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LOCAL BOUNDED COMMUTATIVE RESIDUATED ℓ-MONOIDS

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Abstract. Bounded commutative residuated lattice ordered monoids ($R\ell$-monoids) are a common generalization of, e.g., BL-algebras and Heyting algebras. In the paper, the properties of local and perfect bounded commutative $R\ell$-monoids are investigated.

Keywords: residuated ℓ-monoid, residuated lattice, BL-algebra, MV-algebra, local $R\ell$-monoid, filter

MSC 2000: 06D35, 06F05

1. Introduction

Commutative residuated lattice ordered monoids ($R\ell$-monoids) were introduced (in the dual form) by Swamy [15] as a common generalization of Abelian lattice ordered groups and Heyting algebras. Moreover, bounded commutative $R\ell$-monoids are in very close connections with algebras of fuzzy logics, i.e., with BL-algebras, and consequently, with MV-algebras, which can be viewed as particular cases of such $R\ell$-monoids. Many of important properties of BL-algebras are also satisfied in all bounded commutative $R\ell$-monoids. Therefore bounded commutative $R\ell$-monoids could be taken as an algebraic semantics of a more general logic than Hájek's basic fuzzy logic. Hence it is natural to study filters of those $R\ell$-monoids because from the logical point of view they correspond to sets of provable formulas.

Local BL-algebras which are characterized e.g. by the property that they contain a unique maximal filter, were studied by Turunen and Sessa [18]. In [12], we have analogously introduced the notion of a local bounded commutative $R\ell$-monoid. In the present paper, we study the properties of those $R\ell$-monoids in connection with the properties of their filters.

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For the notions and results concerning BL-algebras and MV-algebras see e.g. [3], [4], [7], [17].

2. Addition in $R\ell$-monoids

Commutative dually residuated lattice ordered monoids ($DR\ell$-monoids) were introduced by Swamy in [15] as a common generalization of Abelian ℓ-groups and Brouwerian algebras. In [9], [10], [11], it was shown that also algebras of fuzzy logics can be viewed as particular cases of bounded commutative $DR\ell$-monoids. For instance, MV-algebras coincide with bounded commutative $DR\ell$-monoids satisfying the double negation law, and BL-algebras are exactly the duals of subdirect products of linearly ordered bounded commutative $DR\ell$-monoids.

In this paper we deal with a generalization of local BL-algebras, hence we use the duals of $DR\ell$-monoids that are called $R\ell$-monoids.

A *commutative $R\ell$-monoid* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, 1)$ of type $\langle 2, 2, 2, 2, 0 \rangle$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a commutative monoid.
(ii) $(M; \vee, \wedge)$ is a lattice.
(iii) The operation $\odot$ distributes over the operations $\vee$ and $\wedge$.
(iv) $x \odot y \leqslant z$ if and only if $x \leqslant y \rightarrow z$, for any $x, y, z \in M$.
(v) $((x \rightarrow y) \wedge 1) \odot x = x \wedge y$, for any $x, y \in M$.

By [15], commutative $R\ell$-monoids form a variety of algebras of the indicated type. In the paper we will deal with bounded commutative $R\ell$-monoids. It is known that an $R\ell$-monoid M is bounded if and only if it is lower bounded. In such a case, 1 is the greatest element in M and identity (v) is in the form $(x \rightarrow y) \odot x = x \wedge y$. Let us denote by 0 the least element in a bounded $R\ell$-monoid, and consider such $R\ell$-monoids as algebras $M = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$.

It is possible to show that bounded commutative $R\ell$-monoids are exactly the bounded commutative integral generalized BL-algebras in the sense of [8] and [1], and that, according to [2] and [8], condition (iii) in the definition of an $R\ell$-monoid is then for bounded cases superfluous. (See also [5] or [6].) Therefore we can modify the definition of a bounded commutative $R\ell$-monoid as follows.

A *bounded commutative $R\ell$-monoid* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a commutative monoid.
(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.
(iii) $x \odot y \leqslant z$ if and only if $x \leqslant y \rightarrow z$, for any $x, y, z \in M$.
(v) $x \odot (x \rightarrow y) = x \wedge y$, for any $x, y \in M$.

For example, both BL-algebras and Heyting algebras are special cases of bounded commutative $R\ell$-monoids, hence the class of bounded commutative $R\ell$-monoids is essentially larger than that of BL-algebras.

In the sequel, by an *$R\ell$-monoid* we will mean a *bounded commutative $R\ell$-monoid.*

On any $R\ell$-monoid M let us define a unary operation negation $^-$ by $x^- := x \to 0$ for any $x \in M$. Further, put $x^1 := x$ and $x^{n+1} := x^n \odot x$ for each $n \in \mathbb{N}$.

Lemma 2.1 ([15], [13]). *In any bounded commutative $R\ell$-monoid M we have for any $x, y \in M$:*

(1) $x \leqslant y \Longleftrightarrow x \to y = 1$.
(2) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.
(3) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.
(4) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.
(5) $(x \vee y) \odot (x \wedge y) = x \odot y$.
(6) $(x \to y) \odot (y \to z) \leqslant x \to z$.
(7) $1^{--} = 1$, $0^{--} = 0$.
(8) $x \leqslant x^{--}$, $x^- = x^{---}$.
(9) $x \leqslant y \Longrightarrow y^- \leqslant x^-$.
(10) $(x \vee y)^- = x^- \wedge y^-$.
(11) $(x \wedge y)^{--} = x^{--} \wedge y^{--}$.
(12) $(x \odot y)^- = y \to x^- = y^{--} \to x^- = x \to y^- = x^{--} \to y^-$.
(13) $(x \odot y)^{--} \geqslant x^{--} \odot y^{--}$.
(14) $(x \to y)^{--} = x^{--} \to y^{--}$.

Remark 2.2. By Lemma 2.1 (8), $x \leqslant x^{--}$ for any $x \in M$. In [9], [10] it is proved that M satisfies the identity $x^{--} = x$ if and only if M is an MV-algebra.

Lemma 2.3. *If M is an $R\ell$-monoid then $x \to y \leqslant (y \to z) \to (x \to z)$, for any $x, y, z \in M$.*

Proof. From the definition of an $R\ell$-monoid and from the fact that M is a lattice ordered monoid we have

$$x \odot (x \to y) \odot (y \to z) = (x \wedge y) \odot (y \to z) \leqslant y \odot (y \to z) = y \wedge z \leqslant z.$$

Thus $(x \to y) \odot (y \to z) \leqslant x \to z$, therefore $x \to y \leqslant (y \to z) \to (x \to z)$. □

Corollary 2.4. *For any $x, y \in M$, $x \to y \leqslant y^- \to x^-$.*

Proposition 2.5. *For any $x, y \in M$, $x^- \to y^- = y^{--} \to x^{--}$.*

Proof. By Corollary 2.4 and Lemma 2.1 (8), $x^- \to y^- \leqslant y^{--} \to x^{--} \leqslant x^{---} \to y^{---} = x^- \to y^-$. □

Proposition 2.6. *For any $x, y \in M$, $(x^- \odot y^-)^- = y^- \to x^{--} = x^- \to y^{--}$.*

Proof. It follows from Lemma 2.1 (12). □

In any MV-algebra there is a binary operation "$\oplus$" dual to the operation "$\odot$". Now we will introduce an operation "$\oplus$" also for arbitrary $R\ell$-monoids and study its properties.

If $M = (M; \odot, \vee, \wedge, \to, 0, 1)$ is an $R\ell$-monoid, then we define a binary operation "$\oplus$" on M as follows:

$$\forall\, x, y \in M\colon\ x \oplus y := (x^- \odot y^-)^-.$$

Lemma 2.7. *For any $x, y \in M$, $(x \oplus y)^- \geqslant x^- \odot y^-$.*

Proof. By Lemma 2.1 (8) and (12), $(x \oplus y)^- = (x^- \odot y^-)^{--} \geqslant x^- \odot y^-$. □

We say that an $R\ell$-monoid M is *normal* if M satisfies the identity

$$(x \odot y)^{--} = x^{--} \odot y^{--}.$$

Remark 2.8. By [13, Proposition 5], every BL-algebra and every Heyting algebra is normal, hence the variety of normal $R\ell$-monoids is considerably wide.

Proposition 2.9. *Let M be a normal $R\ell$-monoid. Then for any $x, y \in M$,*

$$(x \oplus y)^- = x^- \odot y^-.$$

Proof. By the normality and Lemma 2.1 (8), $(x \oplus y)^- = (x^- \odot y^-)^{--} = x^- \odot y^-$. □

Proposition 2.10. *If M is any $R\ell$-monoid, then $(M; \oplus)$ is a semigroup.*

Proof. Let $x, y, z \in M$. Then by Proposition 2.6 and Lemma 2.1 (2),

$$\begin{aligned} x \oplus (y \oplus z) &= x \oplus (y^- \odot z^-)^- = (x^- \odot (y^- \odot z^-)^{--})^- = x^- \to (y^- \odot z^-)^- \\ &= x^- \to (z^- \to y^{--}) = z^- \to (x^- \to y^{--}) = z^- \to (x^- \odot y^-)^- \\ &= ((x^- \odot y^-)^{--} \odot z^-)^- = (x^- \odot y^-)^- \oplus z = (x \oplus y) \oplus z. \end{aligned}$$

□

Now we can put $1 \cdot x = x$, $(n+1)x = nx \oplus x$ for each $n \in \mathbb{N}$.

Let us denote by $R(M) = \{x \in M\colon x^{--} = x\}$ the set of all *regular elements* in M. Obviously, $0, 1 \in R(M)$. If $M = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$ is any $R\ell$-monoid, then by [13, Proposition 4], $R(M)$ is a subalgebra of the reduct $(M; \wedge, \rightarrow, 1)$. We will show further properties of the set $R(M)$.

Lemma 2.11. *If M is an $R\ell$-monoid and $x, y \in M$, then*
(a) $x \oplus 0 = x^{--}$;
(b) $(x \oplus y)^{--} = x^{--} \oplus y^{--} = x \oplus y$.

Proof. (a) $x \oplus 0 = x \oplus 1^- = (x^- \odot 1^{--})^- = (x^- \odot 1)^- = x^{--}$.

(b) $(x \oplus y)^{--} = (x^- \odot y^-)^{---} = (x^- \odot y^-)^- = x \oplus y$, $x^{--} \oplus y^{--} = (x^{---} \odot y^{---})^- = (x^- \odot y^-)^- = x \oplus y$. □

Remark 2.12.
a) By the previous lemma and Remark 2.2, 0 is a neutral element of $(M; \oplus)$ if and only if M is an MV-algebra.
b) The sum $x \oplus y$ of any elements $x, y \in M$ belongs to $R(M)$.

Proposition 2.13. *If M is an $R\ell$-monoid, then $R(M)$ is a subsemigroup of $(M; \oplus)$ and $(R(M); \oplus, 0)$ is a commutative monoid which, moreover, satisfies the identity $(x \odot y)^- = x^- \oplus y^-$.*

Proof. By Lemma 2.11, it is sufficient to prove that $(x \odot y)^- = x^- \oplus y^-$. (It is obvious that $(x \odot y)^-$, x^- and y^- belong to $R(M)$.) Let $x, y \in R(M)$. Then $(x \odot y)^- = (x^{--} \odot y^{--})^- = x^- \oplus y^-$. □

Remark 2.14. Let an $R\ell$-monoid be normal. Then by [13, Theorem 7], $R(M) = (R(M); \odot, \vee_{R(M)}, \wedge, \rightarrow, 0, 1)$, where $y \vee_{R(M)} z =\colon (y \vee z)^{--}$ for any $y, z \in R(M)$ and the other operations are restrictions of the operations on M, is an MV-algebra. In such a case, the operation "$\oplus$" on $R(M)$ is the dual operation to the operation "$\odot$".

Proposition 2.15 ([13, Proposition 2]). *If M is an $R\ell$-monoid, then the following conditions are equivalent for any $x, y \in M$.*
(1) $(x \vee y)^{--} = x^{--} \vee y^{--}$.
(2) $(x \wedge y)^- = x^- \vee y^-$.
(3) $(x \wedge y)^- \odot ((x \rightarrow y) \vee (y \rightarrow x)) = (x \wedge y)^-$.

Every BL-algebra satisfies the identity $(x \rightarrow y) \vee (y \rightarrow x) = 1$, therefore it also satisfies the identities (1), (2) and (3) from the previous proposition. (See also [13, Proposition 2].)

Proposition 2.16. *If an $R\ell$-monoid M satisfies the identities from Proposition 2.15, then the operation "$\oplus$" distributes over the operations "$\vee$" and "$\wedge$", hence $(M;\oplus,\vee,\wedge)$ is a lattice ordered monoid.*

Proof. If $x,y,z\in M$ then by Lemma 2.1 (10),

$$\begin{aligned} x\oplus(y\vee z) &= (x^-\odot(y\vee z)^-)^- = (x^-\odot(y^-\wedge z^-))^- = ((x^-\odot y^-)\wedge(x^-\odot z^-))^- \\ &= (x^-\odot y^-)^-\vee(x^-\odot z^-)^- = (x\oplus y)\vee(x\oplus z), \\ x\oplus(y\wedge z) &= (x^-\odot(y\wedge z)^-)^- = (x^-\odot(y^-\vee z^-))^- = ((x^-\odot y^-)\vee(x^-\odot z^-))^- \\ &= (x^-\odot y^-)^-\wedge(x^-\odot z^-)^- = (x\oplus y)\wedge(x\oplus z). \end{aligned}$$

□

3. Properties of local $R\ell$-monoids

If M is an $R\ell$-monoid and $\emptyset\neq F\subseteq M$, then F is called a *filter* of M if

(i) $x,y\in F\Longrightarrow x\odot y\in F$;

(ii) $x\in F$, $y\in M$, $x\leqslant y\Longrightarrow y\in F$.

By [5], the filters of M are exactly all *deductive systems* of M, i.e. $F\subseteq M$ is a filter of M if and only if

(1) $1\in F$;

(2) $x\in F$, $x\to y\in F\Longrightarrow y\in F$.

Furthermore, by [16], the filters of $R\ell$-monoids coincide with the kernels of their congruences. If F is a filter of M then F is the kernel of the unique congruence $\theta(F)$ such that $\langle x,y\rangle\in\theta(F)$ if and only if $(x\to y)\wedge(y\to x)\in F$ for any $x,y\in M$. Hence we will consider quotient $R\ell$-monoids M/F of $R\ell$-monoids M with respect to their filters F.

If for a filter F the quotient $R\ell$-monoid is an MV-algebra, then F is called an *MV-filter*.

An element $x\in M$ is called *dense* if $x^{--}=1$. Denote by $D(M)$ the set of all dense elements in M. By [13, Theorem 8] and [14, Remark to Theorem 10], or by [5, Proposition 3.3], $D(M)$ is a proper MV-filter of M. Moreover, a filter F of an $R\ell$-monoid M is an MV-filter if and only if $D(M)\subseteq F$.

Let us recall that an $R\ell$-monoid M is called *local* if M contains a unique maximal filter. (See [12].)

Let us put

$$A(M):=\{x\in M\colon\ x^n\neq 0\ \text{ for every }\ n\in\mathbb{N}\}.$$

Define $\operatorname{ord}(x)$, the *order of an element* $x \in M$, as follows: $\operatorname{ord}(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$; otherwise $\operatorname{ord}(x) = \infty$. Hence $A(M)$ is the set of all elements $x \in M$ such that $\operatorname{ord}(x) = \infty$. We have $0 \notin A(M)$, thus $A(M) \neq M$.

Proposition 3.1 ([12, Theorem 3.9]). *If M is an $R\ell$-monoid then the following conditions are equivalent.*

(1) *M is local.*
(2) *$A(M)$ is a filter of M.*
(3) *$A(M)$ is the unique maximal filter of M.*
(4) *If $x^n \neq 0 \neq y^n$ for every $n \in \mathbb{N}$, then $x^n \odot y^n \neq 0$ for all $n \in \mathbb{N}$.*

Corollary 3.2. *If M is a local $R\ell$-monoid, then for any element $x \in M$, $\operatorname{ord}(x) < \infty$ or $\operatorname{ord}(x^-) < \infty$.*

Denote

$$A(M)^- := \{x \in M\colon\ x \leqslant y^- \ \text{ for some } \ y \in A(M)\}.$$

Let us define now the notion of an ideal of an $R\ell$-monoid M. If M is an $R\ell$-monoid and $\emptyset \neq I \subseteq M$, then I is called an *ideal* of M if

(i) $x, y \in I \Longrightarrow x \oplus y \in I$;
(ii) $x \in I$, $z \in M$, $z \leqslant x \Longrightarrow z \in I$.

Proposition 3.3. *If M is a local $R\ell$-monoid then $A(M)^-$ is an ideal of M and $A(M) \cap A(M)^- = \emptyset$.*

Proof. $0 \in A(M)^-$, hence $A(M)^- \neq \emptyset$. Let $x, y \in A(M)^-$. Then $x \leqslant v^-$ and $y \leqslant w^-$ for some elements $v, w \in A(M)$. Thus by Lemma 2.1 (8) and (9),

$$x \oplus y \leqslant v^- \oplus w^- = (v^{--} \odot w^{--})^- \leqslant (v \odot w)^-,$$

and since $A(M)$ is by Proposition 3.1 a filter of M, we have $x \oplus y \in A(M)^-$.

Let $x \in M$, $y \in A(M)^-$, $x \leqslant y$ and $y \leqslant z^-$, where $z \in A(M)$. Then $x \leqslant z^-$, hence $x \in A(M)$.

Therefore $A(M)^-$ is an ideal of M. □

Let M be an $R\ell$-monoid and let F be a filter of M. Then F is called a *primary filter* if it is satisfied for any $x, y \in M$: If there is $n \in \mathbb{N}$ such that $n(x \oplus y) \in F$, then there is $m \in \mathbb{N}$ such that $mx \in F$ or $my \in F$.

Proposition 3.4. *For any $R\ell$-monoid M and any MV-filter F of M, the following conditions are equivalent.*

(1) *M/F is a local $R\ell$-monoid.*

(2) *F is a primary filter.*

Proof. (1) $\Rightarrow$ (2): Let F be a filter of M such that M/F is local. Let us suppose that $x, y \in M$, $n \in \mathbb{N}$ and $n(x \oplus y) \in F$, i.e., $n(x \oplus y)/F$ is the greatest element 1 in M/F. Then $(x^- \odot y^-)^n/F$ is the smallest element 0 in M/F, and since M/F is local, there exists $m \in \mathbb{N}$ such that $(x^-/F)^m = 0$ or $(y^-/F)^m = 0$. Since F is an MV-filter, this implies that there is $m \in \mathbb{N}$ such that $mx \in F$ or $my \in F$. Therefore F is a primary filter.

(2) $\Rightarrow$ (1): Let F be a primary MV-filter. Suppose that $x, y \in M$ and that there exists $n \in \mathbb{N}$ such that $(x/F \odot y/F)^n = 0$. Then $n(x^-/F \oplus y^-/F) = F$, i.e., $n(x^- \oplus y^-) \in F$, hence there is $m \in \mathbb{N}$ such that $mx^- \in F$ or $my^- \in F$. This yields $(x/F)^m = 0$ or $(y/F)^m = 0$, and thus M/F is local. □

Theorem 3.5. *Let M be an $R\ell$-monoid. Then the following conditions are equivalent.*

(1) *Every MV-filter of M is primary.*

(2) *$D(M)$ is a primary filter.*

(3) *$M/D(M)$ is a local MV-algebra.*

Proof. (1) $\Rightarrow$ (2): It follows from the fact that $D(M)$ is the least MV-filter of M.

(2) $\Leftrightarrow$ (3): By Proposition 3.4.

(3) $\Rightarrow$ (1): If F is an MV-filter of M, then $D(M) \subseteq F$, hence by the isomorphism theorems for algebras we get that M/F also contains a unique maximal filter, which means F is primary. □

Proposition 3.6. *Let M be an $R\ell$-monoid.*

a) *If M is local then it satisfies the equivalent conditions from Theorem* 3.5.

b) *If $\{1\}$ is a primary MV-filter then M is a local MV-algebra.*

Proof. a) Let an $R\ell$-monoid M be local, let F be a filter of M, $x, y \in M$, $n \in \mathbb{N}$ and let $n(x \oplus y) \in F$. Then $\operatorname{ord}(n(x \oplus y)) = \infty$, hence $\operatorname{ord}((x^- \odot y^-)^n) < \infty$. Since M is local, we get $\operatorname{ord}(x^-) < \infty$ or $\operatorname{ord}(y^-) < \infty$. That is, there is $m \in \mathbb{N}$ such that $(x^-)^m = 0$ or $(y^-)^m = 0$.

Therefore, if F is an MV-filter then $mx = 1 \in F$ or $my = 1 \in F$ for some $m \in \mathbb{N}$, and thus F is a primary filter of M.

b) If $\{1\}$ is an MV-filter then $D(M) = \{1\}$. Hence the assertion is a direct consequence of Theorem 3.5. □

Proposition 3.7. *Every linearly ordered $R\ell$-monoid is a local BL-algebra.*

Proof. Let M be a linearly ordered $R\ell$-monoid. By [11], BL-algebras are exactly all $R\ell$-monoids which are subdirect products of linearly ordered $R\ell$-monoids. Hence M is a BL-algebra.

Let $x, y \in M$, $n \in \mathbb{N}$ and let $(x \odot y)^n = 0$. Since $x \leqslant y$ or $y \leqslant x$, we have $(x \odot y)^n \geqslant x^{2n}$ or $(x \odot y)^n \geqslant y^{2n}$, thus $\mathrm{ord}(x) < \infty$ or $\mathrm{ord}(y) < \infty$. Therefore by [12, Theorem 3.9], M is local. □

Let M be a local $R\ell$-monoid. Then M is called

a) *perfect* if for any $x \in M$, $\mathrm{ord}(x) < \infty$ implies $\mathrm{ord}(x^-) = \infty$;

b) *singular* if there exist $x, y \in M$ such that $\mathrm{ord}(x) < \infty$, $\mathrm{ord}(y) < \infty$ and $\mathrm{ord}(x \oplus y) = \infty$.

Proposition 3.8. *Every local $R\ell$-monoid M is either perfect or singular and there is no M having both properties.*

Proof. (a) Let M be a local $R\ell$-monoid which is not singular. Then $\mathrm{ord}(y) = \infty$ or $\mathrm{ord}(z) = \infty$ or $\mathrm{ord}(y \oplus z) < \infty$ for every $y, z \in M$.

If x is any element in M then

$$\mathrm{ord}(x \oplus x^-) = \mathrm{ord}((x^- \odot x^{--})^-) = \mathrm{ord}(0^-) = \mathrm{ord}(1) = \infty,$$

hence $\mathrm{ord}(x) = \infty$ or $\mathrm{ord}(x^-) = \infty$.

Therefore M is perfect.

(b) Let now M be a local $R\ell$-monoid that is simultaneously perfect and singular. Then there exist $x, y \in M$ such that $\mathrm{ord}(x) < \infty$, $\mathrm{ord}(y) < \infty$ and $\mathrm{ord}(x \oplus y) = \infty$, and hence $\mathrm{ord}(x^-) = \mathrm{ord}(y^-) = \infty$ and $\mathrm{ord}((x \oplus y)^-) < \infty$. By Proposition 2.9, $(x \oplus y)^- = x^- \odot y^-$, hence we get, because M is local, $\mathrm{ord}(x^-) < \infty$ or $\mathrm{ord}(y^-) < \infty$, a contradiction. □

Let M be an $R\ell$-monoid and F a filter of M. Then F is called a *perfect filter* if it is primary and if, for each $x \in M$, there is $n \in \mathbb{N}$ with $nx \in F$ if and only if $mx^- \notin F$ for each $m \in \mathbb{N}$.

Theorem 3.9. *Let M be an $R\ell$-monoid and F an MV-filter of M. Then the following conditions are equivalent.*

(1) *M/F is a perfect $R\ell$-monoid.*

(2) *F is a perfect filter.*

Proof. (1) ⇒ (2): Let F be an MV-filter of M and let M/F be a perfect $R\ell$-monoid. Then M/F is local by definition, and thus, by Proposition 3.4, F is a primary filter.

Let $x \in M$, $n \in \mathbb{N}$ and $nx \in F$. Then $nx/F = 1$ and $(x^-)^n/F = 0$ in M/F. Hence $\operatorname{ord}(x^-/F) < \infty$, and since M/F is perfect, $\operatorname{ord}(x^{--}/F) = \infty$. Moreover, F is an MV-filter, thus also $\operatorname{ord}(x/F) = \infty$, therefore $x^n/F \neq 0$ for each $n \in \mathbb{N}$. This implies $nx^-/F \neq 1$, thus $nx^- \notin F$ for each $n \in \mathbb{N}$.

The converse implication can be proved analogously, and therefore F is perfect.

(2) $\Rightarrow$ (1): Let F be perfect. Then F is primary, and since it is an MV-filter, we get, by Proposition 3.4, that M/F is a local $R\ell$-monoid. Let $x \in M$ and $\operatorname{ord}(x^-/F) < \infty$. Then there is $n \in \mathbb{N}$ such that $(x^-)^n/F = 0$ in M/F, hence $nx/F = 1$. Thus there exists $n \in \mathbb{N}$ such that $nx \in F$, therefore $mx^- \notin F$ for every $m \in \mathbb{N}$. This implies $mx^-/F \neq 1$ and $x^m/F \neq 0$ for every $m \in \mathbb{N}$. Therefore $\operatorname{ord}(x/F) = \infty$ in M/F. That is, M/F is perfect. □

Theorem 3.10. *Let M be a local $R\ell$-monoid. Then the following conditions are equivalent.*

(a) *M is perfect.*

(b) $M = A(M) \cup A(M)^-$.

Proof. (a) $\Rightarrow$ (b): Let M be perfect and $x \in M \setminus A(M)$. Then $x^- \in A(M)$. We have $x \leqslant x^{--} = (x^-)^-$ and $x^- \in A(M)$, hence $x \in A(M)^-$. Therefore $M = A(M) \cup A(M)^-$.

(b) $\Rightarrow$ (a): Since M is local, $A(M)$ is by [12, Theorem 3.9] a filter of M, and by Proposition 3.3, $A(M)^-$ is an ideal of M and $A(M) \cap A(M)^- = \emptyset$. Thus by the assumption, we get $A(M)^- = \{y \in M\colon \operatorname{ord}(y) < \infty\}$. Let $x \in M$.

If $\operatorname{ord}(x) = \operatorname{ord}(x^-) = \infty$, then $x, x^- \in A(M)$, thus $0 \in A(M)$, a contradiction.

If $\operatorname{ord}(x) < \infty$ and $\operatorname{ord}(x^-) < \infty$, then $x, x^- \in A(M)^-$, and hence $1 \in A(M)^-$, a contradiction.

Therefore $\operatorname{ord}(x) < \infty$ if and only if $\operatorname{ord}(x^-) = \infty$, and this means that M is perfect. □

If M is an $R\ell$-monoid and F is a proper filter of M, set (analogously as for $A(M)$)

$$F^- := \{x \in M\colon\ x \leqslant y^- \ \text{ for some } \ y \in F\}.$$

Obviously $F \cap F^- = \emptyset$.

An $R\ell$-monoid M is called *bipartite* if $M = F \cup F^-$ for some maximal filter of M, and it is called *strongly bipartite* if $M = F \cup F^-$ for every maximal filter of M.

A filter F of M is called a *Boolean filter* if $x \vee x^- \in F$ for any $x \in M$ (or, equivalently, if M/F is a Boolean algebra [12, Theorem 3.2]).

Theorem 3.11. *Let M be a local $R\ell$-monoid. Then the following conditions are equivalent.*

(1) *M is perfect.*
(2) *M is (strongly) bipartite.*
(3) *$A(M)$ is a Boolean filter.*
(4) *For any element $x \in M$, $x \in A(M)$ or $x^- \in A(M)$.*

Proof. (1) $\Leftrightarrow$ (2): By [12, Theorem 3.9], $A(M)$ is a unique maximal filter of M, hence the equivalence follows from Theorem 3.10.

(2) $\Leftrightarrow$ (3): By [12, Theorem 3.8], any $R\ell$-monoid M is strongly bipartite if and only if every maximal filter of M is Boolean.

(3) $\Leftrightarrow$ (4): If M is an $R\ell$-monoid and F is a filter of M then by [12, Theorem 3.3], F is maximal and Boolean if and only if F is a proper filter such that $x \in F$ or $x^- \in F$ for every $x \in M$. □

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