underline{\mathcal{P}}$,

$$\mathcal{B}^m_{\underline{\mathcal{P}}}(0) \cong \mathbf{s}(\underline{\mathcal{P}} \otimes \underline{\mathcal{P}}^!)(a), \quad m \geqslant 0.$$

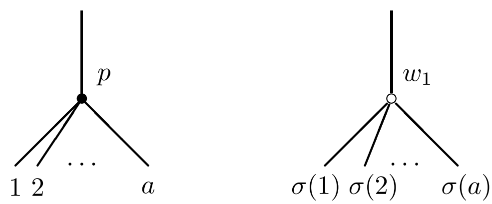


Figure 5. The tree defining a constant in $\mathcal{B}^m_{\mathcal{P}}(0)$ (left) and a unary operation in $\mathcal{B}^0_{\mathcal{P}}(1)$ (right), where $m = a + 1$ as always.

Example 36 (Unary operations of degree 0)**.** Now T is an a-corolla with one white planar vertex and no black vertices, with input legs labelled $\sigma(1), \dots, \sigma(a)$, $\sigma \in \Sigma_a$, as shown in Fig. 5, and $\Phi\colon\ \mathbf{s}\mathcal{P}^!(a) \to \mathbf{s}\mathcal{P}^!(a)$ is a linear map. If

$$f = \sum_\kappa \phi^\kappa \otimes q_\kappa \in [Lin(A^{\otimes a}, A) \otimes \mathbf{s}\mathcal{P}^!(a)]^{\Sigma_a} \cong C^m_{\mathcal{P}}(A; A),$$

then $U(f) = Aver(\sum_\kappa \phi^\kappa \sigma^{-1} \otimes \Phi(q_\kappa))$. Since $f = \sum \phi^\kappa \otimes q_\kappa$ is Σ_a-stable,

$$U(f) = \sum \phi^\kappa \otimes Aver(\Phi_\kappa)(q_\kappa).$$

Therefore $U(f)$ is given by a Σ_a-equivariant map $\Psi :=\downarrow^{a-1} Aver(\Phi)\colon\ \mathcal{P}^!(a) \to \mathcal{P}^!(a)$, thus

$$\mathcal{B}^0_{\mathcal{P}}(1) \cong Lin_\Sigma(\mathcal{P}^!, \mathcal{P}^!), \tag{21}$$

the space of all collections $\{\psi_n\colon \mathcal{P}^!(n) \to \mathcal{P}^!(n)\}_{n\geqslant 0}$ of equivariant maps. We leave as an exercise to verify that, for a non-Σ quadratic Koszul operad $\underline{\mathcal{P}}$,

$$\mathcal{B}^0_{\underline{\mathcal{P}}}(1) \cong Lin(\underline{\mathcal{P}}^!, \underline{\mathcal{P}}^!).$$

Example 37 (Projections)**.** Let $p_m \in \mathcal{B}^0_{\mathcal{P}}(1)$ be given, in identification (21), by $\Psi \in Lin_\Sigma(\mathcal{P}^!, \mathcal{P}^!)$ defined as

$$\Psi|_{\mathcal{P}^!(a)} = \begin{cases} \mathbb{1}_{\mathcal{P}^!(a)}, & \text{for } a = m+1 \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, p_m is the projection $C^*_{\mathcal{P}}(A;A) \twoheadrightarrow C^m_{\mathcal{P}}(A;A)$. The system of all these projections makes $\mathcal{B}_{\mathcal{P}}$ an $\mathbb{Z}^{\geqslant 0}$-colored operad, where $\mathbb{Z}^{\geqslant 0}$ is the set of non-negative integers. Since these projections do not commute with $d_{\mathcal{P}}$ (that is $\delta_{\mathcal{P}}(p_m) \neq 0$ for a generic $\mathcal{P}$), $(\mathcal{B}_{\mathcal{P}}, \delta_{\mathcal{P}})$ *is not* a *dg* $\mathbb{Z}^{\geqslant 0}$-colored operad.

Example 38 (Cup products)**.** In this example we explain how an element

$$t = p \otimes \mathbf{s}q \in \mathcal{P}(n) \otimes \mathbf{s}\mathcal{P}^!(n) \cong \mathbf{s}(\mathcal{P} \otimes \mathcal{P}^!)(n)$$

determines a natural operation in $\mathcal{B}^{n-1}_{\mathcal{P}}(n)$. Let T be as in Fig. 6, with the black vertex decorated by $p \in \mathcal{P}(n)$, and let the linear map $\Phi\colon \mathbf{s}\mathcal{P}^!(a_1) \otimes \ldots \otimes \mathbf{s}\mathcal{P}^!(a_n) \to \mathbf{s}\mathcal{P}^!(a)$ be given by the operadic composition in $\mathbf{s}\mathcal{P}^!$:

$$\Phi(\mathbf{s}q_1, \ldots, \mathbf{s}q_n) := \mathbf{s}q(\mathbf{s}q_1, \ldots, \mathbf{s}q_n), \quad q_i \in \mathcal{P}^!(a_i), \quad 1 \leqslant i \leqslant n.$$

It is more or less clear that the natural operation $U_{(T,\Phi)}$ determined by the above data agrees with the cup product $cup(t)$ of Theorem 23. We recommend as another exercise to verify that also the intrinsic bracket described in (9) is given by natural operations in the sense of this section.

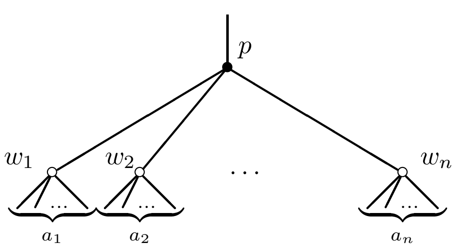


Figure 6. The tree defining the cup product.

Example 39. Let us describe all natural operations $C^1_{\mathcal{P}}(A;A) \otimes C^1_{\mathcal{P}}(A;A) \to C^2_{\mathcal{P}}(A;A)$ for some particular choices of $\mathcal{P}$.

(i) Hochschild cohomology. For $\mathcal{P} = \mathcal{A}ss$, $C^1_{\mathcal{P}}(A;A) = Lin(A^{\otimes 2}, A)$, $C^2_{\mathcal{P}}(A;A) = Lin(A^{\otimes 3}, A)$, and the only natural operations $C^1_{\mathcal{P}}(A;A) \otimes C^1_{\mathcal{P}}(A;A) \to C^2_{\mathcal{P}}(A;A)$ are linear combinations of

$$f, g \to (f \circ_1 g)\sigma,\ \ f, g \to (f \circ_2 g)\sigma,\ \ f, g \to (g \circ_1 f)\sigma,\ \ f, g \to (g \circ_2 f)\sigma,\ \ \sigma \in \Sigma_3,$$

where $\circ_1$, $\circ_2$ are Gerstenhaber-type products [9] given by

$$(u \circ_1 v)(a, b, c) := u(v(a, b), c)), \quad (u \circ_2 v)(a, b, c) := u(a, v(b, c)), \tag{22}$$

for $u, v \in C^1_{\mathcal{P}}(A;A)$, $a, b, c \in A$, and $\sigma \in \Sigma_3$ permutes the factors of $A^{\otimes 3}$ in the usual way. Operations belonging to $\mathcal{B}^0_{\underline{\mathcal{A}ss}}(2)$ are linear combinations of the operations (22) with $\sigma = \mathbb{1}_3$, the unit of Σ_3.

(ii) Chevalley-Eilenberg cohomology. If $\mathcal{P} = \mathcal{L}ie$, then $C^1_{\mathcal{P}}(A;A) = Lin(\wedge^2 A, A)$, $C^2_{\mathcal{P}}(A;A) = Lin(\wedge^3 A, A)$, where $\wedge^n A$ denotes the nth exterior power. The only natural operations $C^1_{\mathcal{P}}(A;A) \otimes C^1_{\mathcal{P}}(A;A) \to C^2_{\mathcal{P}}(A;A)$ are linear combinations of

$$f, g \to f \circ g \quad \text{and} \quad f, g \to g \circ f,$$

where

$$(u \circ v)(a, b, c) := u(v(a, b), c) + u(v(b, c), a) + u(v(c, a), b)$$

for $u, v \in C^2_{\mathcal{P}}(A;A)$ and $a, b, c \in A$.

(iii) Harrison cohomology. If $\mathcal{P} = \mathcal{C}om$, then

$$C^1_{\mathcal{P}}(A;A) = \{u \in Lin(A^{\otimes 2}, A);\ u(a, b) - u(b, a) = 0\}$$

and $C^2_{\mathcal{P}}(A;A)$ consists of all $w \in Lin(A^{\otimes 3}, A)$ such that

$$w(a, b, c) - w(b, a, c) + w(b, c, a) = w(a, b, c) - w(a, c, b) + w(c, a, b) = 0,$$

for $a, b, c \in A$. Natural operations $C^1_{\mathcal{P}}(A;A) \otimes C^1_{\mathcal{P}}(A;A) \to C^2_{\mathcal{P}}(A;A)$ are linear combinations of

$$f, g \to f \circ g \quad \text{and} \quad f, g \to g \circ f,$$

where

$$u \circ v := u(v(a, b), c) - u(v(b, c), a),$$

for $u, v \in C^2_{\mathcal{P}}(A;A)$ and $a, b, c \in A$.

A.1. In this note, an *operad* means an operad in the category of differential graded (dg) vector spaces, that is, a sequence $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geqslant 0}$ of right Σ_n-modules with structure operations

$$\gamma\colon\ \mathcal{P}(n)\otimes\mathcal{P}(k_1)\otimes\ldots\otimes\mathcal{P}(k_n)\to\mathcal{P}(k_1+\ldots+k_n),$$

for $n\geqslant 1$ and $k_1,\ldots,k_n\geqslant 0$, and a unit map $\eta\colon\ \mathbf{k}\to\mathcal{P}(1)$ that satisfy the usual axioms [27], [21]. Instead of $\gamma(p\otimes p_1\otimes\ldots\otimes p_n)$ we will often write $\gamma(p,p_1,\ldots,p_n)$ or $p(p_1,\ldots,p_n)$. Recall [24] that operads can be equivalently defined using the $\circ_i$-operations

$$\circ_i\colon\ \mathcal{P}(m)\otimes\mathcal{P}(n)\to\mathcal{P}(m+n-1)$$

defined, for $m,n\geqslant 0$, $1\leqslant i\leqslant m$, by

$$p\circ_i q := \gamma(p\otimes e^{\otimes(i-1)}\otimes q\otimes e^{\otimes m-i}),$$

where $e := \eta(1)$.

If we remove from the above definition all references to the symmetric group actions, we get a definition of a *non-Σ operad.* Each non-Σ operad $\underline{\mathcal{P}}$ generates a unique (usual) operad $\mathcal{P}$ such that $\mathcal{P}(n)\cong\underline{\mathcal{P}}(n)\otimes\mathbf{k}[\Sigma_n]$, $n\geqslant 0$.

A.2. For each set of operations E, there exists the *free operad* $\Gamma(E)$ generated by E [26, Proposition II.1.92]. Let μ denote a bilinear operation placed in degree 0. The operad $\mathcal{A}ss$ for *associative algebras* is the quotient

$$\mathcal{A}ss := \Gamma(\mu)/(\mu\circ_1\mu-\mu\circ_2\mu),$$

where $(\mu\circ_1\mu-\mu\circ_2\mu)$ denotes the operadic ideal generated by the associativity axiom for μ.

If λ is a skew-symmetric bilinear operation, then the operad for *Lie algebras* is the quotient

$$\mathcal{L}ie := \Gamma(\lambda)/(Jacobi(\lambda)),$$

where

$$Jacobi(\lambda) := \sum_{\sigma\in C_3}(\lambda\circ_1\lambda)\sigma$$

with the summation taken over the order 3 cyclic subgroup C_3 of Σ_3, denotes the Jacobi identity for λ.

Finally, for an arbitrary differential graded vector space V, there is the *endomorphism operad* $\mathcal{E}nd_V = \{Lin(V^{\otimes n}, V)\}_{n \geqslant 0}$, with structure operations given as the usual composition of multilinear maps. A $\mathcal{P}$-*algebra* is then a homomorphism $\alpha\colon \mathcal{P} \to \mathcal{E}nd_V$. We sometimes call α also an *action* of $\mathcal{P}$ on V.

A.3. The *suspension* $\mathbf{s}A = \{\mathbf{s}A(n)\}_{n\geqslant 0}$ of a Σ-module $A = \{A(n)\}_{n\geqslant 0}$ is defined by

$$\mathbf{s}A(n) := \uparrow^{n-1} A(n) \otimes sgn_n,$$

where sgn_n denotes the signum representation of Σ_n, see [26, Definition II.3.15]. If $\mathcal{P}$ is an operad, then the collection $\mathbf{s}\mathcal{P}$ carries a canonical induced operad structure and the operad $\mathbf{s}\mathcal{P}$ is called the *operadic suspension* of $\mathcal{P}$. For any two operads $\mathcal{P}$ and $\mathcal{Q}$,

$$\mathbf{s}(\mathcal{P} \otimes \mathcal{Q}) \cong \mathbf{s}\mathcal{P} \otimes \mathcal{Q} \cong \mathcal{P} \otimes \mathbf{s}\mathcal{Q}.$$

A.4. An (m, n)-*algebra* is [8, Example 9.4] a graded vector space A together with two bilinear maps, $- \cup - : A \otimes A \to A$ of degree m, and $[-,-]\colon A \otimes A \to A$ of degree n (m and n are natural numbers), such that, for any homogeneous $a, b, c \in A$,

(i) $a \cup b = (-1)^{|a|\cdot|b|+m} \cdot b \cup a$,

(ii) $[a, b] = -(-1)^{|a|\cdot|b|+n} \cdot [b, a]$,

(iii) $- \cup -$ is associative in the sense that

$$a \cup (b \cup c) = (-1)^{m\cdot(|a|+1)} \cdot (a \cup b) \cup c,$$

(iv) $[-,-]$ satisfies the following form of the Jacobi identity:

$$(-1)^{|a|\cdot(|c|+n)} \cdot [a, [b, c]] + (-1)^{|b|\cdot(|a|+n)} \cdot [b, [c, a]] + (-1)^{|c|\cdot(|b|+n)} \cdot [c, [a, b]] = 0,$$

(v) the operations $- \cup -$ and $[-,-]$ are compatible in the sense that

$$(-1)^{m\cdot|a|}[a, b \cup c] = [a, b] \cup c + (-1)^{(|b|\cdot|c|+m)}[a, c] \cup b.$$

$(0, 1)$-algebras were considered in [12] under the name 2-*algebras* or *braid algebras*. The corresponding operad $\mathcal{B}raid$ is isomorphic to the homology of the little discs operad $\mathcal{D}_2$, $\mathcal{B}raid \cong H_*(\mathcal{D}_2)$. Following [11, Section 10], we call $(1, 0)$-algebras *Gerstenhaber algebras*, though the terminology is not unique, compare for instance [10, Subsection 10.2] where a Gerstenhaber algebra means a $(0, -1)$-algebra.

A.5. Let M be a right module over a finite group G. We denote, as usual

$$M^G := \{m \in M;\ mg = g \text{ for all } g \in G\} \text{ and } M_G := \frac{M}{(m - mg;\ m \in M,\ g \in G)}.$$

Let $Aver\colon M \to M^G$ be the "averaging" map given by

$$Aver(m) := \frac{1}{|G|} \sum_{g \in G} mg.$$

It is a standard fact that the composition $\pi\iota$ of the projection $\pi\colon M \twoheadrightarrow M_G$ with the inclusion $\iota\colon M^G \hookrightarrow M$ is the identity and that $Aver$ is a right inverse to ι.

A.6. Let us recall the operadic cochain complex and introduce some useful notations. As a graded vector space, the operadic cochain complex is defined by [26, Definition II.3.99]:

$$C^{n-1}_{\mathcal{P}}(A;A) = [Lin((\downarrow A)^{\otimes n}, \downarrow A) \otimes \mathcal{P}^!(n)]^{\Sigma_n}, \quad n \geqslant 1, \tag{23}$$

where $\downarrow A$ denotes the desuspension of the graded vector space A. It will be convenient to denote

$$\widetilde{C}^{n-1}_{\mathcal{P}}(A;A) := Lin((\downarrow A)^{\otimes n}, \downarrow A) \otimes \mathcal{P}^!(n),$$

so that $C^{n-1}_{\mathcal{P}}(A;A) \cong \widetilde{C}^{n-1}_{\mathcal{P}}(A;A)^{\Sigma_n}$. The averaging over the Σ_n-action defines an epimorphism

$$Aver\colon \widetilde{C}^{n-1}_{\mathcal{P}}(A;A) \twoheadrightarrow C^{n-1}_{\mathcal{P}}(A;A)$$

of graded modules which is a left inverse to the inclusion

$$\iota\colon C^{n-1}_{\mathcal{P}}(A;A) \hookrightarrow \widetilde{C}^{n-1}_{\mathcal{P}}(A;A).$$

We will often use the following canonical isomorphisms of graded Σ_n-modules:

$$\begin{aligned}\widetilde{C}^{n-1}_{\mathcal{P}}(A;A) &= Lin((\downarrow A)^{\otimes n}, \downarrow A) \otimes \mathcal{P}^!(n) \cong \uparrow^{\otimes n-1}(Lin(A^{\otimes n}, A) \otimes \mathcal{P}^!(n) \otimes sgn_n) \\ &\cong \mathbf{s}(Lin(A^{\otimes n}, A) \otimes \mathcal{P}^!(n)) \cong \mathbf{s}\mathcal{E}nd_A(n) \otimes \mathcal{P}^!(n) \\ &\cong \mathcal{E}nd_A(n) \otimes \mathbf{s}\mathcal{P}^!(n) \cong \mathcal{E}nd_{\downarrow A}(n) \otimes \mathcal{P}^!(n).\end{aligned}$$

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