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WEIGHTED ENDPOINT ESTIMATES FOR COMMUTATORS OF FRACTIONAL INTEGRALS

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Abstract. Given α, $0 < \alpha < n$, and $b \in \mathrm{BMO}$, we give sufficient conditions on weights for the commutator of the fractional integral operator, $[b, I_\alpha]$, to satisfy weighted endpoint inequalities on $\mathbb{R}^n$ and on bounded domains. These results extend our earlier work [3], where we considered unweighted inequalities on $\mathbb{R}^n$.

Keywords: fractional integrals, commutators, BMO, weights, Orlicz spaces, maximal functions

MSC 2000: 42B20, 42B25

1. Introduction

Given α, $0 < \alpha < n$, define the fractional integral operator I_α by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \,\mathrm{d}y;$$

for $b \in \mathrm{BMO}$, define the commutator $[b, I_\alpha]$ by

$$[b, I_\alpha] f(x) = b(x) I_\alpha f(x) - I_\alpha(bf)(x) = \int_{\mathbb{R}^n} (b(x) - b(y)) \frac{f(y)}{|x-y|^{n-\alpha}} \,\mathrm{d}y.$$

Commutators were first introduced by Chanillo [1] who proved L^p estimates, $1 < p < \infty$. In [3] we proved the following endpoint estimate:

$$|\{x \in \mathbb{R}^n \colon |[b, I_\alpha] f(x)| > t\}| \leqslant C \Psi\biggl(\int_{\mathbb{R}^n} B\Bigl(\|b\|_{\mathrm{BMO}} \frac{|f(x)|}{t}\Bigr) \,\mathrm{d}x\biggr),$$

where $B(t) = t \log(\mathrm{e} + t)$ and $\Psi(t) = [t \log(\mathrm{e} + t^{\alpha/n})]^{n/(n-\alpha)}$.

We initially conjectured that the corresponding weighted inequality was

$$w(\{x \in \mathbb{R}^n \colon |[b, I_\alpha]f(x)| > t\}) \leqslant C\Psi\Big(\int_{\mathbb{R}^n} B\Big(\|b\|_{\mathrm{BMO}}\frac{|f(x)|}{t}\Big)w(x)^{1/q}\,\mathrm{d}x\Big),$$

where $w \in A_1$ and $q = n/(n-\alpha)$. However, this conjecture proved to be false, and we gave a counterexample [3, Example 1.8].

At the time, we were not able to make a new conjecture. One difficulty we had was that locally, our original conjecture appeared to be true; the counterexample works because it exploits the decay of the weight at infinity. Careful consideration of this behavior yielded two results. First, we found the correct weighted endpoint inequality for the commutator on $\mathbb{R}^n$. Second, we showed that if we restrict ourselves to a bounded domain, then there is a sharper endpoint inequality which is even simpler than our original conjecture.

Theorem 1.1. *Given α, $0 < \alpha < n$, and a function $b \in \mathrm{BMO}$, let $B(t) = t\log(\mathrm{e}+t)$, $\Psi(t) = [t\log(\mathrm{e}+t^{\alpha/n})]^{n/(n-\alpha)}$, $\Theta(t) = t^{1-\alpha/n}\log(\mathrm{e}+t^{-\alpha/n})$, and $q = n/(n-\alpha)$. Then for each weight $w \in A_1$, there exists a constant C such that*

$$w(\{x \in \mathbb{R}^n \colon |[b, I_\alpha]f(x)| > t\}) \leqslant C\Psi\Big(\int_{\mathbb{R}^n} B\Big(\|b\|_{\mathrm{BMO}}\frac{|f(x)|}{t}\Big)\Theta(w(x))\,\mathrm{d}x\Big). \tag{1.1}$$

But, given any bounded domain $\Omega \subset \mathbb{R}^n$,

$$w(\{x \in \Omega \colon |[b, I_\alpha]f(x)| > t\}) \leqslant C\Big(\int_{\Omega} B\Big(\|b\|_{\mathrm{BMO}}\frac{|f(x)|}{t}\Big)w(x)^{1/q}\,\mathrm{d}x\Big)^q. \tag{1.2}$$

Remark 1.2. If we replace f by $f\chi_\Omega$, Ω unbounded, (1.1) yields a nominally more general result for unbounded domains. However, in both (1.1) and (1.2), the A_1 weights are still defined on all of $\mathbb{R}^n$.

Theorem 1.1 is best understood by comparing inequalities (1.1) and (1.2) to the weighted endpoint inequality for the fractional integral operators due to Muckenhoupt and Wheeden [7]. They showed that if $w \in A_1$, then

$$w(\{x \in \mathbb{R}^n \colon |I_\alpha f(x)| > t\}) \leqslant C\Big(\frac{1}{t}\int_{\mathbb{R}^n} |f(x)|w(x)^{1/q}\,\mathrm{d}x\Big)^q. \tag{1.3}$$

Intuitively, Theorem 1.1 shows that the commutator $[b, I_\alpha]$ is more singular than I_α itself, and that its singularity is worse at infinity.

Theorem 1.1 should also be compared to the analogous result for commutators of singular integral operators (formally corresponding to the case $\alpha = 0$) due to Pérez [9]. If T is a singular integral operator, $b \in \mathrm{BMO}$ and $w \in A_1$, then

$$w(\{x \in \mathbb{R}^n :\ |[b,T]f(x)| > t\}) \leqslant C \int_{\mathbb{R}^n} B\Big(\|b\|_{\mathrm{BMO}} \frac{|f(x)|}{t}\Big) w(x)\,\mathrm{d}x.$$

The proof of Theorem 1.1 follows the same outline as the proof of the unweighted result on $\mathbb{R}^n$ [3, Theorem 1.1]. The first step is to prove that the associated Orlicz fractional maximal operator,

$$M_{\alpha,B}f(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \|f\|_{B,Q},$$

satisfies inequalities analogous to (1.1) and (1.2).

Theorem 1.3. *With the same notation and hypotheses as in Theorem* 1.1, *we have that*

$$w(\{x \in \mathbb{R}^n :\ M_{\alpha,B}f(x) > t\}) \leqslant C\Psi\Big(\int_{\mathbb{R}^n} B\Big(\frac{|f(x)|}{t}\Big)\Theta(w(x))\,\mathrm{d}x\Big) \tag{1.4}$$

and

$$w(\{x \in \Omega :\ M_{\alpha,B}f(x) > t\}) \leqslant C\Big(\int_{\Omega} B\Big(\frac{|f(x)|}{t}\Big) w(x)^{1/q}\,\mathrm{d}x\Big)^q. \tag{1.5}$$

We prove Theorem 1.3 in Section 2 below; since we can do so with essentially no more work, we prove a generalization that holds for a large class of Young functions.

The proof of Theorem 1.1 now proceeds as in the unweighted case. Here we sketch the main steps, and we refer the reader to [3] for details. On the left-hand side of (1.1) (or (1.2)) we replace $|[b, I_\alpha]f(x)|$ by $M^d([b, I_\alpha]f)(x)$, where M^d is the dyadic maximal operator. We then use the good-λ inequality relating M^d and the sharp maximal operator $M^{\#}$ (Lemma 6.1 in [3], which remains true in the weighted case since $w \in A_1$) to replace this with $M^{\#}([b, I_\alpha]f)(x)$. Next, we apply the inequality

$$M^{\#}([b, I_\alpha]f)(x) \leqslant C\|b\|_{\mathrm{BMO}}[I_\alpha f(x) + M_{\alpha,B}f(x)]$$

(Theorem 1.3 in [3]), which then reduces the estimate to endpoint inequalities for I_α and $M_{\alpha,B}$. To complete the proof we apply inequality (1.3) and Theorem 1.3, and use the fact that $w^{1/q} \leqslant \Theta(w)$.

2. Endpoint results for Orlicz fractional maximal operators

In this section we state and prove two endpoint inequalities for the Orlicz fractional maximal operator. Theorem 1.3 will be an immediate consequence of these results. Hereafter we will assume that the reader is familiar with the basic facts about Orlicz spaces, maximal operators and Muckenhoupt A_p weights, and we refer the reader to [4], [5], [6], [10] for further information. Also, we will draw heavily on our work [3], and we urge the reader to consult that paper.

Remark 2.1. The conclusions of our main results in this section, Theorems 2.3 and 2.5, remain true if $\alpha = 0$. (See Pérez [8].) However, to avoid technical difficulties, we restrict ourselves to $\alpha > 0$.

To state our results, we need one definition.

Definition 2.2. Given an increasing function φ, define a function h_φ by

$$h_\varphi(s) = \sup_{t>0} \frac{\varphi(st)}{\varphi(t)}, \quad 0 \leqslant s < \infty.$$

The function h_φ could be infinite if $s > 1$, but if φ is doubling, then it is finite for all $0 < s < \infty$. (See Maligranda [6, Theorem 11.7].) If φ is submultiplicative, then $h_\varphi \approx \varphi$. Also, for all $s, t > 0$, $\varphi(st) \leqslant h_\varphi(s)\varphi(t)$.

Theorem 2.3. *Given α, $0 < \alpha < n$, let B be a Young function such that $B(t)/t^{n/\alpha}$ is decreasing for all $t > 0$, and let $w \in A_1$. Then there exists a constant C depending only on B and the A_1 constant of w such that for all $t > 0$, $M_{\alpha,B}$ satisfies the modular weak-type inequality*

$$\Phi(w(\{x \in \mathbb{R}^n \colon M_{\alpha,B} f(x) > t\})) \leqslant C \int_{\mathbb{R}^n} B\Big(\frac{f(x)}{t}\Big) h_\Phi(w(x))\,\mathrm{d}x \tag{2.1}$$

for all non-negative $f \in L_B(\mathbb{R}^n)$, where

$$\Phi(s) = \begin{cases} 0 & \text{if } s = 0, \\ \dfrac{s}{h_B(s^{\alpha/n})} & \text{if } s > 0. \end{cases} \tag{2.2}$$

Inequality (1.4) follows easily from Theorem 2.3. Since B is submultiplicative, $h_B \approx B$, so

$$\Phi(t) \approx \frac{t^{1-\alpha/n}}{\log(\mathrm{e} + t^{\alpha/n})}.$$

The function Φ is invertible with

$$\Phi^{-1}(t) \approx \Psi(t) = [t \log(\mathrm{e} + t^{\alpha/n})]^{n/(n-\alpha)}.$$

Thus, inequality (2.1) yields (1.4), provided that

$$h_\Phi(t) \leqslant C\Theta(t) = Ct^{1-\alpha/n} \log(\mathrm{e} + t^{-\alpha/n}).$$

However, this follows from the definition: since h_B is submultiplicative and $h_B \approx B$,

$$h_\Phi(s) = \sup_{t>0} \frac{\Phi(st)}{\Phi(t)} = s \sup_{t>0} \frac{h_B(t^{\alpha/n})}{h_B((st)^{\alpha/n})} \leqslant s h_B(s^{-\alpha/n}) \leqslant CsB(s^{-\alpha/n}) = C\Theta(s).$$

Remark 2.4. Note that if we let $t = s^{-1}$ and use the fact that $h_B(t) \geqslant cB(t)$, we get $h_\Phi(s) \geqslant c\Theta(s)$. Hence, Θ is the best possible function we can get by this means.

Theorem 2.5. *Given a bounded domain $\Omega \subset \mathbb{R}^n$ and given α, $0 < \alpha < n$, let B be a Young function such that $B(t)/t^{n/\alpha}$ is decreasing for all $t > 0$. Suppose further that there exists r, $1 \leqslant r \leqslant n/\alpha$, such that $h_B(t) \leqslant Ct^r$ for all $t \leqslant 6\,\mathrm{diam}(\Omega)^\alpha$. Then for each $w \in A_1$ there exists a constant C, depending on B, $\mathrm{diam}(\Omega)$ and the A_1 constant of w, such that for all non-negative $f \in L_B(\Omega)$,*

$$w(\{x \in \Omega\colon\ M_{\alpha,B}f(x) > t\})^{1-r\alpha/n} \leqslant C \int_\Omega B\Big(\frac{f(x)}{t}\Big) w(x)^{1-r\alpha/n}\,\mathrm{d}x. \tag{2.3}$$

Inequality (1.5) is an immediate consequence of Theorem 2.5. Since $B(t) = t\log(\mathrm{e} + t)$, $h_B \approx B$, so if $t \leqslant 6\,\mathrm{diam}(\Omega)$, then $h_B(t) \leqslant C\log(\mathrm{e} + 6\,\mathrm{diam}(\Omega))t$. Thus inequality (2.3) holds with $r = 1$, and this yields (1.5).

Proof of Theorem 2.3. Our proof is very similar to the proof of Theorem 3.3 in [3], and we refer the reader there for many lemmas and technical details.

Fix a non-negative function f and $t > 0$. Define $E_t = \{x \in \mathbb{R}^n :\ M_{\alpha,B}f(x) > t\}$.

For each $x \in E_t$ there exists a cube $Q_x \ni x$ such that

$$|Q_x|^{\alpha/n} \|f\|_{B,Q_x} > t.$$

The collection $\{Q_x\}_{x\in E_t}$ covers E_t. Thus, by Lemma 3.14 in [3], there exists $\beta > 0$ and a collection of disjoint dyadic cubes $\{P_j\}$ such that $E_t \subset \bigcup\limits_j 3P_j$ and

$$|P_j|^{\alpha/n} \|f\|_{B,P_j} > \beta t.$$

By the properties of the Luxemburg norm on Orlicz spaces and by Definition 2.2,

$$1 < \frac{1}{|P_j|}\int_{P_j} B\Big(\frac{|P_j|^{\alpha/n} f(x)}{\beta t}\Big)\,\mathrm{d}x \tag{2.4}$$

$$\leqslant \frac{C h_B(|3P_j|^{\alpha/n})}{|3P_j|}\int_{P_j} B\Big(\frac{f(x)}{t}\Big)\,\mathrm{d}x$$

$$= \frac{C}{\Phi(|3P_j|)}\int_{P_j} B\Big(\frac{f(x)}{t}\Big)\,\mathrm{d}x.$$

The growth conditions assumed on B imply (see Lemma 3.12 in [3]) that

$$\Phi(w(E_t)) \leqslant \Phi\Big(\sum_j w(3P_j)\Big) \leqslant \sum_j \Phi(w(3P_j)).$$

Hence, if we combine the two inequalities above and apply Definition 2.2, we get

$$\Phi(w(E_t)) \leqslant C\sum_j \frac{\Phi(w(3P_j))}{\Phi(|3P_j|)}\int_{P_j} B\Big(\frac{f(x)}{t}\Big)\,\mathrm{d}x$$

$$\leqslant C\sum_j h_\Phi\Big(\frac{w(3P_j)}{|3P_j|}\Big)\int_{P_j} B\Big(\frac{f(x)}{t}\Big)\,\mathrm{d}x;$$

since $w \in A_1$ and the P_j's are disjoint,

$$\Phi(w(E_t)) \leqslant C\sum_j \int_{P_j} B\Big(\frac{f(x)}{t}\Big) h_\Phi(w(x))\,\mathrm{d}x$$

$$\leqslant C\int_{\mathbb{R}^n} B\Big(\frac{f(x)}{t}\Big) h_\Phi(w(x))\,\mathrm{d}x.$$

This completes the proof. □

The proof of Theorem 2.5 requires two lemmas.

Lemma 2.6. *Given α, $0<\alpha<n$, let B be a Young function such that $B(t)/t^{n/\alpha}$ is decreasing for all $t>0$. Then $h_B(s) \leqslant s^{n/\alpha}$ for all $s \geqslant 1$.*

Proof. Fix $s \geqslant 1$. Then for all $t>0$,

$$\frac{B(st)}{(st)^{n/\alpha}} \leqslant \frac{B(t)}{t^{n/\alpha}}, \quad \text{or equivalently,} \quad \frac{B(st)}{B(t)} \leqslant s^{n/\alpha}.$$

Taking the supremum over all t we get the desired inequality. □

Lemma 2.7. *Given α, $0<\alpha<n$, let B be a Young function such that $B(t)/t^{n/\alpha}$ is decreasing for all $t>0$. If Q and $\bar{Q}$ are cubes and f is a function such that $\operatorname{supp}(f)\subset Q\subset\bar{Q}$, then*

$$|\bar{Q}|^{\alpha/n}\|f\|_{B,\bar{Q}}\leqslant|Q|^{\alpha/n}\|f\|_{B,Q}.$$

Proof. Let $s=|\bar{Q}|/|Q|\geqslant 1$. Then by the definition of the Luxemburg norm and by Lemma 2.6,

$$\begin{aligned}|\bar{Q}|^{\alpha/n}\|f\|_{B,\bar{Q}}&=|Q|^{\alpha/n}\|s^{\alpha/n}f\|_{B,\bar{Q}}\\&=|Q|^{\alpha/n}\inf\Big\{\lambda>0\colon\ \frac{1}{|\bar{Q}|}\int_{\bar{Q}}B\Big(\frac{s^{\alpha/n}|f(x)|}{\lambda}\Big)\,\mathrm{d}x\leqslant 1\Big\}\\&\leqslant|Q|^{\alpha/n}\inf\Big\{\lambda>0\colon\ \frac{1}{|\bar{Q}|}\int_{Q}h_B(s^{\alpha/n})B\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x\leqslant 1\Big\}\\&\leqslant|Q|^{\alpha/n}\inf\Big\{\lambda>0\colon\ \frac{1}{|Q|}\int_{Q}B\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x\leqslant 1\Big\}\\&=|Q|^{\alpha/n}\|f\|_{B,Q}.\end{aligned}$$

□

Proof of Theorem 2.5. The proof of this result is nearly the same as the proof of Theorem 2.3. The major difference is that we must show that we can restrict the size of the cubes used to compute $M_{\alpha,B}$. Fix $x\in\Omega$, and let Q be any cube containing x. If $\ell(Q)\geqslant\operatorname{diam}(\Omega)$, then we can maximize the quantity $|Q|^{\alpha/n}\|f\|_{B,Q}$ by taking Q to be such that $\Omega\subset Q$. Further, by Lemma 2.7 we can increase this quantity by choosing Q such that $\ell(Q)\leqslant\operatorname{diam}(\Omega)$. Consequently, in computing $M_{\alpha,B}$ we can restrict ourselves to cubes whose sidelength is at most the diameter of Ω.

Thus, in the proof of Theorem 2.3, when we cover the set E_t by cubes Q_x, we may assume that each of these cubes has sidelength bounded by $\operatorname{diam}(\Omega)$. We then use Lemma 3.14 in [3] to replace these cubes by a collection of dyadic cubes P_j. But from the proof of this lemma (see [2], [8]) we see that the sidelength of each P_j is bounded by twice the largest sidelength of the Q_x's; therefore $\ell(P_j)\leqslant 2\operatorname{diam}(\Omega)$ for all j.

By assumption, $h_B(t)\leqslant Ct^r$ for $t\leqslant 6\operatorname{diam}(\Omega)^\alpha$. Therefore, $h_B(|3P_j|^{\alpha/n})\leqslant|3P_j|^{r\alpha/n}$ for all j. Hence, we can modify the argument that yielded (2.4) to get

$$1<\frac{C}{|3P_j|^{1-r\alpha/n}}\int_{P_j}B\Big(\frac{f(x)}{t}\Big)\,\mathrm{d}x.$$

If we now let $\Phi(t)=t^{1-r\alpha/n}$, then $h_\Phi(t)=\Phi(t)$ and we can repeat the remainder of the argument in the proof of Theorem 2.3 to get (2.3). □

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