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SPACES WITH LARGE RELATIVE EXTENT

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Abstract. In this paper, we prove the following statements: (1) For every regular uncountable cardinal κ, there exist a Tychonoff space X and Y a subspace of X such that Y is both relatively absolute star-Lindelöf and relative property (a) in X and $e(Y,X) \geqslant \kappa$, but Y is not strongly relative star-Lindelöf in X and X is not star-Lindelöf. (2) There exist a Tychonoff space X and a subspace Y of X such that Y is strongly relative star-Lindelöf in X (hence, relative star-Lindelöf), but Y is not absolutely relative star-Lindelöf in X.

Keywords: relative topological property, Lindelöf, star-Lindelöf, relative extent, relative property (a)

MSC 2000: 54D15, 54D20

1. Introduction

By a space, we mean a topological space. Let X be a space and Y a subspace of X. Recall from [1], [2], [8] that Y is *Lindelöf* in X if for every open cover $\mathscr{U}$ of X, there exists a countable subfamily covering Y. A space X is *star-Lindelöf* (for different names, see [5], [6], [10], [19]) if for every open cover $\mathscr{U}$ of X, there exists a countable subset F of X such that $\mathrm{St}(F,\mathscr{U}) = X$, where $\mathrm{St}(F,\mathscr{U}) = \bigcup\{U \in \mathscr{U} \colon U \cap F \neq \emptyset\}$. A space X is *absolutely star-Lindelöf* (see [4], [10]) if for every open cover $\mathscr{U}$ of X and every dense subspace $D \subseteq X$, there exists a countable subset F of D such that $\mathrm{St}(F,\mathscr{U}) = X$. A space X has *property* (a) (see [8], [10]) if for every open cover $\mathscr{U}$ of X and every dense subspace $D \subseteq X$, there exists a closed (in X) and discrete subset F of D such that $\mathrm{St}(F,\mathscr{U}) = X$. Now, following the general idea of relativization of topological properties [1], it is natural to introduce the following definitions:

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Definition 1.1 ([18]). A subspace Y of a space X is called *relative star-Lindelöf* (*strongly relative star-Lindelöf*) in X if for every open cover $\mathscr{U}$ of X, there exists a countable subset $F \subseteq X$ (respectively, $F \subseteq Y$) such that $Y \subseteq \mathrm{St}(F, \mathscr{U})$.

Definition 1.2 ([13]). A subspace Y of a space X is called *relatively absolute star-Lindelöf* in X if for every open cover $\mathscr{U}$ of X and every dense subspace $D \subseteq X$, there exists a countable subset F of D such that $Y \subseteq \mathrm{St}(F, \mathscr{U})$.

Definition 1.3 ([13]). A subspace Y of a space X is called *relative property* (a) in X if for every open cover $\mathscr{U}$ of X and every dense subspace $D \subseteq X$, there exists a closed (in X) and discrete subset $F \subseteq D$ such that $Y \subseteq \mathrm{St}(F, \mathscr{U})$.

From the above definitions, it is not difficult to see that if a subspace Y of X is strongly relative star-Lindelöf in X, then Y is relative star-Lindelöf in X and if a subspace Y of X is relatively absolute star-Lindelöf in X, then Y is relative star-Lindelöf in X. But the converses do not hold (see below Examples 2.3 and 2.4).

Recall that the *extent* $e(X)$ of a space X is the smallest cardinal number κ such that the cardinality of every discrete closed subset of X is not greater than κ; moreover, Arhangel'skii [2] defined the *extent* $e(Y, X)$ of Y in X as the smallest cardinal number κ such that the cardinality of every closed in X discrete subspace of Y is not greater than κ. It is well-known that the extent of a Lindelöf space is countable. Arhangel'skii [2] proved that if Y is Lindelöf in X, then $e(Y, X)$ is countable. Matveev [14] proved that the extent of a Tychonoff star-Lindelöf space can be arbitrarily large. Matveev [11] asked if the extent of a star-Lindelöf space with the property (a) is greater than $\mathfrak{c}$. Song [16] answered this question positively. It is natural for us to consider the following question:

Question. Do there exist a Tychonoff space X and a subspace Y of X such that Y is both relatively absolute star-Lindelöf and relative property (a) in X and $e(Y, X)$ is greater than $\mathfrak{c}$, but Y is not strongly relative star-Lindelöf in X and X is not star-Lindelöf.

The purpose of this paper is to answer the questions positively and to clarify the relations among these star-Lindelöf spaces by constructing two examples stated in the abstract.

The cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal and $\mathfrak{c}$ the cardinality of the continuum. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For a pair of ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma\colon\ \alpha < \gamma < \beta\}$ and $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leqslant \beta\}$. Other terms and symbols that we do not define will be used as in [7].

2. Two examples on relative star-Lindelöf spaces

In this section, we first construct an example with properties of statement 1 stated in the abstract. The example uses Matveev's space. Recall that a space X is *discretely star-Lindelöf* (for different names, see [17], [18]) if for every open cover $\mathscr{U}$ of X, there exists a countable discrete closed subset F of X such that $\operatorname{St}(F, \mathscr{U}) = X$. It is clear that every discretely star-Lindelöf space is star-Lindelöf. We now sketch the construction of Matveev's space M defined in [14], [15]. Let κ be an infinite cardinal and $D = \{0, 1\}$ be the discrete space. For every $\alpha < \kappa$, let z_α be the point of D^κ defined by $z_\alpha(\alpha) = 1$ and $z_\alpha(\beta) = 0$ for $\beta \neq \alpha$. Put $Z = \{z_\alpha \colon \alpha < \kappa\}$. Matveev's space M is defined to be the subspace

$$M = (D^\kappa \times \omega) \cup (Z \times \{\omega\})$$

of the product space $D^\kappa \times (\omega + 1)$. Then, M is a Tychonoff discretely star-Lindelöf space and $e(M) \geqslant \kappa$, since $Z \times \{\omega\}$ is a discrete closed set in M.

We need the following lemma:

Lemma 2.1 ([15], [16]). *Assume that there exists a family $\{V_\alpha \colon \alpha < \kappa\}$ of open sets in D^κ such that $z_\alpha \in V_\alpha$ for each $\alpha < \kappa$. Then, there exists a countable set $S \subseteq D^\kappa$ such that $S \cap V_\alpha \neq \emptyset$ for each $\alpha < \kappa$ and $\operatorname{cl}_{D^\kappa} S \cap Z = \emptyset$.*

For constructing the example, we use the Alexandroff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup (U \times \{1\} \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X. It is well-known that $A(X)$ is countably compact iff X is countably compact. Recall that a space is *absolutely countably compact* (see [9], [10]) if for every open cover $\mathscr{U}$ of X and every dense subspace D of X, there exists a finite subset F of D such that $\operatorname{St}(F, \mathscr{U}) = X$. In the next example, we use the following lemma from [20].

Lemma 2.2. *If X is countably compact, then $A(X)$ is absolutely countably compact.*

For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 2.3. For every regular uncountable cardinal κ, there exist a Tychonoff space X and a subspace Y of X such that Y is both relatively absolute star-Lindelöf and relative (a) in X and $e(Y, X) \geqslant \kappa$, but Y is not strongly relative star-Lindelöf in X and X is not star-Lindelöf.

P r o o f. Let κ be a regular uncountable cardinal and let

$$S_1 = M = (D^\kappa \times \omega) \cup (Z \times \{\omega\})$$

be a subspace of the product space $D^\kappa \times (\omega + 1)$. Then, S_1 is a Tychonoff space. Note that $e(S_1) \geqslant \kappa$, since $Z \times \{\omega\}$ is discrete closed in S_1.

Let B be the discrete space of cardinality κ and let

$$S_2 = (\beta B \times (\kappa + 1)) \setminus ((\beta B \setminus B) \times \{\kappa\})$$

be a subspace of the product space $\beta B \times (\kappa + 1)$.

We assume that $S_1 \cap S_2 = \emptyset$. Since $|Z \times \{\omega\}| = \kappa$ and $|B \times \{\kappa\}| = \kappa$, we can enumerate $Z \times \{\omega\}$ and $B \times \{\kappa\}$ as $\{\langle z_\alpha, \omega\rangle \colon \alpha < \kappa\}$ and $\{\langle b_\alpha, \kappa\rangle \colon \alpha < \kappa\}$ respectively. Let $\varphi\colon Z \times \{\omega\} \to B \times \{\kappa\}$ be the bijection defined by

$$\varphi(\langle z_\alpha, \omega\rangle) = \langle b_\alpha, \kappa\rangle$$

for each $\alpha < \kappa$. Let X' be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying $\langle z_\alpha, \omega\rangle$ with $\varphi(\langle z_\alpha, \omega\rangle)$ for each $\alpha < \kappa$. Let $\pi\colon S_1 \oplus S_2 \to X'$ be the quotient map.

Let

$$X = A(X') \quad \text{and} \quad Y = A(\pi(S_2)) \setminus (\pi(B \times \{\kappa\}) \times \{1\}).$$

Clearly, X is a Tychonoff space. Note that $e(Y, X) \geqslant \kappa$, since $\pi(Z \times \{\omega\}) \times \{0\}$ is a closed (in X) discrete subspace of Y. We show that Y is both relatively absolute star-Lindelöf and relative property (a) in X. For this end, let $\mathscr{U}$ be an open cover of X. Let

$$X'_\omega = \pi(Z \times \{\omega\}) \times \{0\}; \quad X''_\omega = \pi(Z \times \{\omega\}) \times \{1\};$$
$$X_n = A(\pi(D^\kappa \times \{n\})) \quad \text{for each } n \in \omega$$

and

$$X'' = A(\pi(\beta B \times \kappa)).$$

Then,

$$X = X'' \cup X'_\omega \cup X''_\omega \cup \bigcup_{n<\omega} X_n.$$

Let S be the set of all isolated points of κ and let $D' = B \times S$. If we put

$$D_0 = (\pi(D') \times \{0\}) \cup (\pi(X') \times \{1\}),$$

then D_0 is dense in X and every dense subspace of X includes D_0. Thus, it suffices to show that there exists a countable $F \subseteq D_0$ such that F is discrete and closed in X and $Y \subseteq \mathrm{St}(F, \mathscr{U})$. By refining $\mathscr{U}$, we may assume that $\mathscr{U}$ is cover of the from

$$\mathscr{U} = \mathscr{U}_0 \cup \mathscr{U}'_\omega \cup \mathscr{U}_\omega \cup \bigcup_{n\in\omega} \mathscr{U}_n,$$

where $\mathscr{U}_0$, $\mathscr{U}'_\omega$, $\mathscr{U}_\omega$ and $\mathscr{U}_n$, $n \in \omega$ are defined as follows:

$$\mathscr{U}_0 = \{U \cap A(\pi(\beta B \times \kappa))\colon\ U \in \mathscr{U}\}; \quad \mathscr{U}'_\omega = \{\pi(\langle z_\alpha, \omega\rangle) \times \{1\}\colon\ \alpha < \kappa\};$$

$\mathscr{U}_\omega = \{U_\alpha\colon\ \alpha < \kappa\}$, where each U_α is of the form

$$U_\alpha = A(\pi(V_\alpha \times (n_\alpha, \omega)) \cup \{\pi(\langle z_\alpha, \omega\rangle) \times \{0\}\} \cup \{A(\pi\langle z_\alpha, \omega\rangle) \times (\beta_\alpha, \kappa))\}$$

for some open neighborhood V_α of z_α in D^κ, $n_\alpha < \omega$ and $\beta_\alpha < \kappa$; and $\mathscr{U}_n = \{U_{n,x}\colon\ x \in D^\kappa\} \cup \{\pi(\langle x, n\rangle) \times \{1\}\colon\ x \in D^\kappa\}$, where $U_{n,x}$ is of the from

$$U_{n,x} = (\pi(V_{n,x} \times \{n\}) \times \{0,1\}) \setminus (\pi(\langle x, n\rangle) \times \{1\}),$$

for some open neighborhood $V_{n,x}$ of x in D^κ.

By applying Lemma 2.1 to the family $\{V_\alpha\colon\ \alpha < \kappa\}$, we can find a countable set $S = \{s_i\colon\ i \in \omega\} \subseteq D^\kappa$ such that $S \cap V_\alpha \neq \emptyset$ for all $\alpha < \kappa$ and $\mathrm{cl}_{D^\kappa} S \cap Z = \emptyset$. Define

$$E = \bigcup_{i<\omega} \{\pi(\langle s_i, j\rangle) \times \{1\}\colon\ i < j < \omega\}.$$

Since $\mathrm{cl}_{D^\kappa} S \cap Z = \emptyset$ and $|E \cap X_n| < \omega$ for each $n \in \omega$, E is discrete closed in X. Moreover, since $S \cap V_\alpha \neq \emptyset$ for all $\alpha < \kappa$,

$$X''_\omega \subseteq \mathrm{St}(E, \mathscr{U}_\omega) \subseteq \mathrm{St}(E, \mathscr{U}).$$

Since κ is locally compact and countably compact, it follows from [7, Theorem 3.10.13] that $\beta B \times \kappa$ is countably compact, hence $\pi(\beta B \times \kappa)$ is countably compact. By applying Lemma 2.2, there exists a finite subset $E' \subseteq (\pi(D') \times \{0\}) \cup (\pi(\beta B \times \kappa) \times \{1\})$ such that

$$A(\pi(\beta B \times \kappa)) \subseteq \mathrm{St}(E', \mathscr{U}).$$

If we put $E_0 = E \cup E'$, then

$$Y \subseteq \mathrm{St}(E_0, \mathscr{U}),$$

which shows that Y is both relatively absolute star-Lindelöf and relative property (a) in X.

Next, we show that Y is not strongly star-Lindelöf in X. For each $\alpha < \kappa$, let V_α be an open neighborhood of z_α in D^κ. Let

$$U_\alpha = A(\pi(\{z_\alpha\} \times (\alpha, \kappa]) \cup V_\alpha \times (0, \omega)) \quad \text{for each } \alpha < \kappa$$

and

$$W_n = A(\pi(D^\kappa \times \{n\})) \quad \text{for each } n \in \omega.$$

Let us consider the open cover

$$\mathscr{U} = \{A(\pi(\beta B \times \kappa)\} \cup \{U_\alpha \colon \alpha < \kappa\} \cup \{W_n \colon n \in \omega\} \cup \{\langle\langle z_\alpha, \omega\rangle, 1\rangle \colon \alpha < \kappa\}$$

of X and let F be any countable subset of Y. It suffices to show that $Y \nsubseteq \mathrm{St}(F, \mathscr{U})$. Since F is countable, there exist $\alpha_1, \alpha_2 < \kappa$ such that

$$F \cap A(\pi(\beta B \times (\alpha, \kappa)) = \emptyset$$

and

$$F \cap \{\pi(\langle z_\alpha, \kappa\rangle) \times \{0\} \colon \alpha > \alpha_2\} = \emptyset.$$

If we pick $\alpha_0 > \max\{\alpha_1, \alpha_2\}$, then

$$\langle\langle z_{\alpha_0}, \kappa\rangle, 0\rangle \notin \mathrm{St}(F, \mathscr{U}),$$

since U_{α_0} is the only element of $\mathscr{U}$ containing $\langle\langle z_{\alpha_0}, \kappa\rangle, 0\rangle$ and $U_{\alpha_0} \cap F = \emptyset$, since F is countable, which shows that Y is not strongly relative star-Lindelöf in X.

Finally, we show that X is not star-Lindelöf. Since $\pi(Z \times \{\omega\}) \times \{1\}$ is a discrete closed and open subset of X with cardinality κ and star-Lindelöfness is preserved by closed and open subsets, X is not star-Lindelöf, which completes the proof. □

Remark 1. In Example 2.3, it is not difficult to see that Y is absolutely relative star-Lindelöf in X. Example 2.3 shows that there exist a Tychonoff space X and a subspace Y of X such that Y is relative star-Lindelöf in X, but Y is not strongly relative star-Lindelöf in X.

Example 2.4. There exist a Tychonoff space X and a subspace Y of X such that Y is strongly relative star-Lindelöf in X, but Y is not relatively absolute star-Lindelöf in X.

P r o o f. Let $X=\omega_1\times(\omega_1+1)$ be the product of ω_1 and ω_1+1 and $Y=\omega_1\times\{\omega_1\}$. Then, Y is strongly relative star-Lindelöf in X, since Y is homeomorphic with ω_1.

Next, we show that Y is not absolutely relative star-Lindelöf in X. Let $D=\omega_1\times\omega_1$. Then, D is dense in X.

Let

$$U_\alpha=\{\langle\beta,\gamma\rangle\colon\ \gamma>\alpha,\ \beta<\alpha\}\ \text{ for each }\ \alpha<\omega_1.$$

Let us consider the open cover

$$\mathscr{U}=\{U_\alpha\colon\ \alpha<\omega_1\}\cup\{D\}$$

of X and a dense subset D of X. Let F be any countable subset of D.

Let

$$\alpha_0=\sup\{\beta\colon\ \exists\alpha<\omega_1\quad\text{such that }\langle\alpha,\beta\rangle\in F\}.$$

Then, $\alpha_0<\omega_1$, since F is countable. Choose $\alpha'>\alpha_0$. Then, $\langle\alpha',\omega_1\rangle\notin\operatorname{St}(F,\mathscr{U})$, since for every $U\in\mathscr{U}$, if $\langle\alpha',\omega_1\rangle\in U$, then $U\cap F=\emptyset$. This shows that Y is not strongly relative star-Lindelöf in X, which completes the proof. □

Remark 2. If a subspace Y of X is strongly relative star-Lindelöf in X, then Y is relative star-Lindelöf in X. Thus, Example 2.3 shows that there exist a Tychonoff space X and a subspace Y of X such that Y is relative star-Lindelöf in X, but Y is not absolutely relative star-Lindelöf in X.

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