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APPROACH REGIONS FOR THE SQUARE ROOT OF THE POISSON KERNEL AND BOUNDARY FUNCTIONS IN CERTAIN ORLICZ SPACES

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Abstract. If the Poisson integral of the unit disc is replaced by its square root, it is known that normalized Poisson integrals of L^p and weak L^p boundary functions converge along approach regions wider than the ordinary nontangential cones, as proved by Rönning and the author, respectively. In this paper we characterize the approach regions for boundary functions in two general classes of Orlicz spaces. The first of these classes contains spaces L^Φ having the property $L^\infty \subset L^\Phi \subset L^p$, $1 \leqslant p < \infty$. The second contains spaces L^Φ that resemble L^p spaces.

Keywords: square root of the Poisson kernel, approach regions, almost everywhere convergence, maximal functions, Orlicz spaces

MSC 2000: 42B25, 42A99, 43A85

1. Introduction

Let $P(z,\varphi)$ be the standard Poisson kernel in the unit disc U,

$$P(z,\varphi) = \frac{1}{2\pi} \cdot \frac{1-|z|^2}{|z-\mathrm{e}^{\mathrm{i}\varphi}|^2}$$

where $z \in U$ and $\varphi \in \partial U = \mathbb{T} \cong (-\pi,\pi]$.

Let

$$Pf(z) = \int_{\mathbb{T}} P(z,\varphi) f(\varphi)\,\mathrm{d}\varphi,$$

the Poisson integral of $f \in C(\mathbb{T})$. Then $Pf(z) \to f(\theta)$ as $z \to \mathrm{e}^{\mathrm{i}\theta}$, as was first shown by Schwarz [12].

For any function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ let

$$(1) \qquad \mathcal{A}_h(\theta) = \{z \in U\colon\ |\arg z - \theta| \leqslant h(1-|z|)\}.$$

We refer to $\mathcal{A}_h(\theta)$ as the (natural) approach region determined by h at $\theta \in \mathbb{T}$. Note that, even though we use the word "region", we have not imposed any openness assumptions on $\mathcal{A}_h(\theta)$. It is natural, but not necessary, to think of h as an increasing and continuous function, with $h(t) \to 0$ as $t \to 0$. Later, we shall let $z \in U$ approach the boundary ($z \to \mathrm{e}^{\mathrm{i}\theta}$) within $\mathcal{A}_h(\theta)$. We may think of the function h as a parameter that measures the maximal admissible tangency a curve along which z approaches the boundary may have.

If we only assume that $f \in L^1(\mathbb{T})$, the convergence properties are different than in the case of continuous functions. Fatou [7] proved in 1906 that if $h(t) = \alpha t$, $\alpha > 0$, then $Pf(z) \to f(\theta)$ a.e. as $z \to \mathrm{e}^{\mathrm{i}\theta}$ and $z \in \mathcal{A}_h(\theta)$, i.e. the convergence is non-tangential. To prove this, one establishes a weak type $(1,1)$ estimate for the corresponding maximal operator. The result then follows via standard techniques. Littlewood [8] proved that the theorem, in a certain sense, is best possible:

Theorem (Littlewood, [8]). *Let $\gamma_0 \subset U \cup \{1\}$ be a simple closed curve, having a common tangent with the circle at the point 1. Let γ_θ be the rotation of γ_0 by the angle θ. Then there exists a bounded harmonic function f in U with the property that, for a.e. $\theta \in \mathbb{T}$, the limit of f along γ_θ does not exist.*

Littlewood's result has been generalized, in different directions. For example, given a curve $\gamma_0 \subset U \cup \{1\}$ that touches $\mathbb{T}$ tangentially at the point 1, Aikawa [1] constructs a bounded harmonic function f in U such that, for any point $\theta \in \mathbb{T}$, the limit $\lim\limits_{z \to \mathrm{e}^{\mathrm{i}\theta}} f(z)$ does not exist along the curve γ_θ, where γ_θ is the rotation of γ_0 by the angle θ.

It is worth noting that one could consider more general approach regions, not necessarily given in the form (1). This is done, for instance, in [9] by Nagel and Stein. The essence of that paper is to prove that, whereas tangential *curves* are not good for convergence (Littlewood), tangential *sequences* may be.

For a more complete treatise on the theorems and the general theory mentioned so far, see [6].

For $z = x + \mathrm{i}y$ let

$$L_z = \frac{1}{4}(1-|z|^2)^2(\partial_x^2 + \partial_y^2),$$

the hyperbolic Laplacian. Then

$$u(z) = P_\lambda f(z) = \int_{\mathbb{T}} P(z,\varphi)^{\lambda+1/2} f(\varphi)\,\mathrm{d}\varphi,$$

for $\lambda \geqslant 0$, defines a solution of the equation

$$L_z u = (\lambda^2 - \tfrac{1}{4})u.$$

In connection with representation theory of the group $SL(2,\mathbb{R})$, one uses the powers $P(z,\varphi)^{\mathrm{i}\alpha+1/2}$, $\alpha \in \mathbb{R}$, of the Poisson kernel.

We shall use the notation $f \lesssim g$, for positive functions f and g, if there exists a constant $C > 0$ such that $f \leqslant Cg$ at all points, and we write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

Since

$$P_0 1(z) \sim (1-|z|)^{1/2} \log \frac{1}{1-|z|},$$

as $|z| \to 1$, one sees that the one has to normalize P_0 in order to get boundary convergence ($P_0 1(z)$ does not converge to 1). Thus, the operator that we shall be concerned with is defined by

$$\mathcal{P}_0 f(z) = \frac{P_0 f(z)}{P_0 1(z)}.$$

For $\lambda > 0$ one has that

$$P_\lambda 1(z) \sim (1-|z|)^{1/2-\lambda},$$

and if one considers normalized λ-Poisson integrals for $\lambda > 0$, i.e. $\mathcal{P}_\lambda f(z) = P_\lambda f(z)/P_\lambda 1(z)$, the convergence properties are the same as for the ordinary Poisson integral. This is because the kernels essentially behave in the same way.

We summarise the known convergence results in the following table. It should be read from left to right as "For all $f \in$ [**Function space**] one has for almost all $\theta \in \mathbb{T}$ that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to \mathrm{e}^{\mathrm{i}\theta}$ and $z \in \mathcal{A}_h(\theta)$ [**Conv.**] [$\mathcal{A}_h(\theta)$ determined by]." In the table it is assumed that $1 \leqslant p < \infty$ and $1 < p_1 < \infty$, and

$$\sigma_k = \sup_{2^{-2^k} \leqslant s \leqslant 2^{-2^{k-1}}} \frac{h(s)}{s(\log 1/s)^{p_1}}.$$

By $L^{p,\infty}$ we mean weak L^p (standard notation).

A few comments are in order. First of all, the convergence for continuous functions is at all points, not only almost every point. This is because $\mathcal{P}_0$ is a convolution operator with a kernel which behaves like an approximate identity in $\mathbb{T}$.

The results for $L^p(\mathbb{T})$, for finite values of p, are proved via weak type (p,p) estimates for the corresponding maximal operators. To do this, in [11], Rönning uses a quite technical machinery. In [5], a significantly easier proof is given (relying basically only on Hölder's inequality), and the sharpness of the result is proved (without

Function space	Conv.	$\mathcal{A}_h(\theta)$ determined by	Ref.
$C(\mathbb{T})$	if	$h(t) = +\infty$	–
$L^1(\mathbb{T})$	iff	$\limsup_{t\to 0} \dfrac{h(t)}{t \log 1/t} < \infty$	[13]
$L^p(\mathbb{T})$	iff	$\limsup_{t\to 0} \dfrac{h(t)}{t(\log 1/t)^p} < \infty$	[11], [5]
$L^\infty(\mathbb{T})$	iff	$\limsup_{t\to 0} \dfrac{h(t)}{t^{1-\varepsilon}} = 0 \ \forall \varepsilon > 0$	[14]
$L^{p_1,\infty}(\mathbb{T})$	iff	$\sum_{k\geqslant 0} \sigma_k < \infty$	[3]

the assumption that h should be monotone, which Rönning assumed). Actually, it is proved that $M_0 f \leqslant (M_{\mathrm{HL}} f^p)^{1/p}$, where

$$M_0 f(\theta) = \sup_{\substack{|\arg z - \theta| < h(1-|z|) \\ |z| > 1/2}} |\mathcal{P}_0 f(z)|,$$

the relevant maximal operator, and M_{HL} is the classical Hardy-Littlewood maximal operator.

In $L^p(\mathbb{T})$ one concludes the proofs with a standard approximation argument with continuous functions, for which convergence is known to hold. However, this is not an option in the case of boundary functions in $L^\infty(\mathbb{T})$, since the continuous functions are not dense in this space. The result by Sjögren, [14], is therefore deeper in its nature. It relies on a theorem of Bellow and Jones, [2], "A Banach principle for L^∞". Basically, the Bellow-Jones result for L^∞ states that a.e. convergence is equivalent to continuity of the maximal operator at 0, when restricted to the unit ball in L^∞, in the topology of convergence in measure. Actually, what Sjögren had to show was that for all $\varepsilon > 0$ and all $\kappa > 0$ there exists $\delta > 0$ such that

$$\|f\|_1 < \delta \Rightarrow |\{\theta \in \mathbb{T} \colon\ M_0 f(\theta) > \varepsilon\}| < \kappa,$$

for any function f in the unit ball of L^∞, where M_0 is the maximal operator defined above. (It is easy to see that, in the unit ball in L^∞, the topology of convergence in measure is equivalent with the L^1-topology.)

In [3], the author used a method similar to Sjögren's to determine the approach regions for boundary functions in $L^{p,\infty}$ (weak L^p), $1 < p < \infty$. It relied on a Banach principle for $L^{p,\infty}$, proved in the paper.

The author has also, with essential help and an original idea from professor Mizuta, Hiroshima University, established a result for the corresponding "square root operator" in the half space $\mathbb{R}^{n+1}_+$ with boundary functions $f \in L^p(G)$, where $G \subset \mathbb{R}^n$ is nonempty, bounded and open. For this result, see [4].

To understand better the significant difference in approach regions for L^p and L^∞ we consider, in this paper, two distinct classes of Orlicz spaces L^Φ. Firstly, Orlicz spaces where $\log\Phi$ grows at least as some positive power, thus possessing the property that $L^\infty \subset L^\Phi \subset L^p$ for any $p \geqslant 1$. Secondly, Orlicz spaces that resemble L^p spaces. As a special case, with $\Phi(x) = x^p$, $L^\Phi = L^p$. To make this more precise, we shall now define these two classes of functions, ∇ and Δ, from which we then define the corresponding Orlicz spaces:

Definition 1. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing C^2-function with $\Phi(0) = 0$ and define $M(x) = \log\Phi'(x)$. Then, Φ is said to satisfy the ∇ *condition*, denoted $\Phi \in \nabla$, if the following conditions hold:
(i) $M'(x) > 0$ for all $x \in (0,\infty)$.
(ii) $M((0,\infty)) = \mathbb{R}$.
(iii) $\liminf\limits_{x\to\infty} M(2x)/M(x) = m_0 > 1$ (possibly $m_0 = \infty$).

We note immediately that the conditions in Definition 1 imply that, for sufficiently small $\alpha > 0$, one has

$$\lim_{x\to\infty} \frac{M(x)}{x^\alpha} = \infty. \tag{2}$$

The space L^Φ, $\Phi \in \nabla$, that we shall define below (Definition 3) does not depend on the behaviour of Φ close to 0. Thus, without loss of generality, we impose one further convenient assumption on M:

$$\int_0^1 xM'(x)\,\mathrm{d}x < \infty. \tag{3}$$

Definition 2. A function $\Phi\colon [0,\infty) \to [0,\infty)$ is said to satisfy the Δ *condition*, denoted $\Phi \in \Delta$, if the following conditions hold:
(i) $\Phi \in C^2(0,\infty)$ with $\Phi''(x) > 0$ for $x > 0$.
(ii) $\lim\limits_{x\to 0} \Phi(x) = \lim\limits_{x\to 0} \Phi'(x) = 0$.
(iii) $x\varphi'(x)/\varphi(x) \sim 1$, uniformly for $x > x_0$ for some $x_0 \geqslant 0$, where $\varphi(x) = \Phi'(x)$.

Definition 3. For $\Phi \in \nabla$ we define

$$L^\Phi = \{f \in L^1(\mathbb{T})\colon\ \Phi(c|f|) \in L^1(\mathbb{T}) \quad \text{for some } c > 0\}.$$

Definition 4. Let $\Phi \in \Delta$. For $f \in L^1(\mathbb{T})$ define $\|f\|_\Phi = \|\Phi(|f|)\|_1$ and let

$$L^\Phi = \{f \in L^1(\mathbb{T})\colon \ \|f\|_\Phi < \infty\}.$$

It is readily checked that L^Φ is a vector space, regardless of if $\Phi \in \nabla$ or $\Phi \in \Delta$. For further reading on Orlicz spaces, we refer to [10].

In this paper we shall prove the following two theorems:

Theorem 1. *Let $\Phi \in \nabla$ be given. Then, the following conditions are equivalent for any function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$:*

(i) *For any $f \in L^\Phi$ one has for almost all $\theta \in \mathbb{T}$ that $\mathcal{P}_0 f(z) \to f(\theta)$ a.e. as $z \to \mathrm{e}^{\mathrm{i}\theta}$ and $z \in \mathcal{A}_h(\theta)$.*

(ii) *$M\big(\frac{\log 1/t}{\log g(t)}\big)/\log g(t) \to \infty$ as $t \to \infty$ for all $C > 0$, where $g(t) = h(t)/t$.*

Theorem 2. *Let $\Phi \in \Delta$ be given. Then the following conditions are equivalent for any function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$:*

(i) *For any $f \in L^\Phi$ one has for almost all $\theta \in \mathbb{T}$ that $\mathcal{P}_0 f(z) \to f(\theta)$ a.e. as $z \to \mathrm{e}^{\mathrm{i}\theta}$ and $z \in \mathcal{A}_h(\theta)$.*

(ii) *$\limsup\limits_{t\to 0} g(t)/\Phi(\log 1/t) < \infty$, where $g(t) = h(t)/t$.*

We conclude this section with some examples of $\Phi \in \nabla$ and $\Phi \in \Delta$, indicating what condition (ii) in the theorems reduces to in these cases.

Let $L_1(x) = \log x$ and, for $n \geqslant 2$, let $L_n(x) = L_{n-1}(\log x)$.

The convergence condition (ii) in Theorem 1 and Theorem 2 only takes large arguments of M and Φ into account, respectively. Thus, it is clearly sufficient to know the order of magnitude of $M(x)$ and $\Phi(x)$ as $x \to \infty$.

Example 1 ($\Phi \in \nabla$)**.** Our first example is $M(x) \sim x^p$, $p > 0$, as $x \to \infty$. This example covers all spaces L^Φ, where $\Phi(x) \sim x^\alpha \exp[x^p]$ as $x \to \infty$, $\alpha \in \mathbb{R}$ and $p > 0$.

Since $M(x) \sim x^p$ as $x \to \infty$, we may (in this context) assume that $M(x) = x^p$. We now have

$$M\Big(C\frac{\log 1/t}{\log g(t)}\Big)\Big/\log g(t) = C^p\bigg(\frac{(\log 1/t)^{p/(p+1)}}{\log g(t)}\bigg)^{p+1}.$$

Clearly, this expression tends to ∞ (for all $C > 0$) if and only if

$$\frac{\log g(t)}{(\log 1/t)^{p/(p+1)}} \to 0,$$

as $t \to 0$. Note that the convergence is independent of $\alpha > 0$.

Obviously, there is no optimal approach region. Specific examples of admissible functions h determining $\mathcal{A}_h(\theta)$ are $h(t) = t\exp[C(\log 1/t)^s(L_n(1/t))^{s'}]$, for $0 < s < p/(p+1)$, $n \geqslant 2$ and arbitrary C, $s' > 0$.

Example 2 ($\Phi \in \nabla$)**.** In this example we assume that $M(x) \sim \exp[x^p]$, $p > 0$, as $x \to \infty$. As above, we may assume that we have equality, i.e. $M(x) = \exp[x^p]$. We get

$$\begin{aligned} M\Big(C\frac{\log 1/t}{\log g(t)}\Big)\Big/\log g(t) &= \exp\Big[\Big(C\frac{\log 1/t}{\log g(t)}\Big)^p - L_2(g(t))\Big] \\ &= \exp\Big[L_2(g(t))\Big(\Big(C\frac{\log 1/t}{L_2(g(t))^{1/p}\log g(t)}\Big)^p - 1\Big)\Big]. \end{aligned}$$

Clearly this expression tends to ∞ as $t \to 0$, for *all* $C > 0$, if and only if

$$\frac{\log 1/t}{L_2(g(t))^{1/p}\log g(t)} \to \infty$$

as $t \to 0$.

Again, there is no optimal approach region. Specific examples of admissible functions h determining $\mathcal{A}_h(\theta)$ are

$$h(t) = t\exp\Big[\frac{\log 1/t}{L_n(1/t)^{\alpha}L_2(1/t)^{1/p}}\Big],$$

where $\alpha \in (0,1)$ if $n = 1$ and $\alpha > 0$ if $n \geqslant 2$.

Example 3 ($\Phi \in \Delta$)**.** The natural example here is $\Phi(x) = x^p$, $p > 1$, which obviously gives $L^\Phi = L^p$. It is easily seen that we, in this case, recover the convergence result by Rönning. More generally, if $\Phi \in \Delta$, we have convergence along approach regions specified by $h(t) = Ct\Phi(\log 1/t)$, but not along any essentially wider approach regions. This should be compared to the result in Theorem 1, where in general no largest possible approach region exists.

2. Preliminaries, $\Phi \in \nabla$

In this section we assume that $\Phi \in \nabla$, without further notice. For $c, \beta > 0$ define $\varphi_{\beta,c}(x) = \beta\exp[M(cx)]$. Furthermore, let

- $\Phi_{\beta,c}(x) = \int_0^x \varphi_{\beta,c}(y)\,\mathrm{d}y$.
- $\psi_{\beta,c}(y) = (\varphi_{\beta,c})^{-1}(y)$.
- $\Psi_{\beta,c}(y) = \int_0^y \psi_{\beta,c}(t)\,\mathrm{d}t$.

For abbreviation, if $\beta = c = 1$, we write φ, Φ, ψ and Ψ instead of $\varphi_{1,1}$, $\Phi_{1,1}$, $\psi_{1,1}$ and $\Psi_{1,1}$, respectively.

Note that, if $\beta = c = 1$, this definition is in agreement with Definition 1, where $M(x) = \log\Phi'(x)$. The pair $(\Phi_{\beta,c}, \Psi_{\beta,c})$ is referred to as a *complementary pair*.

We shall make use of the following standard inequality:

Proposition (Young's inequality). *Let $(\Phi_{\beta,c}, \Psi_{\beta,c})$ be a complementary pair. Then*

$$xy \leqslant \Phi_{\beta,c}(x) + \Psi_{\beta,c}(y),$$

for any positive numbers x and y. Equality holds if and only if $x = \psi_{\beta,c}(y)$.

Lemma 1. *If $f \in L^{\Phi}$ then $\|f\|_1 \leqslant 2\pi \Phi_{1,c}^{-1}(\|\Phi_{1,c}(|f|)\|_1/(2\pi))$.*

P r o o f. Φ is convex, so the result is just a restatement of Jensen's inequality. $\square$

For the concluding approximation argument, in the proof of Theorem 1, we need

Lemma 2. *Assume that $f \in L^{\Phi}(\mathbb{T})$, i.e. assume that $\|\Phi_{1,c}(|f|)\|_1 < \infty$ for some $c > 0$. Then, for $\varepsilon > 0$ given, there exists $g \in L^{\infty}(\mathbb{T})$ such that $\|\Phi_{1,c}(|f - g|)\|_1 < \varepsilon$.*

P r o o f. Let $g(x) = f(x)\chi_{\{|f|<R\}}$ for sufficiently large $R > 0$. $\square$

Next, we prove an elementary lemma:

Lemma 3. *Assume that $\{a_k\}$ and $\{b_k\}$ are two sequences of positive numbers, such that $\lim\limits_{k\to\infty} a_k = 0$ and such that*

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \infty.$$

Then there exists subsequences $\{a_{k_i}\}$ and $\{b_{k_i}\}$ and a sequence $\{N_i\} \subset \mathbb{N}$ such that

$$\sum_i N_i a_{k_i} = \infty, \quad \textit{and} \quad \sum_i N_i b_{k_i} < \infty.$$

P r o o f. For $i \in \mathbb{N}$ choose $k_i \uparrow \infty$ such that $a_{k_i}/b_{k_i} > 2^i$ and $a_{k_i} < 1$. Now, choose $N_i \in \mathbb{N}$ such that $1 \leqslant N_i a_{k_i} < 2$. Then $\sum\limits_i N_i a_{k_i} \geqslant \sum\limits_i 1$ and $\sum\limits_i N_i b_{k_i} \lesssim \sum\limits_i 2^{-i}$. $\square$

The following proposition is a key observation, solving an extremal problem.

Proposition 1. *Let a, c and ε be given positive numbers. Let $g \in L^{\Psi}$ be a nonnegative function, not identically 0, supported in $[-a, a]$. Then there exists a nonnegative and measurable function $\tilde{f}$, supported in $[-a, a]$ and satisfying $\int_{\mathbb{T}} \tilde{f}(\varphi) g(\varphi)\,\mathrm{d}\varphi = \varepsilon$, such that, for all nonnegative functions f such that $\int_{\mathbb{T}} f(\varphi) g(\varphi)\,\mathrm{d}\varphi \geqslant \varepsilon$, one has that*

$$\int_{|\varphi|<a} \Phi_{1,c}(f(\varphi))\,\mathrm{d}\varphi \geqslant \int_{|\varphi|<a} \Phi_{1,c}(\tilde{f}(\varphi))\,\mathrm{d}\varphi.$$

Moreover, $\tilde{f}(\varphi) = \psi_{\beta,c}(g(\varphi))$, where $\beta > 0$ is the unique number determined by $\int_{|\varphi|<a} \psi_{\beta,c}(g(\varphi)) g(\varphi)\,\mathrm{d}\varphi = \varepsilon$.

Proof. By the Young inequality we have, for any $\beta > 0$, that

$$\int_{|\varphi|<a} f(\varphi) g(\varphi)\,\mathrm{d}\varphi \leqslant \int_{|\varphi|<a} \Phi_{\beta,c}(f(\varphi))\,\mathrm{d}\varphi + \int_{|\varphi|<a} \Psi_{\beta,c}(g(\varphi))\,\mathrm{d}\varphi,$$

where equality holds if and only if $f(\varphi) = \tilde{f}(\varphi) = \psi_{\beta,c}(g(\varphi))$. Choose $\beta > 0$ (uniquely) such that

$$\int_{|\varphi|<a} \tilde{f}(\varphi) g(\varphi)\,\mathrm{d}\varphi = \varepsilon.$$

For an arbitrary nonnegative function f with $\int_{\mathbb{T}} f(\varphi) g(\varphi)\,\mathrm{d}\varphi \geqslant \varepsilon$, we then have

$$\begin{aligned}\int_{|\varphi|<a} \Phi_{\beta,c}(f(\varphi))\,\mathrm{d}\varphi &\geqslant \int_{|\varphi|<a} f(\varphi) g(\varphi)\,\mathrm{d}\varphi - \int_{|\varphi|<a} \Psi_{\beta,c}(g(\varphi))\,\mathrm{d}\varphi \\ &\geqslant \varepsilon - \int_{|\varphi|<a} \Psi_{\beta,c}(g(\varphi))\,\mathrm{d}\varphi \\ &= \int_{|\varphi|<a} \Phi_{\beta,c}(\tilde{f}(\varphi))\,\mathrm{d}\varphi,\end{aligned}$$

which is equivalent to

$$\int_{|\varphi|<a} \Phi_{1,c}(f(\varphi))\,\mathrm{d}\varphi \geqslant \int_{|\varphi|<a} \Phi_{1,c}(\tilde{f}(\varphi))\,\mathrm{d}\varphi,$$

as desired. □

3. The proof of Theorem 1

Throughout this section we assume that $g(t) = h(t)/t \to \infty$ as $t \to 0$, without loss of generality.

Before turning to the proofs of the two implications, we introduce a suitable notation. If we write $t = 1 - |z|$ and $z = (1-t)\mathrm{e}^{\mathrm{i}\theta}$, then

$$\mathcal{P}_0 f(z) = R_t * f(\theta),$$

where the convolution is taken in $\mathbb{T}$ and

$$R_t(\theta) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t(2-t)}}{|(1-t)\mathrm{e}^{\mathrm{i}\theta} - 1|} \frac{1}{P_0 1(1-t)}.$$

Here $\theta \in \mathbb{T} \cong (-\pi, \pi]$, as before. We are interested only in small values of t, so we might as well assume from now on that $t < \frac{1}{2}$. Since $P_0 1(1-t) \sim \sqrt{t} \log 1/t$, the order of magnitude of R_t is given by

$$R_t(\theta) \sim Q_t(\theta) = \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta|}.$$

Now let τ_η denote the translation $\tau_\eta f(\theta) = f(\theta - \eta)$. Then the convergence condition (i) in Theorem 1 above means

$$\lim_{\substack{t \to 0 \\ |\eta| < h(t)}} \tau_\eta R_t * f(\theta) = f(\theta).$$

The relevant maximal operator for our problem is

$$M_0 f(\theta) = \sup_{\substack{|\arg z - \theta| < h(1-|z|) \\ |z| > 1/2}} |\mathcal{P}_0 f(z)|.$$

Notice that $M_0 f(\theta)$ is dominated by a constant times

$$Mf(\theta) = \sup_{\substack{|\eta| < h(t) \\ t < 1/2}} \tau_\eta Q_t * |f|(\theta). \tag{4}$$

3.1. Proof of (ii) $\Rightarrow$ (i)

Proof. Let $f \in L^\Phi$ and $\varepsilon > 0$ be given.

We may assume that $f \geqslant 0$, without loss of generality. Write

$$Q_t(\theta) = Q_t(\theta)\chi_{\{|\theta| \leqslant 2h(t)\}} + Q_t(\theta)\chi_{\{|\theta| > 2h(t)\}} = Q_t^1(\theta) + Q_t^2(\theta).$$

By letting

$$M_j f(\theta) = \sup_{\substack{|\eta|<h(t)\\ 0<t<1/2}} \tau_\eta Q_t^j * f(\theta),$$

$j \in \{1,2\}$, we get $Mf \leqslant M_1 f + M_2 f$ and hence

$$\{Mf > 2\varepsilon\} \subset \{M_1 f > \varepsilon\} \cup \{M_2 f > \varepsilon\}.$$

To deal with $M_2 f$, we observe that when $|\eta| < h(t)$

$$\tau_\eta Q_t^2(\theta) = \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta - \eta|} \chi_{\{|\theta-\eta|>2h(t)\}} \leqslant \frac{2}{\log 1/t} \cdot \frac{1}{t+|\theta|}.$$

The last expression is a decreasing function of $|\theta|$, whose integral in $\mathbb{T}$ is bounded uniformly in t. It is well known that convolution by such a function is controlled by the Hardy-Littlewood maximal operator M_{HL}, so that $M_2 f \leqslant C M_{\mathrm{HL}} f$. Since M_{HL} is of weak type $(1,1)$, we obtain

$$|\{M_2 f > \varepsilon\}| \leqslant C\varepsilon^{-1} \|f\|_1.$$

By invoking Lemma 1, we get

$$|\{M_2 f > \varepsilon\}| \leqslant \frac{C \cdot \Phi_{1,c}^{-1}(\|\Phi_{1,c}(f)\|_1/(2\pi))}{\varepsilon}. \tag{5}$$

Let us now turn our attention to M_1. Assume that $M_1 f(\theta) > \varepsilon$. Then there exists $t \in (0, \frac{1}{2})$ and $|\eta| < h(t)$ such that

$$\frac{1}{\log 1/t} \int_{|\varphi|<2h(t)} \frac{f(\theta - \eta - \varphi)}{t + |\varphi|} \, \mathrm{d}\varphi > \varepsilon.$$

It follows then, by Proposition 1, that

$$\int_{|\varphi|<2h(t)} \Phi_{1,c}(f(\theta-\eta-\varphi)) \, \mathrm{d}\varphi \geqslant \int_{|\varphi|<2h(t)} \Phi_{1,c}\Big(\psi_{\beta,c}\Big(\frac{1}{t+|\varphi|}\Big)\Big) \, \mathrm{d}\varphi, \tag{6}$$

where β is chosen such that

$$\int_{|\varphi|<2h(t)} \psi_{\beta,c}\Big(\frac{1}{t+|\varphi|}\Big) \cdot \frac{1}{t+|\varphi|} \, \mathrm{d}\varphi = \varepsilon \log \frac{1}{t}. \tag{7}$$

We shall now use (7) to get an estimate of the size of β. We have

$$\begin{aligned}
\varepsilon \log\frac{1}{t} &= \int_{|\varphi|<2h(t)} \psi_{\beta,c}\Big(\frac{1}{t+|\varphi|}\Big)\cdot\frac{1}{t+|\varphi|}\,\mathrm{d}\varphi \\
&= 2\int_{1/(t+2h(t))}^{1/t} \frac{\psi_{\beta,c}(y)}{y}\,\mathrm{d}y \\
&\leqslant 2\psi_{\beta,c}(1/t)\cdot\log\left(1+2g(t)\right) \\
&\leqslant C\psi_{\beta,c}(1/t)\cdot\log g(t),
\end{aligned}$$

so that

$$\frac{1}{\beta} \geqslant t\varphi_{1,c}\Big(C_\varepsilon \frac{\log 1/t}{\log g(t)}\Big). \tag{8}$$

Now, let $B(s)=\Phi_{1,c}(\psi_{1,c}(s))$. Then it is clear that B is increasing and $\lim\limits_{s\to\infty} B(s) = \infty$. For convenience, let I_t denote the interval $[-2h(t),2h(t)]$. We have

$$\begin{aligned}
\Big\|\Phi_{1,c}\Big(\psi_{\beta,c}\Big(\frac{1}{t+|\varphi|}\Big)\Big)\Big\|_{L^1(I_t)} &= \int_{I_t} B\Big(\frac{1}{\beta(t+|\varphi|)}\Big)\,\mathrm{d}\varphi \\
&\geqslant 4h(t)B\Big(\frac{1}{\beta(t+2h(t))}\Big) \\
&\geqslant 4h(t)B\Big(\frac{1}{3\beta h(t)}\Big).
\end{aligned}$$

We may now invoke (8) to get

$$\begin{aligned}
\Big\|\Phi_{1,c}\Big(\psi_{\beta,c}\Big(\frac{1}{t+|\varphi|}\Big)\Big)\Big\|_{L^1(I_t)} &\geqslant 4h(t)B\left(t\varphi_{1,c}\Big(C_\varepsilon\frac{\log 1/t}{\log g(t)}\Big)\Big/3h(t)\right) \\
&\geqslant 4h(t)B\left(C\exp\Big[M\Big(C_\varepsilon\frac{\log 1/t}{\log g(t)}\Big)-\log g(t)\Big]\right) \\
&\geqslant C(\varepsilon)h(t),
\end{aligned}$$

by condition (ii) in Theorem 1. Thus, we have

$$\frac{h(t)}{\|\Phi_{1,c}(\psi_{\beta,c}((t+|\varphi|)^{-1}))\|_{L^1(I_t)}} \leqslant C,$$

which gives, by (6),

$$h(t) \leqslant C\int_{I_t} \Phi_{1,c}(\tilde f(\varphi))\,\mathrm{d}\varphi \leqslant C\int_{I_t}\Phi_{1,c}(f(\theta-\eta-\varphi))\,\mathrm{d}\varphi.$$

To sum up, we have shown that for each θ with $M_1 f(\theta) > \varepsilon$ there exists a t such that the interval $J(\theta) = [\theta - 3h(t), \theta + 3h(t)]$ has the property

$$\int_{J(\theta)} \Phi_{1,c}(f(\varphi))\,\mathrm{d}\varphi \geqslant Ch(t).$$

A covering argument now yields a sequence (θ_i, t_i) with $M_1 f(\theta_i) > \varepsilon$ such that the corresponding intervals $J(\theta_i)$ are disjoint, and such that the union of the scaled intervals $J'(\theta_i) = [\theta_i - 10h(t_i), \theta_i + 10h(t_i)]$ covers the set $\{M_1 f > \varepsilon\}$. In particular we have

$$\|\Phi_{1,c}(f)\|_1 \geqslant \sum_i \int_{J(\theta_i)} \Phi_{1,c}(f(\varphi))\,\mathrm{d}\varphi \geqslant C \sum_i h(t_i).$$

Thus,

$$|\{M_1 f > \varepsilon\}| \leqslant \sum_i |J'(\theta_i)| \leqslant C \sum_i h(t_i) \leqslant C\|\Phi_{1,c}(f)\|_1.$$

It follows, from the above estimate and from (5), that

$$|\{Mf > 2\varepsilon\}| \leqslant C_1(\varepsilon)\|\Phi_{1,c}(f)\|_1 + C_2(\varepsilon)\Phi_{1,c}^{-1}(\|\Phi_{1,c}(f)\|_1/(2\pi)).$$

For each $\varepsilon > 0$ the right-hand side tends to 0 with $\|\Phi_{1,c}(f)\|_1$. By Lemma 2 we are done (approximation by bounded functions). $\square$

3.2. Proof of (i) $\Rightarrow$ (ii)

Proof. Assume that condition (ii) in Theorem 1 is false. We show that this implies that (i) is false too.

Assume that, for some $C_0 > 0$,

$$\liminf_{t\to 0} M\Big(C_0 \frac{\log 1/t}{\log g(t)}\Big)\Big/\log g(t) = A < \infty.$$

The claim now is that we may assume that

$$\liminf_{t\to 0} M\Big(C_0 \frac{\log 1/t}{\log g(t)}\Big)\Big/\log g(t) = A \in \Big(\frac{1}{4}, \frac{1}{2}\Big). \tag{9}$$

To see that we may assume that $A < \frac{1}{2}$ we note that, by the conditions we have on M, there is a number $m \in (0,1)$ such that $M(x) \leqslant mM(2x)$ for sufficiently large x. Thus we have

$$\liminf_{t\to 0} M\Big(2^{-N} C_0 \frac{\log 1/t}{\log g(t)}\Big)\Big/\log g(t) \leqslant m^N \liminf_{t\to 0} M\Big(C_0 \frac{\log 1/t}{\log g(t)}\Big)\Big/\log g(t) = m^N A.$$

By choosing $N = N(A)$ large enough, we can make $m^N A < \frac{1}{2}$. Thus, we can assume from now on that $A < \frac{1}{2}$.

To see that we may assume that $A > \frac{1}{4}$, note that if for some $t > 0$ we have

$$M\Big(C_0 \frac{\log 1/t}{\log g(t)}\Big) \Big/ \log g(t) \leqslant \frac{1}{4},$$

then we can clearly make $g(t)$ smaller so that the quotient above is greater than $\frac{1}{4}$, say, and still smaller than $\frac{1}{2}$. Then the corresponding approach region for the new function g (at any $\theta \in \mathbb{T}$) is a subset of the original one, and it suffices to disprove convergence in the new one.

Pick a decreasing sequence $\{t_i\}_1^\infty$, converging to 0, such that

$$M\Big(C_0 \frac{\log 1/t_i}{\log g(t_i)}\Big) \Big/ \log g(t_i) \to A, \tag{10}$$

as $i \to \infty$. For convenience, let $s_i = C_0 \frac{\log 1/t_i}{\log g(t_i)}$. We may assume that $\{t_i\}_1^\infty$ is chosen such that

$$\frac{1}{4} \leqslant \frac{M(s_i)}{\log g(t_i)} \leqslant \frac{1}{2}, \tag{11}$$

for all $i \in \mathbb{N}$.

Let

$$f_i(\varphi) = \psi_{\beta_i,1}\Big(\frac{1}{t_i + |\varphi|}\Big) \cdot \chi_{\{|\varphi| < h(t_i)\}},$$

where $\beta_i^{-1} = t_i \varphi(s_i)$.

Note that $\Phi(x) \leqslant x \cdot \varphi(x)$, so that $\Phi(\psi_{\beta,1}(x)) \leqslant (x/\beta) \cdot \psi_{\beta,1}(x) = (x/\beta) \cdot \psi(x/\beta)$, and thus

$$\begin{aligned}
\|\Phi(f_i)\|_1 &\leqslant 2 \int_0^{h(t_i)} \frac{\psi\big((\beta_i(t_i+\varphi))^{-1}\big)}{\beta_i(t_i+\varphi)}\, \mathrm{d}\varphi \\
&= \frac{2}{\beta_i} \int_{1/(\beta_i(t_i+h(t_i)))}^{1/(\beta_i t_i)} \frac{\psi(y)}{y}\, \mathrm{d}y \\
&\leqslant 2t_i \cdot \frac{1}{\beta_i t_i} \int_0^{1/(\beta_i t_i)} \frac{\psi(y)}{y}\, \mathrm{d}y \\
&= 2t_i \cdot \varphi(s_i) \int_0^{\varphi(s_i)} \frac{\psi(y)}{y}\, \mathrm{d}y.
\end{aligned}$$

At this stage we make a change of variables, $y = \varphi(x)$, and use (3) to get

$$\begin{aligned}\|\Phi(f_i)\|_1 &\leqslant 2t_i \cdot \varphi(s_i) \int_0^{s_i} xM'(x)\,\mathrm{d}x \\ &\leqslant 2t_i \cdot \varphi(s_i) \Big(\int_0^1 xM'(x)\,\mathrm{d}x + \int_1^{s_i} xM'(x)\,\mathrm{d}x \Big) \\ &\leqslant 2t_i \cdot \varphi(s_i) \Big(C + s_i \int_1^{s_i} M'(x)\,\mathrm{d}x \Big) \\ &\leqslant 2t_i \cdot \varphi(s_i)(C + s_i M(s_i)) \leqslant Ct_i \cdot \varphi(s_i) \cdot s_i M(s_i).\end{aligned}$$

Now, using the above estimate, we get

$$\begin{aligned}\frac{h(t_i)}{\|\Phi(f_i)\|_1} &\geqslant C \frac{h(t_i)}{t_i \cdot \varphi(s_i) \cdot s_i M(s_i)} \\ &\geqslant \frac{C}{\log 1/t_i} \exp\left[\log g(t_i) - M(s_i)\right] \geqslant \frac{Cg(t_i)^{1/2}}{\log 1/t_i},\end{aligned}$$

the last two inequalities by (11). For all $t > 0$ sufficiently small, we have that

$$\frac{1}{2} \geqslant M\Big(C_0 \frac{\log 1/t}{\log g(t)}\Big) \Big/ \log g(t) \geqslant C_0^{\alpha} \frac{(\log 1/t)^{\alpha}}{(\log g(t))^{1+\alpha}},$$

for some sufficiently small $\alpha > 0$, by (11) and (2).

It follows that

$$\frac{h(t_i)}{\|\Phi(f_i)\|_1} \to \infty, \tag{12}$$

as $i \to \infty$.

It follows from (12), by Lemma 3, that we can pick a subsequence of $\{t_i\}$, with possible repetitions, for simplicity denoted $\{t_i\}$ also, such that

$$\sum_1^{\infty} h(t_i) = \infty, \tag{13}$$

and

$$\sum_1^{\infty} \|\Phi(f_i)\|_1 < \infty. \tag{14}$$

We shall now proceed with the construction of a function that disproves boundary convergence a.e. The idea is to distribute mass on $\mathbb{T}$ over and over again, sufficient to make the relevant Poisson integral larger than some positive constant, at all points in $\mathbb{T}$, and at the same time being able to make the function arbitrarily close to 0 on a set with positive measure.

Let $A_1 = h(t_1)$, and for $n \geqslant 2$ let $A_n = h(t_n) + \sum_{j=1}^{n-1} 2h(t_j)$. By (13) one has that $\lim_{n\to\infty} A_n = \infty$.

Define (on $\mathbb{T}$) $F_j(\varphi) = \tau_{A_j} f_j(\varphi)$, and let

$$F^{(N)}(\varphi) = \sup_{j\geqslant N} F_j(\varphi).$$

It is clear by construction that any given $\varphi \in \mathbb{T}$ lies in the support of infinitely many F_j:s.

Pointwise one obviously has that

$$\Phi(F^{(N)}(\varphi)) \leqslant \sum_{j=N}^{\infty} \Phi(F_j(\varphi)),$$

so that

$$\|\Phi(F^{(N)})\|_1 \leqslant \sum_{j=N}^{\infty} \|\Phi(F_j)\|_1 = \sum_{j=N}^{\infty} \|\Phi(f_j)\|_1 \to 0$$

as $N \to \infty$, by (14). Thus, in particular, $F^{(N)} \in L^\Phi$ for any $N \geqslant 1$.

For $\theta \in \mathbb{T}$ and a given $\xi_0 > 0$ we can, by construction, find $j \in \mathbb{N}$ so that $\theta \in \operatorname{supp}(F_j)$ and so that $t_j \in (0, \xi_0)$. We can then choose η, with $|\eta| < h(t_j)$, so that $\theta - \eta \equiv A_j \bmod 2\pi$. It follows that

$$\limsup_{t\to 0,\, |\eta|<h(t)} \mathcal{P}_0 F^{(N)}((1-t)\mathrm{e}^{\mathrm{i}(\theta-\eta)}) \geqslant \limsup_{j\to\infty} \mathcal{P}_0 F_j((1-t_j)\mathrm{e}^{\mathrm{i}A_j}).$$

We shall now conclude the proof by proving that the right-hand side above is always greater than some positive constant.

We have

$$\begin{aligned}
\mathcal{P}_0 F_j((1-t_j)\mathrm{e}^{\mathrm{i}A_j}) &\geqslant \frac{C}{\log 1/t_j} \int_{|\varphi|<h(t_j)} \frac{F_j(A_j-\varphi)}{t_j+|\varphi|}\,\mathrm{d}\varphi = \frac{C}{\log 1/t_j} \int_{|\varphi|<h(t_j)} \frac{f_j(\varphi)}{t_j+|\varphi|}\,\mathrm{d}\varphi \\
&= \frac{C}{\log 1/t_j} \int_0^{h(t_j)} \frac{\psi\big((\beta_j(t_j+\varphi))^{-1}\big)}{t_j+\varphi}\,\mathrm{d}\varphi \\
&= \frac{C}{\log 1/t_j} \int_{(\beta_j(t_j+h(t_j)))^{-1}}^{(\beta_j t_j)^{-1}} \frac{\psi(y)}{y}\,\mathrm{d}y \geqslant \frac{C}{\log 1/t_j} \int_1^{\varphi(s_j)} \frac{\psi(y)}{y}\,\mathrm{d}y.
\end{aligned}$$

In the last inequality, the lower limit $1/(\beta_j/(t_j+h(t_j)))$ can be replaced by 1, since by (11) we have

$$\beta_j(t_j+h(t_j)) \geqslant \beta_j h(t_j) = \exp[\log g(t_j) - M(s_j)] \geqslant \exp[\tfrac{1}{2}\log g(t_j)] \to \infty,$$

as $j \to \infty$.

We continue the estimate by making the change of variables $y = \varphi(x)$, and we get

$$\begin{aligned}\mathcal{P}_0 F_j((1-t_j)\mathrm{e}^{\mathrm{i}A_j}) &\geqslant \frac{C}{\log 1/t_j}\int_{\psi(1)}^{s_j} \frac{x\varphi'(x)}{\varphi(x)}\,\mathrm{d}x = \frac{C}{\log 1/t_j}\int_{\psi(1)}^{s_j} xM'(x)\,\mathrm{d}x\\ &\geqslant \frac{C}{\log 1/t_j}\int_{s_j/2}^{s_j} xM'(x)\,\mathrm{d}x\\ &\geqslant \frac{Cs_j}{\log 1/t_j}(M(s_j)-M(\tfrac{1}{2}s_j)).\end{aligned}$$

At this point we note that, by Definition 1 (iii), we have $M(s_j)-M(\frac{1}{2}s_j) \geqslant CM(s_j)$ for some positive constant C (depending only on m_0). We may now, finally, continue the estimate to get the desired conclusion. We have

$$\mathcal{P}_0 F_j((1-t_j)\mathrm{e}^{\mathrm{i}A_j}) \geqslant \frac{Cs_jM(s_j)}{\log 1/t_j} = \frac{CM(s_j)}{\log g(t_j)} \geqslant C_1,$$

the last inequality by (11).

To sum up, we have shown that for any $\theta \in \mathbb{T}$ one has

$$\limsup_{t\to 0,\,|\eta|<h(t)} \mathcal{P}_0 F^{(N)}((1-t)\mathrm{e}^{\mathrm{i}(\theta-\eta)}) \geqslant C_1. \tag{15}$$

Take N so large so that $\lambda_{F^{(N)}}(\frac{1}{2}C_1) < \pi$, say, and a.e. convergence is disproved. □

4. The proof of Theorem 2

In this section we assume that $\Phi \in \Delta$, without further notice. We use basically the same notation as we did in the proof of Theorem 1, and we shall carry out only those calculations that differ from that proof. Remember that the parameter c should have the value 1 when applying the other proof to this. The results from Section 2 are easily seen to remain true for $\Phi \in \Delta$ (again with $c = 1$).

For $\beta > 0$, let $\Phi_\beta(x) = \beta\Phi(x)$. Furthermore, let

- $\varphi_\beta(x) = \Phi'_\beta(x)$.
- $\psi_\beta(y) = (\varphi_\beta)^{-1}(y)$.
- $\Psi_\beta(y) = \int_0^y \psi_\beta(t)\,\mathrm{d}t$.

(Φ_β, Ψ_β) is referred to as a *complementary pair*, as before.

For short, if $\beta = 1$, we write φ, Φ, ψ and Ψ instead of φ_1, Φ_1, ψ_1 and Ψ_1, respectively.

Lemma 4. *Assume that $\Phi \in \Delta$. Then the following hold, uniformly in (x_0, ∞):*
(i) *$\varphi(2x) \sim \varphi(x)$ and $\Phi(2x) \sim \Phi(x)$.*
(ii) *$\Phi(x) \sim x\varphi(x)$.*
(iii) *$\int_0^x \psi(y)/y \,\mathrm{d}y \sim \psi(x)$.*

P r o o f. To prove the first part of (i), note that

$$\log \frac{\varphi(2x)}{\varphi(x)} = \int_x^{2x} \frac{\varphi'(t)}{\varphi(t)} \,\mathrm{d}t \sim \int_x^{2x} \frac{\mathrm{d}t}{t} \sim 1,$$

and the statement follows. If we can establish (ii), then the second part of (i) follows by the same techniques used to prove the first part. We have

$$\Phi(x) = \int_0^x \varphi(t) \,\mathrm{d}t \sim \int_0^x t\varphi'(t) \,\mathrm{d}t = x\varphi(x) - \Phi(x),$$

and thus $\Phi(x) \sim x\varphi(x)$, so (ii) is proved. Statement (iii) is trivial, via the change of coordinates given by $y = \varphi(t)$. $\square$

4.1. Proof of (ii) $\Rightarrow$ (i)

P r o o f. All we need to prove, according to the proof of Theorem 1, is that

$$\frac{h(t)}{\|\Phi(\psi_\beta((t + |\varphi|)^{-1}))\|_{L^1(I_t)}} \leqslant C. \tag{16}$$

In fact, all we need to do to show this, is to estimate β slightly differently. Here we have

$$\varepsilon \log 1/t = 2\int_{1/(t+2h(t))}^{1/t} \frac{\psi_\beta(y)}{y} \,\mathrm{d}y \leqslant 2\int_0^{1/t} \frac{\psi_\beta(y)}{y} \,\mathrm{d}y \lesssim \psi_\beta(1/t),$$

the last inequality by Lemma 4 (iii), so that

$$\frac{1}{\beta} \geqslant t\varphi(C_\varepsilon \log 1/t). \tag{17}$$

Now, let $B(s) = \Phi(\psi(s))$. Then, by Lemma 4 (ii), we have $B(s) \sim s\psi(s)$. For convenience, let I_t denote the interval $[-2h(t), 2h(t)]$. We have

$$\begin{aligned}\Big\|\Phi\Big(\psi_\beta\Big(\frac{1}{t + |\varphi|}\Big)\Big)\Big\|_{L^1(I_t)} &= \int_{I_t} B\Big(\frac{1}{\beta(t + |\varphi|)}\Big) \,\mathrm{d}\varphi \\ &\sim C\int_{I_t} \psi_\beta\Big(\frac{1}{t + |\varphi|}\Big) \cdot \frac{1}{\beta(t + |\varphi|)} \,\mathrm{d}\varphi = \frac{C\varepsilon \log 1/t}{\beta},\end{aligned}$$

the last equality by (7). We may now invoke (17) to get

$$
\begin{aligned}
\Big\|\Phi\Big(\psi_\beta\Big(\frac{1}{t+|\varphi|}\Big)\Big)\Big\|_{L^1(I_t)} &\geqslant C_1(\varepsilon)t\log\frac{1}{t}\varphi\Big(C_\varepsilon\log\frac{1}{t}\Big)\\
&\geqslant C_2(\varepsilon)t\Phi\Big(C_\varepsilon\log\frac{1}{t}\Big)\\
&\sim C_3(\varepsilon)t\Phi\Big(\log\frac{1}{t}\Big),
\end{aligned}
$$

where we have used Lemma 4 (i) and (ii). Thus, by assumption (ii) in Theorem 2, the desired inequality (16) follows. □

4.2. Proof of (i) ⇒ (ii)

Proof. Assume that condition (ii) in Theorem 2 is false. We show that this implies that (i) is false too.

Pick a decreasing sequence $\{t_i\}_1^\infty$, converging to 0, such that

$$
\frac{g(t_i)}{\Phi(\log 1/t_i)}\to\infty, \tag{18}
$$

as $i\to\infty$. Let $s_i=\log 1/t_i$, and define

$$
f_i(\varphi)=\psi_{\beta_i}\Big(\frac{1}{t_i+|\varphi|}\Big)\cdot\chi_{\{|\varphi|<h(t_i)\}},
$$

where $\beta_i^{-1}=t_i\varphi(s_i)$.

Using $\Phi(\psi_\beta(x))\sim(x/\beta)\cdot\psi_\beta(x)$, we get

$$
\begin{aligned}
\|f_i\|_\Phi &\lesssim \int_0^{h(t_i)}\frac{\psi_{\beta_i}\big((t_i+\varphi)^{-1})\big)}{\beta_i(t_i+\varphi)}\,\mathrm{d}\varphi\\
&=\frac{1}{\beta_i}\int_{1/(\beta_i(t_i+h(t_i)))}^{1/(\beta_i t_i)}\frac{\psi(y)}{y}\,\mathrm{d}y\\
&\leqslant t_i\cdot\varphi(s_i)\int_0^{\varphi(s_i)}\frac{\psi(y)}{y}\,\mathrm{d}y\\
&\lesssim t_i\cdot\varphi(s_i)s_i\lesssim t_i\cdot\Phi(s_i).
\end{aligned}
$$

Now, using the above estimate, we get

$$
\frac{h(t_i)}{\|f_i\|_\Phi}\geqslant C\frac{g(t_i)}{\Phi(s_i)}.
$$

Thus, by (18), we have

$$\frac{h(t_i)}{\|f_i\|_\Phi} \to \infty,$$

as $i \to \infty$.

Copying the proof of Theorem 1, we now see that it suffices to prove that

$$\frac{1}{\log 1/t_j} \int_{\psi(1)}^{s_j} \frac{x\varphi'(x)}{\varphi(x)}\,\mathrm{d}x \geqslant C,$$

for some constant $C > 0$, to disprove convergence. However, by Definition 2 (iii), we have

$$\frac{1}{\log 1/t_j} \int_{\psi(1)}^{s_j} \frac{x\varphi'(x)}{\varphi(x)}\,\mathrm{d}x \geqslant \frac{1}{\log 1/t_j} \int_0^{s_j} C_0\,\mathrm{d}x = C_0.$$

We are done. □

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